



School of Engineering
Brown University

EN40: Dynamics and Vibrations

Solutions to Differential Equations of Motion for Vibrating Systems

Here, we summarize the solutions to the most important differential equations of motion that we encounter when analyzing single degree of freedom linear systems. The solutions are derived [here](#) if you are curious, but we won't ask you to derive the solutions in exams.

CASE I:
$$\frac{1}{\omega_n^2} \frac{d^2 x}{dt^2} + x = C$$

CASE II
$$\frac{1}{\alpha^2} \frac{d^2 x}{dt^2} - x = -C$$

CASE III
$$\frac{1}{\omega_n^2} \frac{d^2 x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = C$$

CASE IV
$$\frac{1}{\omega_n^2} \frac{d^2 x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = C + KF(t) \text{ with } F(t) = F_0 \sin \omega t$$

CASE V
$$\frac{1}{\omega_n^2} \frac{d^2 x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = C + K \left(y + \frac{2\zeta}{\omega_n} \frac{dy}{dt} \right) \text{ with } y = Y_0 \sin \omega t$$

CASE VI
$$\frac{1}{\omega_n^2} \frac{d^2 x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = C - \frac{K}{\omega_n^2} \frac{d^2 y}{dt^2} \text{ with } y = Y_0 \sin \omega t$$

CASE VII
$$\frac{1}{\omega_n^2} \frac{d^2 x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = K \left(\frac{\lambda^2}{\omega_n^2} \frac{d^2 y}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dy}{dt} + y \right) \text{ with } y = Y_0 \sin \omega t$$

SOLUTION 1:

The equation

$$\frac{1}{\omega_n^2} \frac{d^2 x}{dt^2} + x = C$$

with initial conditions

$$x = x_0 \quad \frac{dx}{dt} = v_0 \quad t = 0$$

has solution

$$x = C + X_0 \sin(\omega_n t + \phi)$$

$$X_0 = \sqrt{(x_0 - C)^2 + v_0^2 / \omega_n^2} \quad \phi = \tan^{-1} \left(\frac{(x_0 - C)\omega_n}{v_0} \right)$$

or, equivalently

$$x(t) = C + (x_0 - C) \cos \omega_n t + \frac{v_0}{\omega_n} \sin \omega_n t$$

SOLUTION 2

The equation

$$\frac{1}{\alpha^2} \frac{d^2 x}{dt^2} - x = -C$$

with initial conditions

$$x = x_0 \quad \frac{dx}{dt} = v_0 \quad t = 0$$

has solution

$$x(t) = C + \frac{1}{2} \left((x_0 - C) + \frac{v_0}{\alpha} \right) \exp(+\alpha t) + \frac{1}{2} \left((x_0 - C) - \frac{v_0}{\alpha} \right) \exp(-\alpha t)$$

SOLUTION 3

The equation

$$\frac{1}{\omega_n^2} \frac{d^2 x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = C$$

with initial conditions

$$x = x_0 \quad \frac{dx}{dt} = v_0 \quad t = 0$$

has the following solutions:

Case I: Overdamped System $\zeta > 1$

$$x(t) = C + \exp(-\zeta\omega_n t) \left\{ \frac{v_0 + (\zeta\omega_n + \omega_d)(x_0 - C)}{2\omega_d} \exp(\omega_d t) - \frac{v_0 + (\zeta\omega_n - \omega_d)(x_0 - C)}{2\omega_d} \exp(-\omega_d t) \right\}$$

where $\omega_d = \omega_n \sqrt{\zeta^2 - 1}$

Case II: Critically Damped System $\zeta = 1$

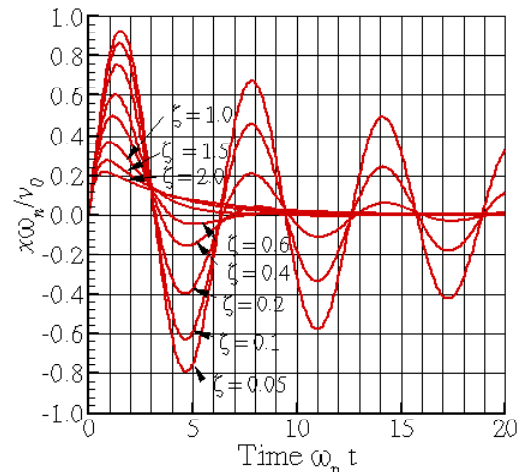
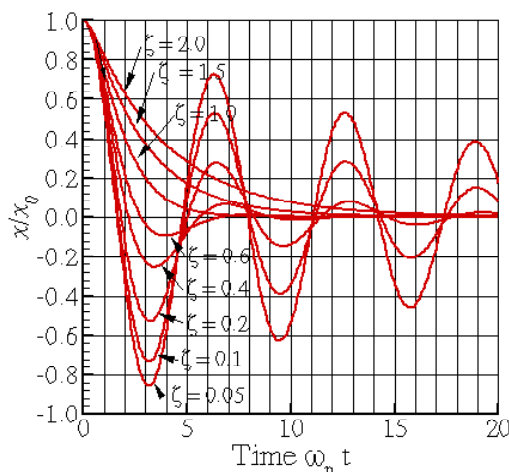
$$x(t) = C + \{(x_0 - C) + [v_0 + \omega_n(x_0 - C)]t\} \exp(-\omega_n t)$$

Case III: Underdamped System $\zeta < 1$

$$x(t) = C + \exp(-\zeta\omega_n t) \left\{ (x_0 - C) \cos \omega_d t + \frac{v_0 + \zeta\omega_n(x_0 - C)}{\omega_d} \sin \omega_d t \right\}$$

where $\omega_d = \omega_n \sqrt{1 - \zeta^2}$

The graphs below show $x(t)$ for two types of initial condition: the first graph shows results with $v_0 = 0$, while the second graph shows results with $x_0 = 0$. Both results are for $C=0$.



Graphs of solutions to ODE governing free vibration of a damped spring-mass system

SOLUTION 4:

$$\frac{1}{\omega_n^2} \frac{d^2 x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = C + KF(t) \quad \text{with } F(t) = F_0 \sin \omega t$$

and initial conditions

$$x = x_0 \quad \frac{dx}{dt} = v_0 \quad t = 0$$

has solution of the form

$$x(t) = C + x_h(t) + x_p(t)$$

where the **steady state solution** (or particular integral) is

$$x_p(t) = X_0 \sin(\omega t + \phi) \quad X_0 = KF_0 M(\omega / \omega_n, \zeta)$$

$$M(\omega / \omega_n, \zeta) = \frac{1}{\left\{ \left(1 - \omega^2 / \omega_n^2\right)^2 + \left(2\zeta\omega / \omega_n\right)^2 \right\}^{1/2}} \quad \phi = \tan^{-1} \left(\frac{-2\zeta\omega / \omega_n}{1 - \omega^2 / \omega_n^2} \right) \quad (-\pi < \phi < 0)$$

while the **transient solution** (or homogeneous solution, or complementary solution) is:

Case I: Overdamped System $\zeta > 1$

$$x_h(t) = \exp(-\zeta\omega_n t) \left\{ \frac{v_0^h + (\zeta\omega_n + \omega_d)x_0^h}{2\omega_d} \exp(\omega_d t) - \frac{v_0^h + (\zeta\omega_n - \omega_d)x_0^h}{2\omega_d} \exp(-\omega_d t) \right\}$$

where $\omega_d = \omega_n \sqrt{\zeta^2 - 1}$

Case II: Critically Damped System $\zeta = 1$

$$x_h(t) = \left\{ x_0^h + \left[v_0^h + \omega_n x_0^h \right] t \right\} \exp(-\omega_n t)$$

Case III: Underdamped System $\zeta < 1$

$$x_h(t) = \exp(-\zeta\omega_n t) \left\{ x_0^h \cos \omega_d t + \frac{v_0^h + \zeta\omega_n x_0^h}{\omega_d} \sin \omega_d t \right\}$$

where $\omega_d = \omega_n \sqrt{1 - \zeta^2}$

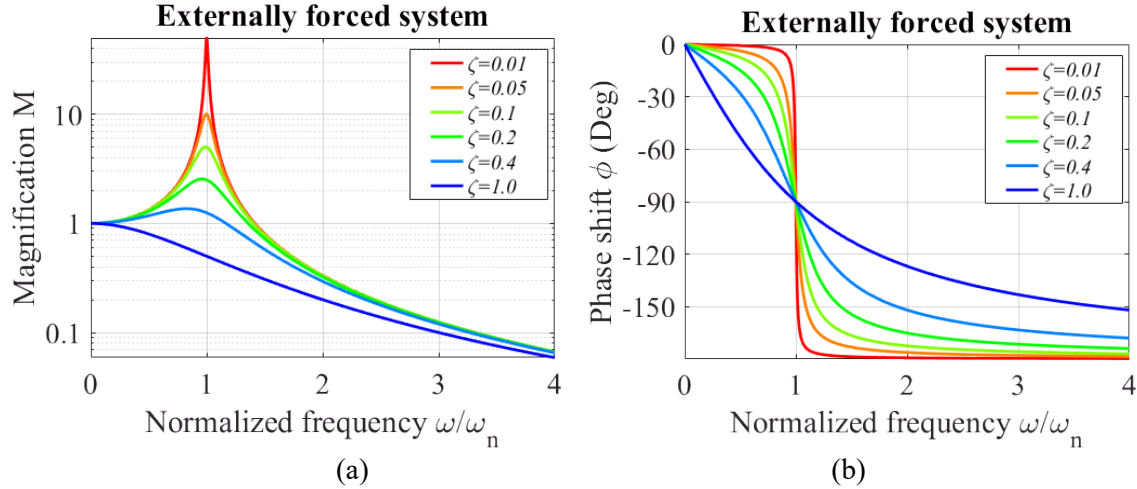
In all three preceding cases, we have set

$$x_0^h = x_0 - C - x_p(0) = x_0 - C - X_0 \sin \phi$$

$$v_0^h = v_0 - \frac{dx_p}{dt} \Big|_{t=0} = v_0 - X_0 \omega \cos \phi$$

Observe that for large time, the transient solution always decays to zero.

The graphs below plot the amplitude of the steady state vibration and the steady state phase lead.



Steady state response of a forced spring—mass system (a) amplitude and (b) phase

SOLUTION 5

The equation

$$\frac{1}{\omega_n^2} \frac{d^2 x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = C + K \left(y + \frac{2\zeta}{\omega_n} \frac{dy}{dt} \right) \text{ with } y(t) = Y_0 \sin \omega t$$

and initial conditions

$$x = x_0 \quad \frac{dx}{dt} = v_0 \quad t = 0$$

has solution of the form

$$x(t) = C + x_h(t) + x_p(t)$$

where the **steady state solution** (or particular integral) is

$$x_p(t) = X_0 \sin(\omega t + \phi) \quad X_0 = KY_0 M(\omega / \omega_n, \zeta)$$

$$M(\omega / \omega_n, \zeta) = \frac{\left\{ 1 + (2\zeta\omega / \omega_n)^2 \right\}^{1/2}}{\left\{ \left(1 - \omega^2 / \omega_n^2 \right)^2 + (2\zeta\omega / \omega_n)^2 \right\}^{1/2}} \quad \phi = \tan^{-1} \frac{-2\zeta\omega^3 / \omega_n^3}{1 - (1 - 4\zeta^2)\omega^2 / \omega_n^2} \quad (-\pi < \phi < 0)$$

while the **transient solution** (or homogeneous solution) is:

Case I: Overdamped System $\zeta > 1$

$$x_h(t) = \exp(-\zeta\omega_n t) \left\{ \frac{v_0^h + (\zeta\omega_n + \omega_d)x_0^h}{2\omega_d} \exp(\omega_d t) - \frac{v_0^h + (\zeta\omega_n - \omega_d)x_0^h}{2\omega_d} \exp(-\omega_d t) \right\}$$

where $\omega_d = \omega_n \sqrt{\zeta^2 - 1}$

Case II: Critically Damped System $\zeta = 1$

$$x_h(t) = \left\{ x_0^h + \left[v_0^h + \omega_n x_0^h \right] t \right\} \exp(-\omega_n t)$$

Case III: Underdamped System $\zeta < 1$

$$x_h(t) = \exp(-\zeta \omega_n t) \left\{ x_0^h \cos \omega_d t + \frac{v_0^h + \zeta \omega_n x_0^h}{\omega_d} \sin \omega_d t \right\}$$

where $\omega_d = \omega_n \sqrt{1 - \zeta^2}$

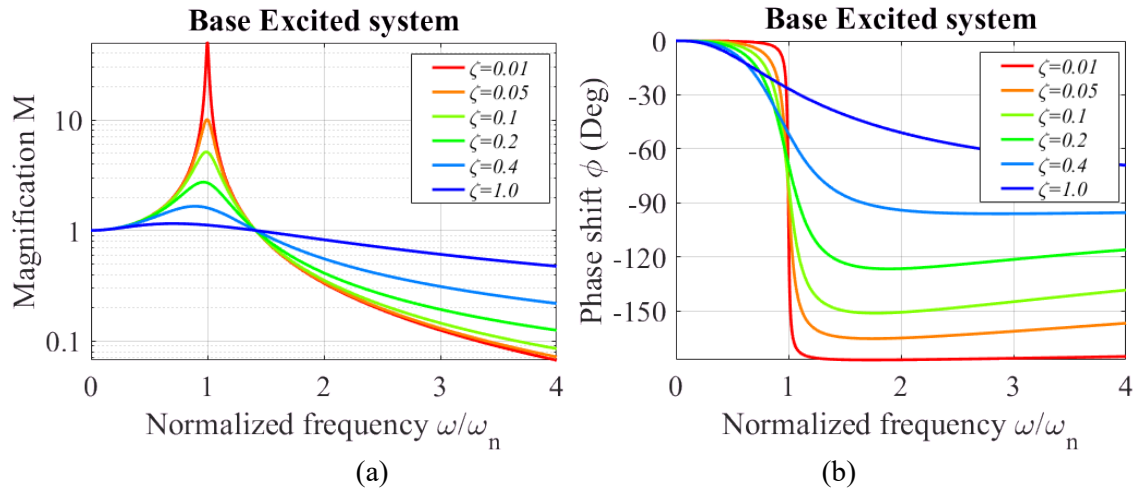
In all three preceding cases, we have set

$$x_0^h = x_0 - C - x_p(0) = x_0 - C - X_0 \sin \phi$$

$$v_0^h = v_0 - \left. \frac{dx_p}{dt} \right|_{t=0} = v_0 - X_0 \omega \cos \phi$$

Observe that for large time, the transient solution always decays to zero.

The graphs below show the steady state amplitude and phase



Steady state response of a base excited spring mass system (a) Amplitude and (b) Phase

SOLUTION 6

The equation

$$\frac{1}{\omega_n^2} \frac{d^2 x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = C - \frac{K}{\omega_n^2} \frac{d^2 y}{dt^2} \quad \text{with } y = Y_0 \sin \omega t$$

and initial conditions

$$x = x_0 \quad \frac{dx}{dt} = v_0 \quad t = 0$$

has solution of the form

$$x(t) = C + x_h(t) + x_p(t)$$

where the **steady state solution** (or particular integral) is

$$x_p(t) = X_0 \sin(\omega t + \phi) \quad X_0 = KY_0 M(\omega / \omega_n, \zeta)$$
$$M(\omega / \omega_n, \zeta) = \frac{\omega^2 / \omega_n^2}{\left\{ \left(1 - \omega^2 / \omega_n^2\right)^2 + \left(2\zeta\omega / \omega_n\right)^2 \right\}^{1/2}} \quad \phi = \tan^{-1} \frac{-2\zeta\omega / \omega_n}{1 - \omega^2 / \omega_n^2} \quad (-\pi < \phi < 0)$$

while the **transient solution** (or homogeneous solution) is:

Case I: Overdamped System $\zeta > 1$

$$x_h(t) = \exp(-\zeta\omega_n t) \left\{ \frac{v_0^h + (\zeta\omega_n + \omega_d)x_0^h}{2\omega_d} \exp(\omega_d t) - \frac{v_0^h + (\zeta\omega_n - \omega_d)x_0^h}{2\omega_d} \exp(-\omega_d t) \right\}$$

where $\omega_d = \omega_n \sqrt{\zeta^2 - 1}$

Case II: Critically Damped System $\zeta = 1$

$$x_h(t) = \left\{ x_0^h + \left[v_0^h + \omega_n x_0^h \right] t \right\} \exp(-\omega_n t)$$

Case III: Underdamped System $\zeta < 1$

$$x_h(t) = \exp(-\zeta\omega_n t) \left\{ x_0^h \cos \omega_d t + \frac{v_0^h + \zeta\omega_n x_0^h}{\omega_d} \sin \omega_d t \right\}$$

where $\omega_d = \omega_n \sqrt{1 - \zeta^2}$

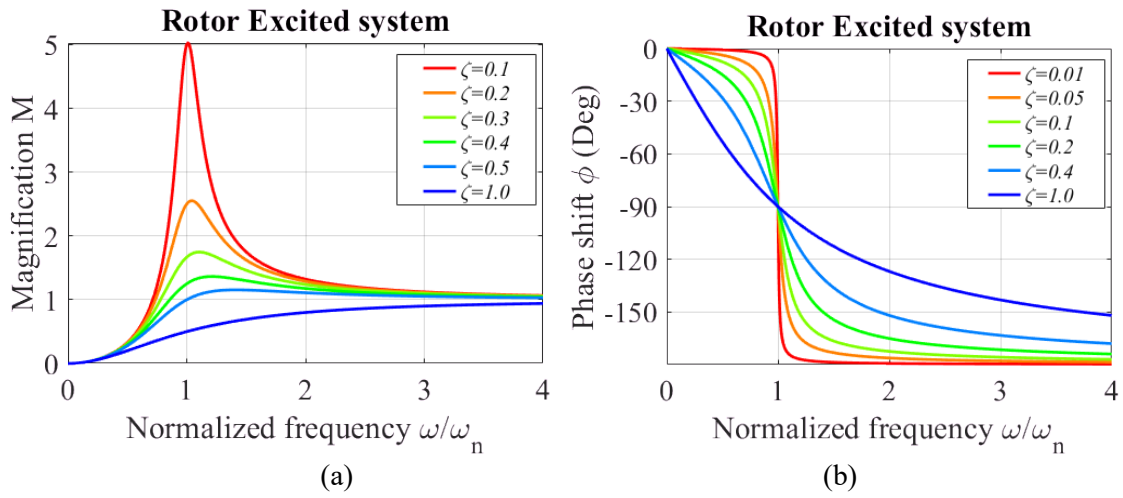
In all three preceding cases, we have set

$$x_0^h = x_0 - C - x_p(0) = x_0 - C - X_0 \sin \phi$$

$$v_0^h = v_0 - \left. \frac{dx_p}{dt} \right|_{t=0} = v_0 - X_0 \omega \cos \phi$$

Observe that for large time, the transient solution always decays to zero.

The graphs below show the steady state amplitude and phase lead for Case 6.



Steady state response of a rotor excited spring—mass system (a) Amplitude; (b) Phase

SOLUTION 7

The equation

$$\frac{1}{\omega_n^2} \frac{d^2 x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = C + K \left(\frac{\lambda^2}{\omega_n^2} \frac{d^2 y}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dy}{dt} + y \right) \quad \text{with } y = Y_0 \sin \omega t$$

and initial conditions

$$x = x_0 \quad \frac{dx}{dt} = v_0 \quad t = 0$$

has solution of the form

$$x(t) = C + x_h(t) + x_p(t)$$

where the **steady state solution** (or particular integral) is

$$x_p(t) = X_0 \sin(\omega t + \phi) \quad X_0 = KY_0 M(\omega / \omega_n, \zeta)$$

$$M(\omega / \omega_n, \zeta) = \frac{\left\{ \left(1 - \lambda^2 \omega^2 / \omega_n^2 \right)^2 + (2\zeta \omega / \omega_n)^2 \right\}^{1/2}}{\left\{ \left(1 - \omega^2 / \omega_n^2 \right)^2 + (2\zeta \omega / \omega_n)^2 \right\}^{1/2}}$$

$$\phi = \cos^{-1} \frac{1 - \lambda^2 \omega^2 / \omega_n^2}{\left\{ \left(1 - \lambda^2 \omega^2 / \omega_n^2 \right)^2 + (2\zeta \omega / \omega_n)^2 \right\}^{1/2}} - \cos^{-1} \frac{1 - \omega^2 / \omega_n^2}{\left\{ \left(1 - \omega^2 / \omega_n^2 \right)^2 + (2\zeta \omega / \omega_n)^2 \right\}^{1/2}} \quad (-\pi < \phi < 0)$$

while the **transient solution** (or homogeneous solution) is:

Case I: Overdamped System $\zeta > 1$

$$x_h(t) = \exp(-\zeta \omega_n t) \left\{ \frac{v_0^h + (\zeta \omega_n + \omega_d) x_0^h}{2\omega_d} \exp(\omega_d t) - \frac{v_0^h + (\zeta \omega_n - \omega_d) x_0^h}{2\omega_d} \exp(-\omega_d t) \right\}$$

where $\omega_d = \omega_n \sqrt{\zeta^2 - 1}$

Case II: Critically Damped System $\zeta = 1$

$$x_h(t) = \left\{ x_0^h + \left[v_0^h + \omega_n x_0^h \right] t \right\} \exp(-\omega_n t)$$

Case III: Underdamped System $\zeta < 1$

$$x_h(t) = \exp(-\zeta \omega_n t) \left\{ x_0^h \cos \omega_d t + \frac{v_0^h + \zeta \omega_n x_0^h}{\omega_d} \sin \omega_d t \right\}$$

where $\omega_d = \omega_n \sqrt{1 - \zeta^2}$

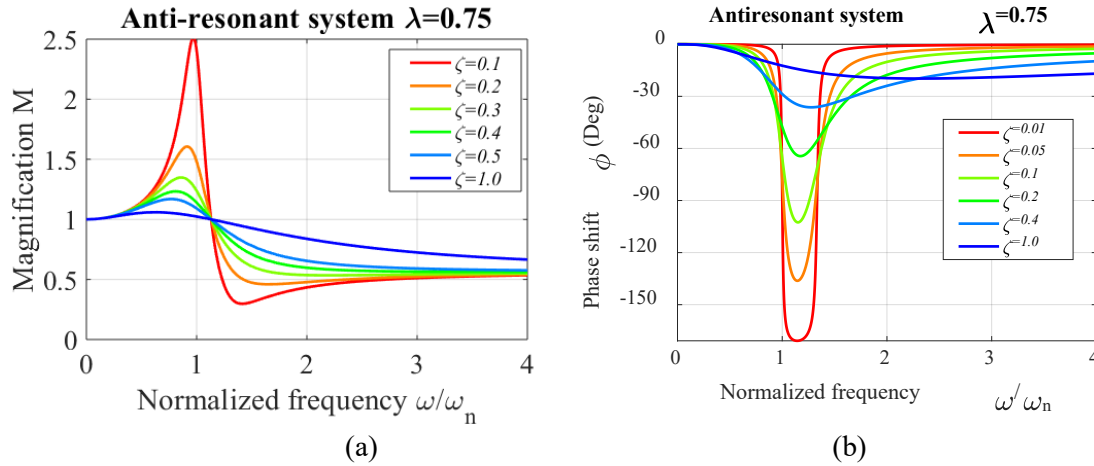
In all three preceding cases, we have set

$$x_0^h = x_0 - C - x_p(0) = x_0 - C - X_0 \sin \phi$$

$$v_0^h = v_0 - \left. \frac{dx_p}{dt} \right|_{t=0} = v_0 - X_0 \omega \cos \phi$$

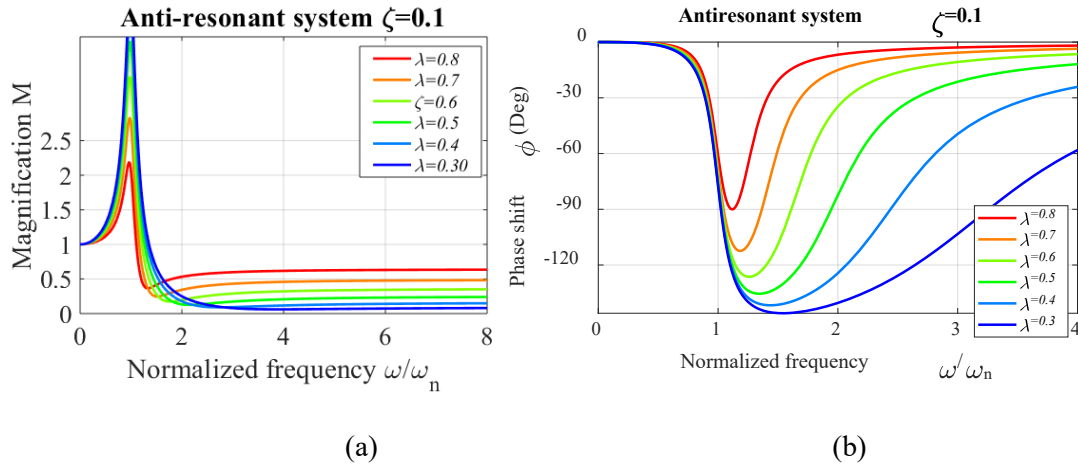
Observe that for large time, the transient solution always decays to zero.

The graphs below show the steady state amplitude and phase lead for Case 7.



Steady state response of an anti-resonant spring mass system as a function of damping ζ

(a) Amplitude; (b) Phase



Steady state response of an anti-resonant spring mass system as a function of lever ratio λ

(a) Amplitude; (b) Phase