EN4 Dynamics and Vibrations

Design Project

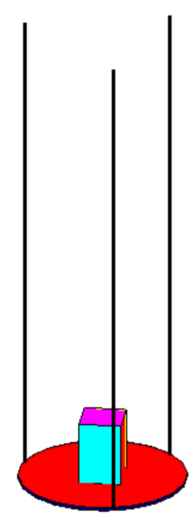
A Trifilar Pendulum to Measure Mass Moments of Inertia

Synopsis

You will design, build and test a device to measure the mass moment of inertia of a complex shaped object. The device consists of a 'trifilar pendulum' (a platform suspended from three cables), which is used to hold the test object. The mass moment of inertia of the object can be determined by measuring the vibration frequency of the pendulum. Your design process will involve (i) Deriving the formula that relates the vibration frequency to the mass moment of inertia; (ii) Designing the pendulum itself; (iii) Constructing the pendulum; (iv) Testing the accuracy of your design; and (v) using the design to measure mass moments of inertia of several objects with complex shapes.

- ***Schedule:***
  - ***April 5: Start analysis of Trifilar Pendulum (part of homework # 7)***
  - ***April 12: Submit analysis (KEEP A COPY)***
  - ***Week of April 14: Construction and testing***
  - ***April 19: Final design testing; report due.***

1. Introduction



Designing an instrument to make accurate measurements is among the most challenging engineering problems you are likely to face. This project will give you some experience in this kind of design problem, and will also provide a practical application of the course material on vibrations and rigid bodies.

Specifically, you will design a device to measure the mass moment of inertia I of a rigid body, and use the device to test a range of objects. The mass moment of inertia around an axis of rotation is a central quantity in the dynamics of rigid bodies. I is the rotational equivalent of mass; it represents the resistance of a rigid body to acceleration under action of an applied moment. The mass moment of inertia depends on the mass and the *shape* of the body, and so is far more difficult to determine than the mass alone. While I can be calculated, it is often easier to measure I experimentally, especially if the mass density or mass distribution within the object are not known or the shape is quite complex

The device to be designed is the "Trifilar Pendulum", shown in the figure. The trifilar pendulum consists of a platform with mass m_0 and moment of inertia $I_{platform}$ about its center of mass axis (CM). Three lightweight, uniformly spaced cables suspend this platform a distance L below a rigid support (not shown). Each of the cables is attached to the platform at a distance R from the platform CM. The object to be measured is placed on the platform with its center of mass directly above the CM of the platform. The platform is given a small initial rotation and released. The platform oscillates, and has a period τ. Measurement of the period τ can be used to deduce the total moment of inertia of the system (platform plus object).

2. Analysis of the trifilar pendulum (part of HW7 – repeated here as a reminder)

The first step of the design process is to determine the formula that relates the measured period of oscillation τ to the moment of inertia of the test object. As in all 'free vibration' problems, the approach will be to derive an equation of motion for the system, and arrange it into the form

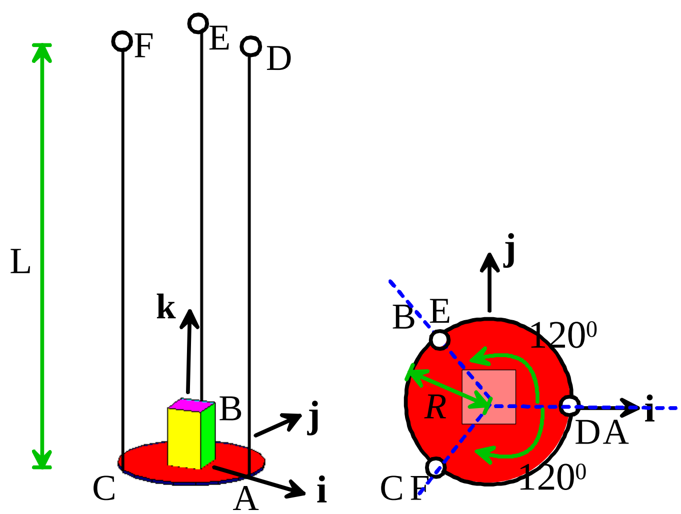
$$\frac{d^2x}{dt^2} + \omega_n^2 x = 0$$

Since we are solving a rigid body problem, this equation will be derived using Newton's law $\mathbf{F} = m\mathbf{a}_{COM}$, and the moment-angular acceleration relation $M\mathbf{k} = I_Z \alpha \mathbf{k}$. Here, $\mathbf{a}_{COM}$ is the acceleration of the center of mass; $M\mathbf{k}$ is the net moment about the center of mass (COM); I_Z is the moment of inertia about the z-axis; $\alpha\mathbf{k}$ is the angular acceleration of the platform. Note that, by symmetry, the center of mass is the center of the platform.

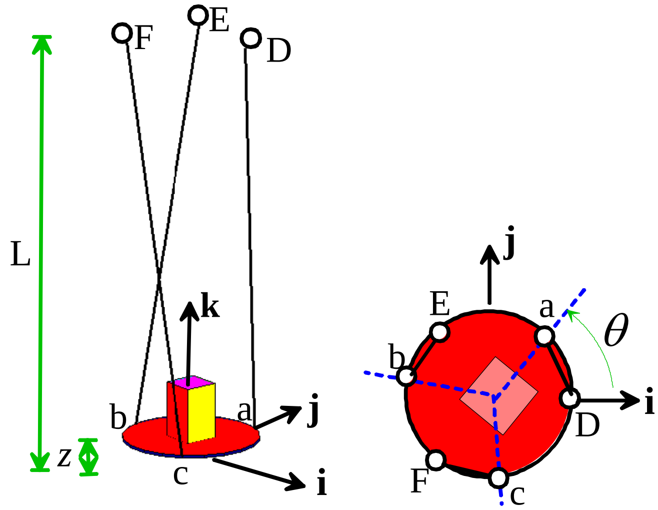
Before starting this problem, watch the animation posted on the EN40 projects page closely. Then, notice that

(i) The table is rotating about its center, without lateral motion

(ii) If you look closely at the platform, you will see that it moves up and down by a very small distance. The platform is at its lowest position when the cables are vertical.



(a) The figure above shows the system in its static equilibrium position. The three cables are vertical, and all have length L. The platform has radius R. Take the origin at the center of the disk in the static equilibrium configuration, and let $\{\mathbf{i,j,k}\}$ be a Cartesian basis as shown in the picture. Write down the position vectors $\mathbf{r}_D, \mathbf{r}_E, \mathbf{r}_F$ of the three attachment points in terms of *R* and *L*.



(b) Now, suppose that the platform rotates about its center through some angle θ, and also rises by a distance *z*, as shown in the figure. Write down the position vectors $\mathbf{r}_a, \mathbf{r}_b, \mathbf{r}_c$ of the three points where the cable is tied to the platform, in terms of *R*, *z* and θ.

(c) Assume that the cables do not stretch. Use the results of (a) and (b) to calculate the distance between *a* and *D*, and show that *z* and θ are related by the equation:

$$2R^2(1-\cos\theta)+z(z-2L)=0$$

Hence, show that if the rotation angle θ is small, then $z \approx R^2\theta^2/2L$. (Hint – use Taylor series). Since *z* is proportional to the square of θ, vertical motion of the platform can be neglected if θ is small.

(d) Write down formulas for unit vectors parallel to each of the deflected cables, in terms of *L*, $\mathbf{r}_a, \mathbf{r}_b, \mathbf{r}_c$ and $\mathbf{r}_D, \mathbf{r}_E, \mathbf{r}_F$. (It is not necessary to express the results in $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ components).

(e) Draw a free body diagram showing the forces acting on the platform and test object together.

(f) Assume that the tension has the same magnitude *T* in each cable. Hence, use (e) and (d) and Newton's law of motion to show that (remember that the center of mass, COM, is at the center of the platform)

$$m\left[a_x\mathbf{i}+a_y\mathbf{j}+\frac{d^2z}{dt^2}\mathbf{k}\right]=T\frac{\{(\mathbf{r}_D+\mathbf{r}_E+\mathbf{r}_F)-(\mathbf{r}_a+\mathbf{r}_b+\mathbf{r}_c)\}}{L}-mg\mathbf{k}$$

(g) Note that $(\mathbf{r}_a+\mathbf{r}_b+\mathbf{r}_c)/3$ is the average position of the three points where the cables connect to the platform. By inspection, this point must be at the center of the platform. Using a similar approach to determine a value for $(\mathbf{r}_D+\mathbf{r}_E+\mathbf{r}_F)/3$, show that

$$m\left[a_x\mathbf{i}+a_y\mathbf{j}+\frac{d^2z}{dt^2}\mathbf{k}\right]=\{3T(1-z/L)-mg\}\mathbf{k}$$

(h) For small θ, we can assume $z\approx 0,\ d^2z/dt^2=0$. Hence, find a formula for the cable tension *T*.

(i) Finally, consider rotational motion of the system. Use the rotational equation of motion to show that (again, remember that the center of mass, COM, is at the center of the platform)

$$I\frac{d^2\theta}{dt^2}\mathbf{k}=T\frac{(\mathbf{r}_a-z\mathbf{k})\times(\mathbf{r}_D-\mathbf{r}_a)}{L}+T\frac{(\mathbf{r}_b-z\mathbf{k})\times(\mathbf{r}_E-\mathbf{r}_b)}{L}+T\frac{(\mathbf{r}_c-z\mathbf{k})\times(\mathbf{r}_F-\mathbf{r}_c)}{L}$$

Either by using mupad to evaluate the cross products, (or if you are mupad-phobic try to find a clever way to evaluate the cross products by inspection – you might like to do this as a challenge even if you love mupad. Then again, you may prefer to have your wisdom teeth pulled.), show that

$$I\frac{d^2\theta}{dt^2}+\frac{3R^2T}{L}\sin\theta=0$$

(j) Hence, find a formula for the frequency of vibration of the system, in terms of m, g, R, L and *I*.

(k) Note that the total mass moment of inertia *I* consists of the inertia of the platform, together with that of the test object, i.e.

$$I=I_{platform}+I_{object}$$

The mass moment of inertia of the platform can be calculated by hand, or measured.
Write down the formula relating I_{object} to the period and $I_{platform}$

3. More sophisticated analysis

The calculation in the preceding section gives the formula that you will use to determine I_{object} . However, this is an approximate result. For example, you assumed that the center of mass of the object is at the center of the disk; you assumed the rotation angle was small, and so on. If you have a nervous disposition (most engineers are completely paranoid) you may want to check the accuracy of your analysis. To do this, we have provided a matlab code that solves the full 3D rigid body equations of motion for the trifilar pendulum. The code animates the motion of the system, and compares the predicted value of the mass moment of inertia of the test object with the actual value. You can edit the code to test the errors caused by various changes in the design. You could also read duBois *et al*, Experimental Mechanics, **49** (4), 1741 (2009). (As usual a first year design project at Brown is graduate research at other schools!)

4. Design of a Trifilar Pendulum

Based on the analysis in Section 2, you must now design a suitable platform system to maximize the accuracy in measuring I_{object} by measuring the period of vibration τ. There are two main aspects to the design:

1. You must make sure you can measure the period of the system sufficiently accurately. For instance, it would probably be quite difficult, given the kinds of measuring equipment available, to measure a very small period such as τ =0.0001s. You can measure the period using a stop watch, or designing the system to accommodate electronic timers that will be made available. Using a stop watch is preferred. Electronic timers are shared and must remain in Prince Lab.

2. You must make the device sensitive to the object of interest, so that you can deduce the moment of inertia of just the object, even though you are measuring the moment of inertia of both the platform and the object.

Additional issues to consider in making your design:

1. Accuracy of alignment of CM, for both platform and objects of interest.
2. Mass and size range of objects to study
3. Support of various shapes
4. Robustness of design

Although the figures in the previous section showed the platform as a flat solid disk hung by vertically strung cables, many other designs are possible. The platform need not be circular; it need not be solid; and the cables need not be vertical, for example (if you do splay the cables, you will need to account for this in your calculations). Since you may be testing spherical objects, you will have to have a way to support them in the device. They must not move relative to the platform during testing.

Your entire device will include the cables and a top support structure. During testing, this support structure will be clamped to another fixed platform in Prince Lab.

Design limitations:

a. The platform must hang no more than 1.5 meters below its support.
b. The pendulum can consist of the platform, cables, and supporting ceiling, and should be fully constructed when presented for the final testing.
c. During the final testing, the supporting structure will be attached to a fixed structure in Prince Lab using c-clamps. You will be given timers and objects and told their masses. You will then place an object in your device and demonstrate its use, measuring the period of your pendulum in action. You will be asked to immediately report the moment of inertia in units of kg-cm^2 based on this measured period. No major adjustments or reconfiguration of the pendulum will be permitted during the testing. You will be able to use a calculator or laptop.

For preliminary testing of your device, we suggest that you weigh and measure some regularly shaped, uniform objects (disks, blocks, rods) and look up or calculate their moments of inertia. Compare the known moments of inertia with those found using your pendulum.

Parameters of potential test objects (we may add others with similar sizes/masses):

1. A bicycle wheel 5-gear cluster has a mass of about 412g and diameter of ~11.5 cm
2. A typical softball has a mass of about 160g and a diameter of ~9 cm
3. A football has a mass of about 420 g and its mid-section diameter is about 16 cm.
4. A spoked wheel has a mass of 1200g and diameter of ~30.5 cm.
5. A flange mount has a mass of 1400g and diameter of ~11.5cm.
6. A basketball has a mass of about 590 g and diameter of ~23.5 cm
7. A roller bearing has a mass of 230g and diameter of 6 cm.

5. Materials and Construction

The pendulum may be constructed from the following materials:

Foam-core board
Perforated Aluminum Sheet
Plywood (12”x12” squares of 1/2”, ¼” thickness should be available)
Plexiglass sheets (12” x 12” squares – check back for thickness – about 0.5cm)

The top support structure can be made of wood – plywood is probably preferable.

Various types of string and cable, hooks, bolts, screws, brackets, and other miscellaneous hardware will be available in Prince Lab.

Construction will take place in Prince Lab. Hand tools, X-Acto knives, and small power tools will be available for cutting and drilling.

Power tools may only be used under the supervision of Mr. Brian Corkum or one of the Lab TAs.

Mr. Corkum will supervise the overall use of the lab, and is an invaluable source of insight and advice, but please do not abuse his generosity of time and effort.

Construction can be performed during daytime hours April 15-18. TAs and faculty will be available during normal section times, and in the afternoon hours.

6. Safety

- **Wear safety glasses when using any power tool.**
- **Tie back all hair and loose clothing.**
- **Clamp all work pieces while being cut or drilled**
- **NO OPEN TOED SHOES IN PRINCE LAB**
- **BE CAREFUL!**

7. Evaluation

The design testing will take place in Prince Lab on Friday April 19, from 9 am-4pm. Design teams will sign up for 30 minute time slots throughout the day.

Each group will make a brief (10 minute max) oral presentation, showing design drawings and design parameters, and explaining the basis for the final design. A short document describing the design methodology and final drawings should be submitted.

All group members must be present at the test time. Absent group members will not receive credit for the lab.

All members will be asked to fill out a "team member participation/performance assessment" form.

Design and performance will be assessed according to the following guidelines:

Design concept	10 points
Performance (reference disk & repeatability)	20 points
Quality/Robustness of construction	5 points
Oral presentation	10 points
Final drawings	5 points

The above elements are of course related. Good design and good quality of construction should lead to good performance!