Universality of Phases in QCD and QCD-like Theories

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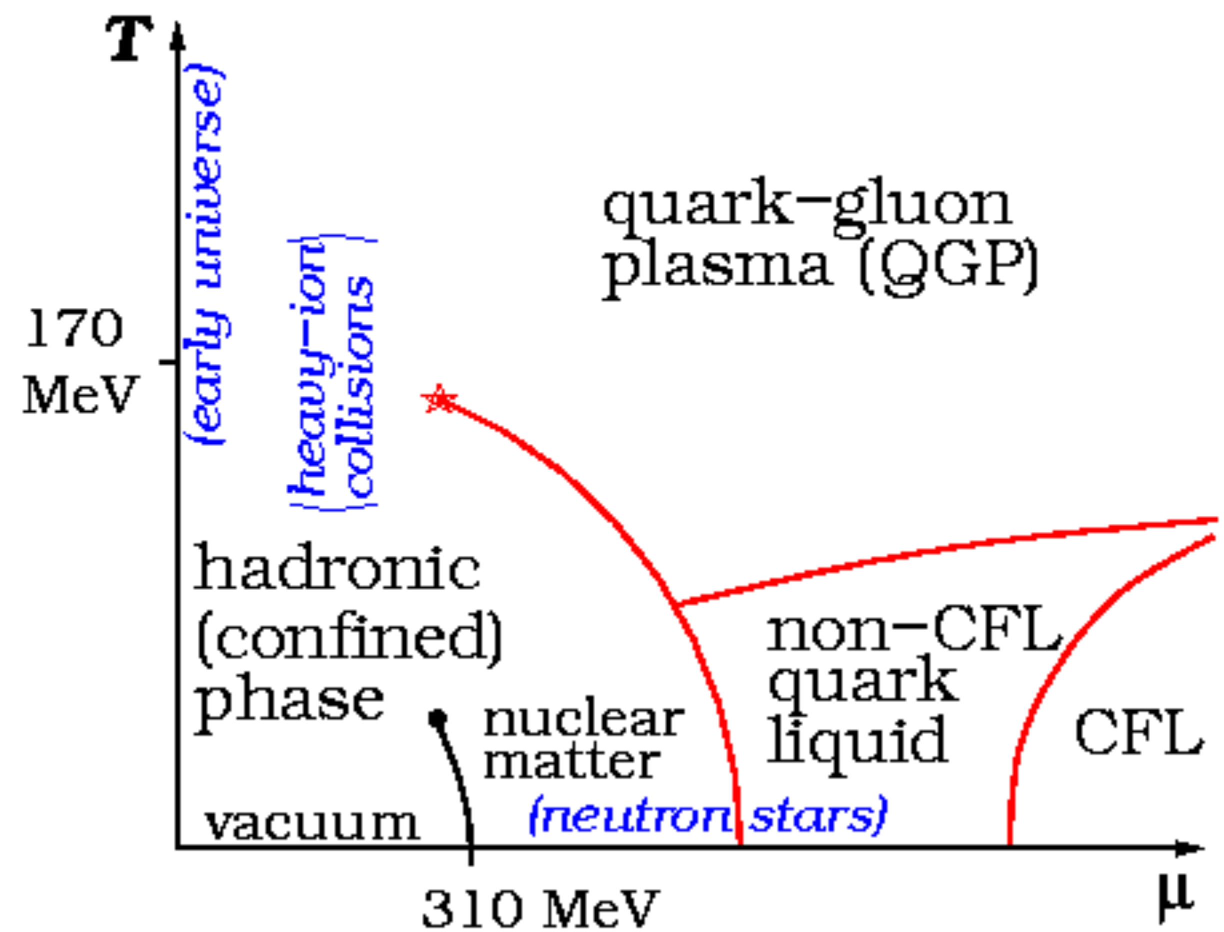
A.Cherman, M.H. and D. Robles-Llana, PRL106, 091603(2011)[1009.1623[hep-th]]
M.H. and N.Yamamoto, 1103.5480[hep-ph].
M.H., C. Hoyos, A. Karch and L.Yaffe, to appear in hep-th.

11th workshop on Non-Perturbative QCD@Paris

QCD phase diagram

- At finite baryon chemical potential, lattice simulation is difficult because of the sign problem.



(from wikipedia)

Our proposal:

Use large-Nc techniques to dodge the sign problem!

QCD with baryonic chemical potential = sign-free theories

large-Nc

What is the "sign problem" ?

- In lattice simulation, one generates many configurations with probability exp(-S)/Z.

$$\lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k}\mathcal{O}[A_\mu^{(i)}]=\frac{1}{Z_{YM}}\int dA_\mu\mathcal{O}[A]e^{-S_{YM}[A]}\equiv\langle\mathcal{O}\rangle$$

- But one cannot regard exp(-S)/Z as "probability" when it is not real positive. It does happen with Euclidean signature, with fermions!

$$\langle\bar{\psi}\psi\rangle=\frac{\int dA_\mu Tr\not{D}^{-1}[A]\cdot\det\not{D}[A]\cdot e^{-S_{YM}[A]}}{\int dA_\mu\det\not{D}[A]\cdot e^{-S_{YM}[A]}}$$

det can be complex

Reweighting method

$$\langle \mathcal{O} \rangle_{full\ theory} = \frac{\langle \mathcal{O} \cdot phase \rangle_{phase\ quench}}{\langle phase \rangle_{phase\ quench}}$$

- R.H.S. is calculable *in principle*
- Difficult in practice -- often <phase> becomes very small, so that the R.H.S. is essentially 0/0.
- "overlapping problem" may appear.

Our proposal:

Use large-Nc techniques to dodge the sign problem!

QCD with baryonic chemical potential $\overset{\text{large-Nc}}{=}$ sign-free theories

Claim

- In the large-Nc limit, QCD μ_B is equivalent to sign-free theories : $SO(2N_c)$YM μ_B/$Sp(2N_c)$ YM μ_B/ QCD μ_I
- So... at large-Nc, the sign problem might be just an *illusion*. (for a class of observables.)
- Rather nice agreement already at finite N_c.
- Similar argument can be the chiral random matrix theory.
- Holographic (AdS/CFT) realization is possible.

Large-N_c orbifold equivalence

Kachru-Silverstein '98, Bershadsky-Kakushadze-Vafa '98, Bershadsky-Johansen '98, ...

- Consider the superstrings on $AdS_5 \times S_5$ and on its orbifold projections such as $AdS_5 \times RP_5$.
- Many correlators don't care about the internal space; only AdS_5 part matters.
- Even if they know the shape of the internal space, "integer spin" modes coincide, while "half-integer" modes disagree.

In fact, it can be proven solely in field theory language.

(Bershadsky-Johansen '98, Kovtun-Unsal-Yaffe '06,...)

1-1 correspondence of the Feynman diagrams, agreement of the Schwinger-Dyson eqs, etc

The large-Nc equivalence

SO(2Nc) + fund + μ_B

sign-free

Equivalent as long as "b-pion" does not condense in SO(2Nc) YM

$$S_{ab} = q_a^T C \gamma^5 q_b$$

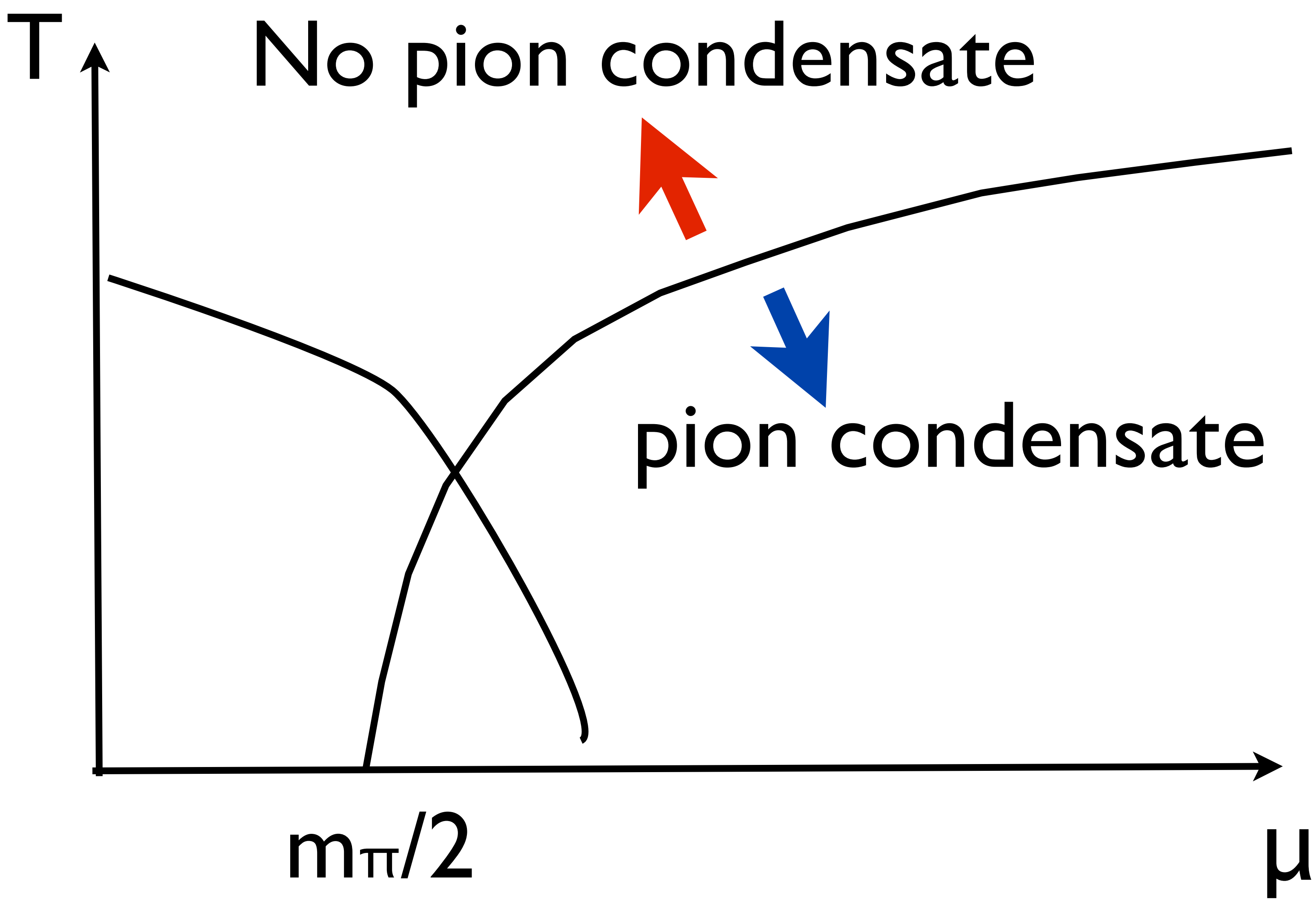
U(Nc) + fund + μ_B

sign problem

U(Nc) + fund + μ_I

sign-free

Equivalent as long as pion does not condense in μ_I side


T
No pion condensate
pion condensate
mπ/2
μ

- Consider SO(2Nc) YM with Nf Dirac fundamental fermions and the "baryon number" chemical potential.

$$\mathcal{L}_{SO} = \frac{1}{4g_{SO}^2}\,\mathrm{Tr}\ F_{\mu\nu}^2 + \sum_{a=1}^{N_f} \bar{q}_a(\gamma^\mu D_\mu + m_q + \mu_B \gamma^4) q_a$$

$$D = \gamma^\mu D_\mu + m_q + \mu_B \gamma^4$$

real

$$\rightarrow \quad C\gamma^5 D (C\gamma^5)^{-1} = D^*$$

$$\rightarrow \quad \det D \geq 0$$

The projection condition:

$$A_\mu \to J A_\mu J^T, \quad q_a \to \underbrace{-i}_{Z_4 \text{ in } U(1)_B} J q_a$$

$$J = \begin{pmatrix} & -\mathbf{1}_N \\ \mathbf{1}_N & \end{pmatrix}$$

$$A_\mu = i \begin{pmatrix} A_\mu^A + B_\mu^A & C_\mu^A - D_\mu^S \\ C_\mu^A + D_\mu^S & A_\mu^A - B_\mu^A \end{pmatrix}$$

$$\longrightarrow \quad J A_\mu J^T = i \begin{pmatrix} A_\mu^A - B_\mu^A & -C_\mu^A - D_\mu^S \\ -C_\mu^A + D_\mu^S & A_\mu^A + B_\mu^A \end{pmatrix}$$

$$A_\mu^{proj} = i \begin{pmatrix} A_\mu^A & -D_\mu^S \\ D_\mu^S & A_\mu^A \end{pmatrix}$$

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1_{N_c} & i1_{N_c} \\ 1_{N_c} & -i1_{N_c} \end{pmatrix}$$

$$P A_\mu^{proj} P^{-1} = \begin{pmatrix} -\mathcal{A}_\mu^T & 0 \\ 0 & \mathcal{A}_\mu \end{pmatrix}$$

$$\mathcal{A}_\mu \equiv D_\mu^S + i A_\mu^A$$

Two copies of the U(Nc) gauge field!

$$
Pq_a = \begin{pmatrix} \lambda_a^+ \\ \lambda_a^- \end{pmatrix} \to \begin{pmatrix} \cancelto{0}{-\lambda_a^+} \\ \lambda_a^- \end{pmatrix}
$$

After the projection, U(Nc) QCD
with baryonic chemical potential is obtained.

$$
\mathcal{L} = \frac{1}{4g_{SU}^2} \operatorname{Tr} \mathcal{F}_{\mu\nu}^2 + \sum_{a=1}^{N_f} \bar{\psi}^a \left(\gamma^\mu \mathcal{D}_\mu + m_q + \mu_B \gamma^4\right) \psi_a
$$

$$
\mathcal{A}_\mu = D_\mu^S + iA_\mu^A, \; \psi_a = \lambda_a^-
$$

$$\langle \mathcal{O}_1^{(p)} \mathcal{O}_2^{(p)} \cdots \rangle_p = \langle \mathcal{O}_1^{(d)} \mathcal{O}_2^{(d)} \cdots \rangle_d$$

parent (SO)

operators invariant under the projection symmetry

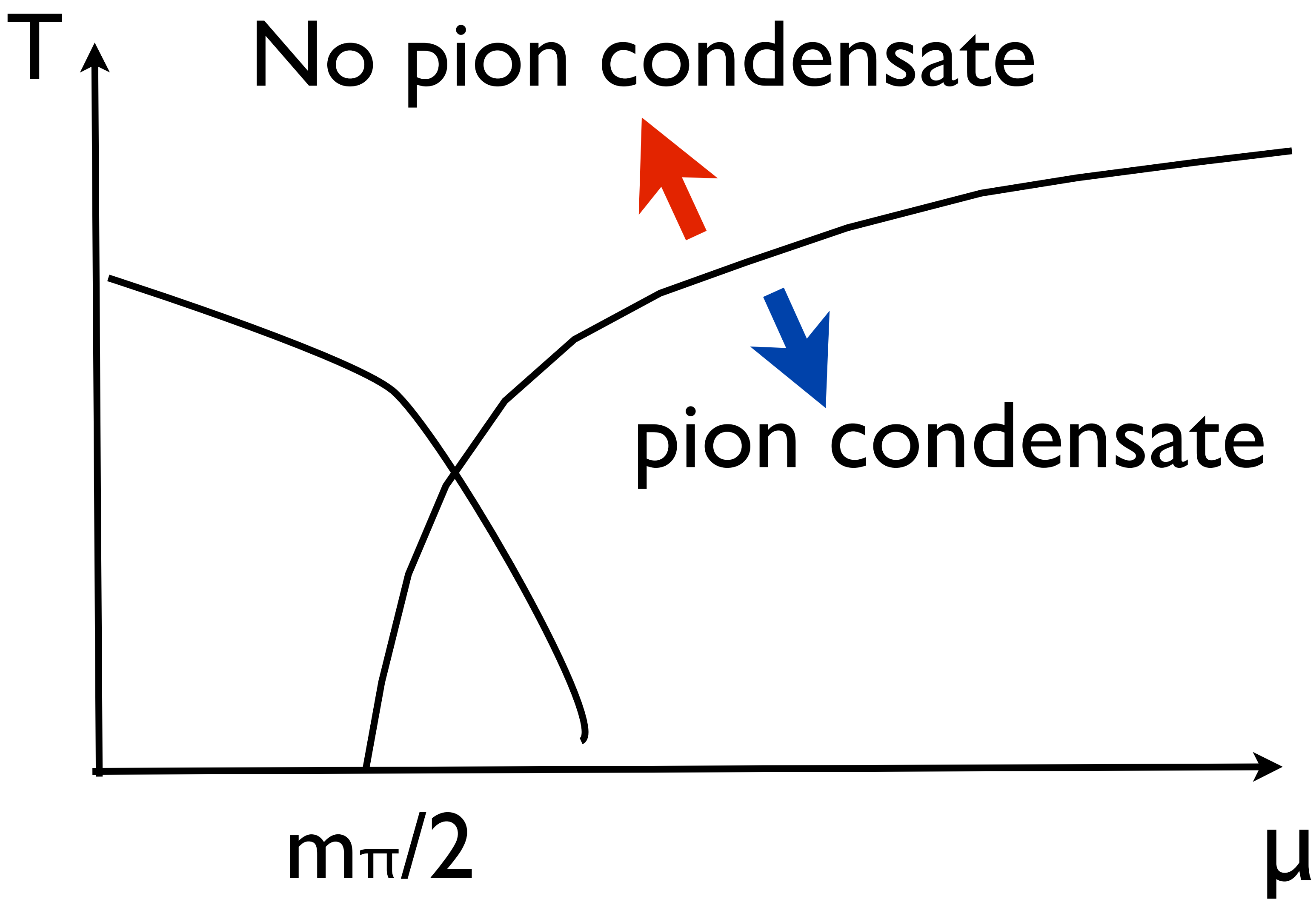
daughter (SU)

operators made of projected fields

- One-to-one correspondence between planar diagrams, up to fermion one-loop.
- Non-planar diagrams and/or diagrams with more than fermion one loops disagree in general.

Equivalence in the 't Hooft large-N_c limit!

- The equivalence holds when the orbifolding symmetry is not broken.
- However... SO theory has "baryon-number-charged" mesons; if they condense, $U(1)_B$ is broken to Z_2; but we need Z_4 for the projection.
- No problem at small μ and/or high temperature.


T
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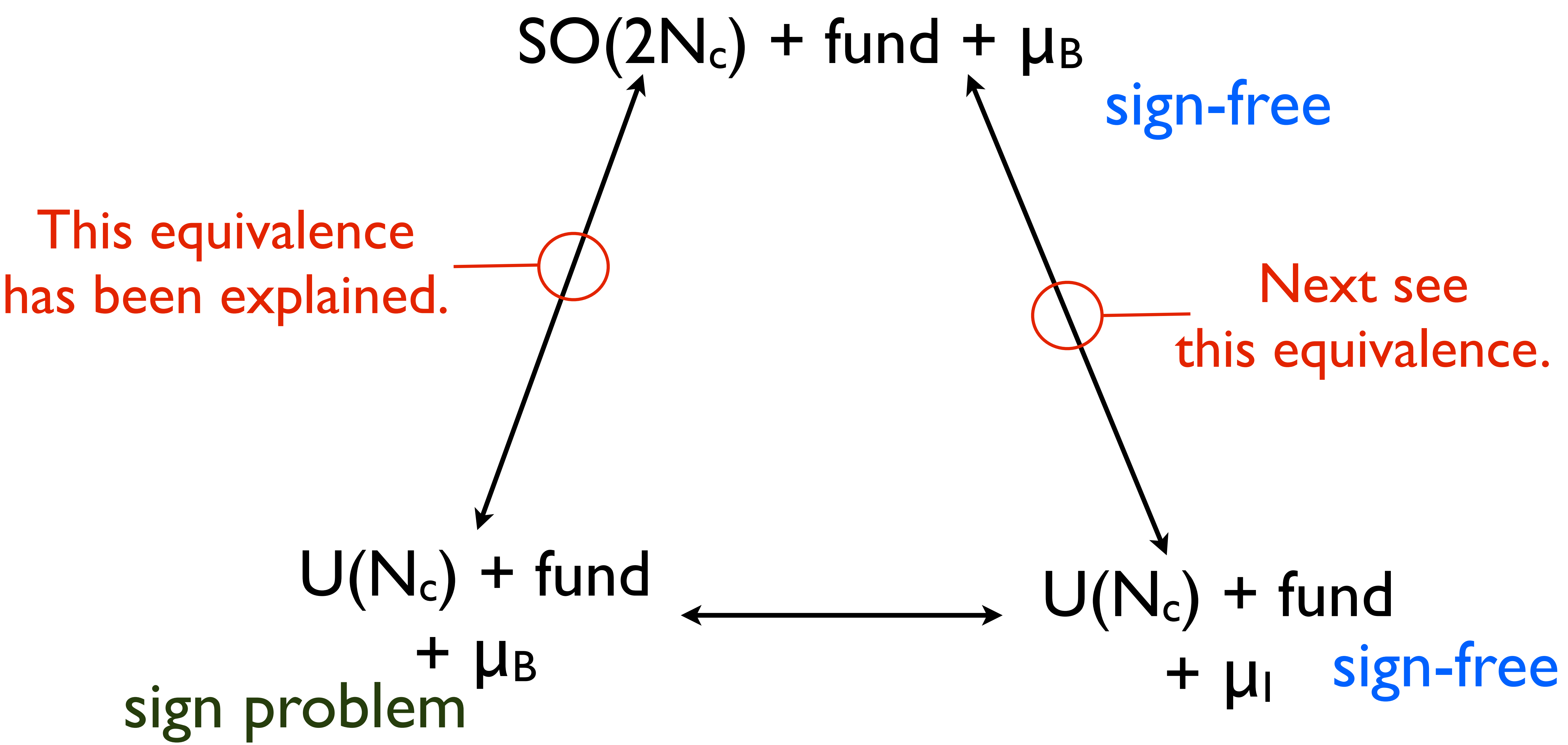
The large-Nc equivalence



Another projection: isospin chemical potential

$$J_c \psi J_i^{-1} = \psi$$

gauge symmetry

flavor(isospin) symmetry

$$\psi = \begin{pmatrix} \psi_1^{(1)} & \psi_1^{(2)} \\ \psi_2^{(1)} & \psi_2^{(2)} \end{pmatrix}$$

$$\psi_\pm^{(1)} = \mp i \psi_\pm^{(2)}$$

The large-Nc equivalence

SO(2Nc) + fund + μ_B

sign-free

Equivalent as long as “b-pion” does not condense in SO(2Nc) YM

U(Nc) + fund + μ_B

sign problem

U(Nc) + fund + μ_I

sign-free

Equivalent as long as pion does not condense in μ_I side

Q. but μ_B and μ_I look different..

A. They *are* different.

They agree only in the neutral sector.
Only leading large-N behaviors agree.

Chiral condensate and baryon/isospin density agree.

Spectrum of π^+/π^- disagree.

Another large-Nc equivalence

$SO(2N_c)$ +
N_f fund + μ_B

$Sp(2N_c)$ +
N_f fund + μ_B

$U(N_c)$ + N_f fund
+ μ_B/μ_I

More generally:

$SO(2N_c)$ +
N_f fund + $\mu_1, \mu_2, \ldots$

$Sp(2N_c)$ +
N_f fund + $\mu_1, \mu_2, \ldots$

$U(N_c)$ + N_f fund
+ $\mu_1, \mu_2, \ldots$

Yet another large-N_c equivalence

$SO(2N_c) + N_f$ adj $+ \mu_B$

Kovtun-Unsal-Yaffe ’03

Kovtun-Unsal-Yaffe ’03

$U(N_c) + N_f$ adj $+ \mu_B$

$U(N_c) + 2N_f$ anti-sym $+ \mu_I$

Armoni-Shifman-Veneziano ’02

Note that all three theories here are sign-free.

1/N correction

How good at SU(3) ?

- SO μ_B/Sp μ_B/SU μ_I are equivalent in the Veneziano limit (nonzero N_f/N_c).
- SO μ_B/Sp μ_B/SU μ_I/SU μ_B are equivalent in the 't Hooft limit ($N_f/N_c \to 0$), because planar diagrams coincide only up to one-fermion-loop.

SU(3) μ_B vs SU(3) μ_I

- Chiral condensate : agree only to the leading order.
- Polyakov loop : the leading corrections coincide.

coincide

N_f/N_c : planar, one-fermion-loop

$(1/N_c^2)^k$ $(k>0)$: nonplanar, no fermion loop

disagree

$(N_f/N_c)^k$ $(k>1)$: fermion, multi-fermion-loop

$(N_f/N_c)^p$ $(1/N_c^2)^q$ $(p>0, q>0)$

: nonplanar with fermion loop

Other models

chiral random matrix theory
holographic models (AdS/CFT)

Chiral random matrix theory

- For each Nc, RMT is a "large-N" theory; N corresponds not to Nc but to the volume.
- There are large-N (not large-Nc) equivalences within the RMT framework.
- In fact a part of the equivalences has been observed by directly calculating the chiral condensate! (e.g. Klein-Toublan-Verbaarschot '03)

$$D = \begin{pmatrix} m\mathbf{1}_{2N} & \Phi + \mu\mathbf{1}_{2N} \\ -\Phi^\dagger + \mu\mathbf{1}_{2N} & m\mathbf{1}_{2N} \end{pmatrix}$$

$$S_B = -NTr\Phi\Phi^\dagger$$

$\beta_D=1$ (SU(2),Sp(2Nc)) sign free

$\beta_D=4$ (SO(2Nc) etc) sign free

$\beta_D=2$ (SU(Nc), Nc>2)

μ_B

sign problem

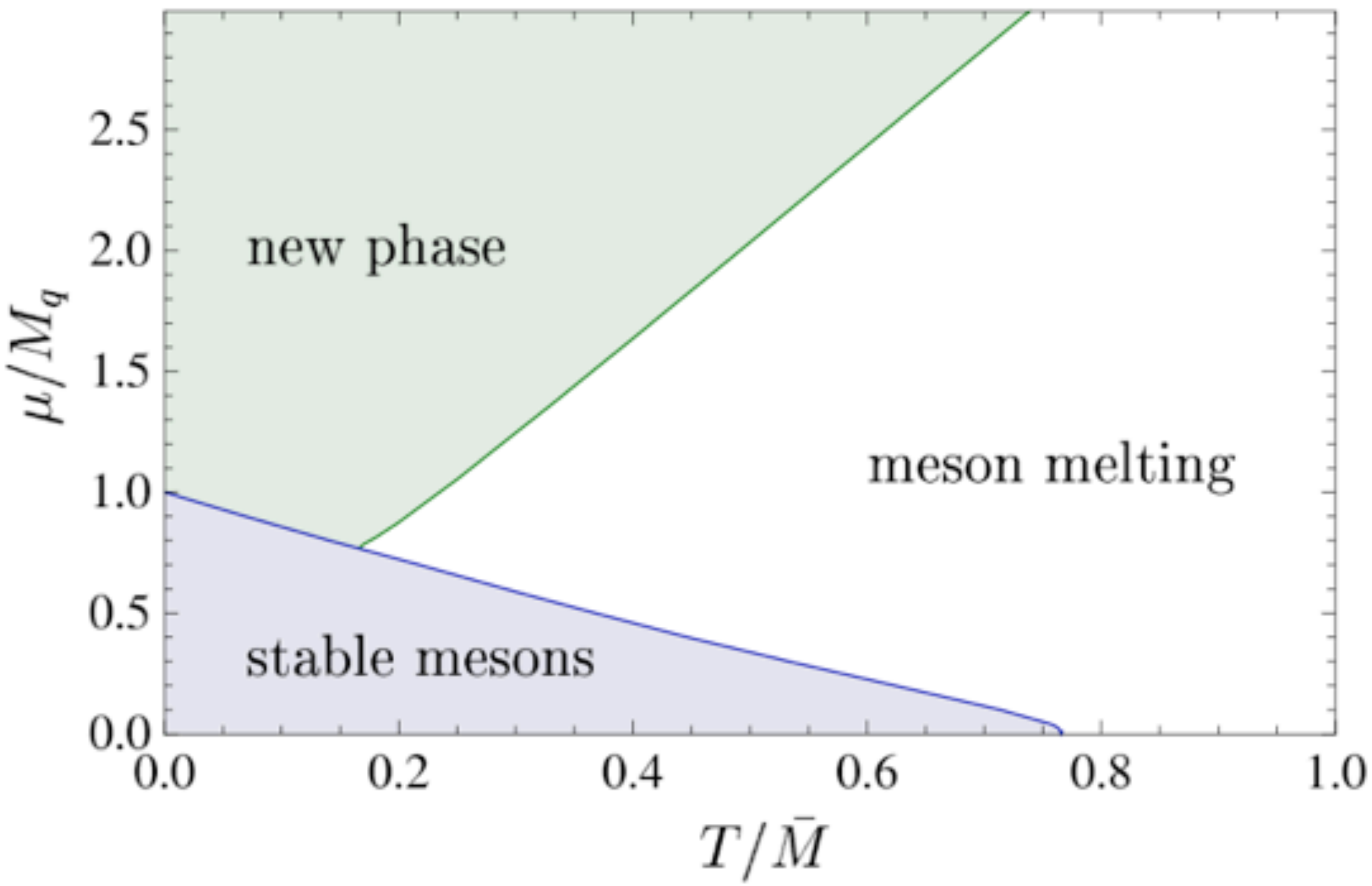
$\beta_D=2$ (SU(Nc), Nc>2)

μ_I sign free

Nonperturbative equivalence can be demonstrated by explicitly solving RMTs.

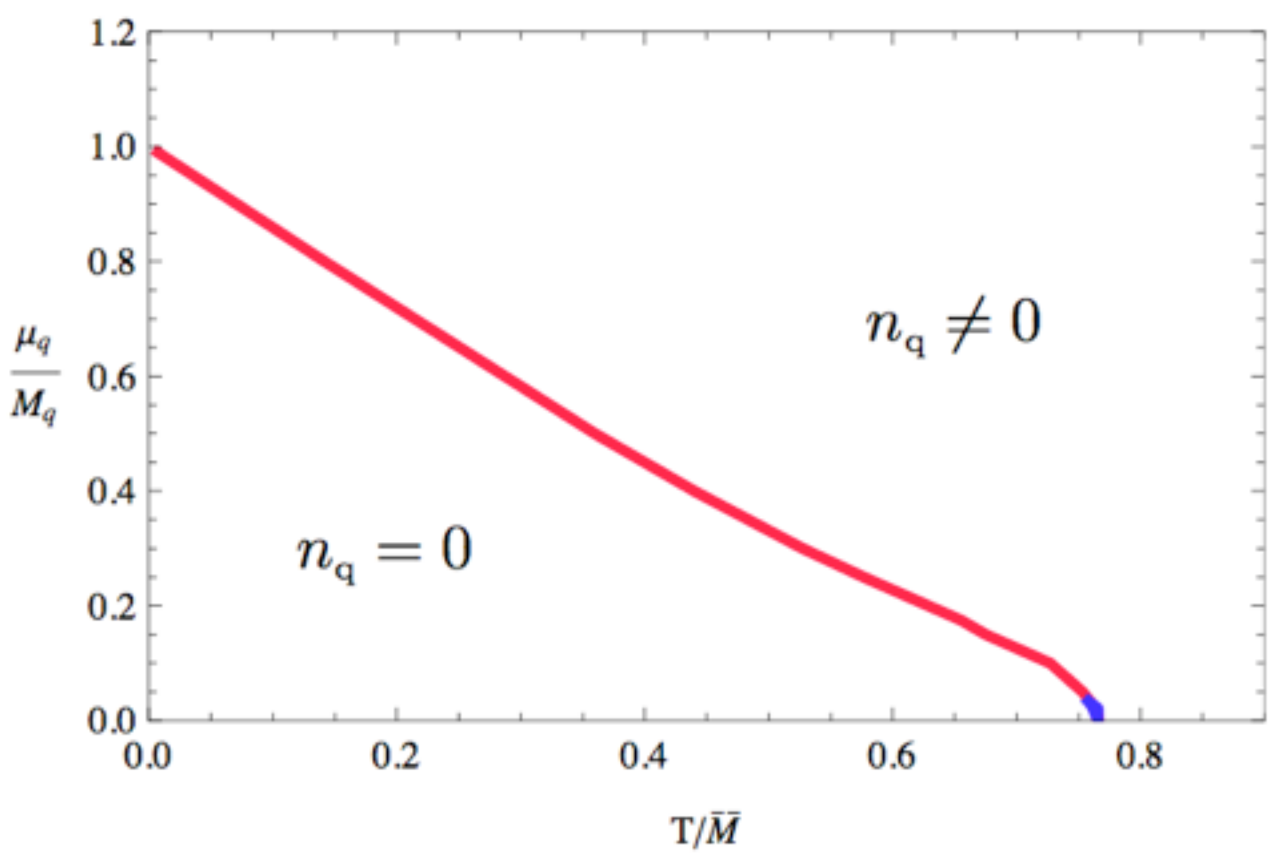
Holographic realization

- "holographic analogue" in D3/D7-system
- Dynamics of mesons are described by the Dirac-Born-Infeld (DBI) action.
- Projections connecting SU μ_B/SU μ_I/SO μ_B exist.
- Equations of motion derived from DBI action agree!
- Actually this "equivalence" had been observed, though the reason was not known...



Ammon-Erdmenger-Kaminski-Kerner, **0903.1864** [hep-th]

(isospin chemical potential)



Mateos-Matsuura-Myers-Thomson, **0709.1225** [hep-th]

(baryon chemical potential)

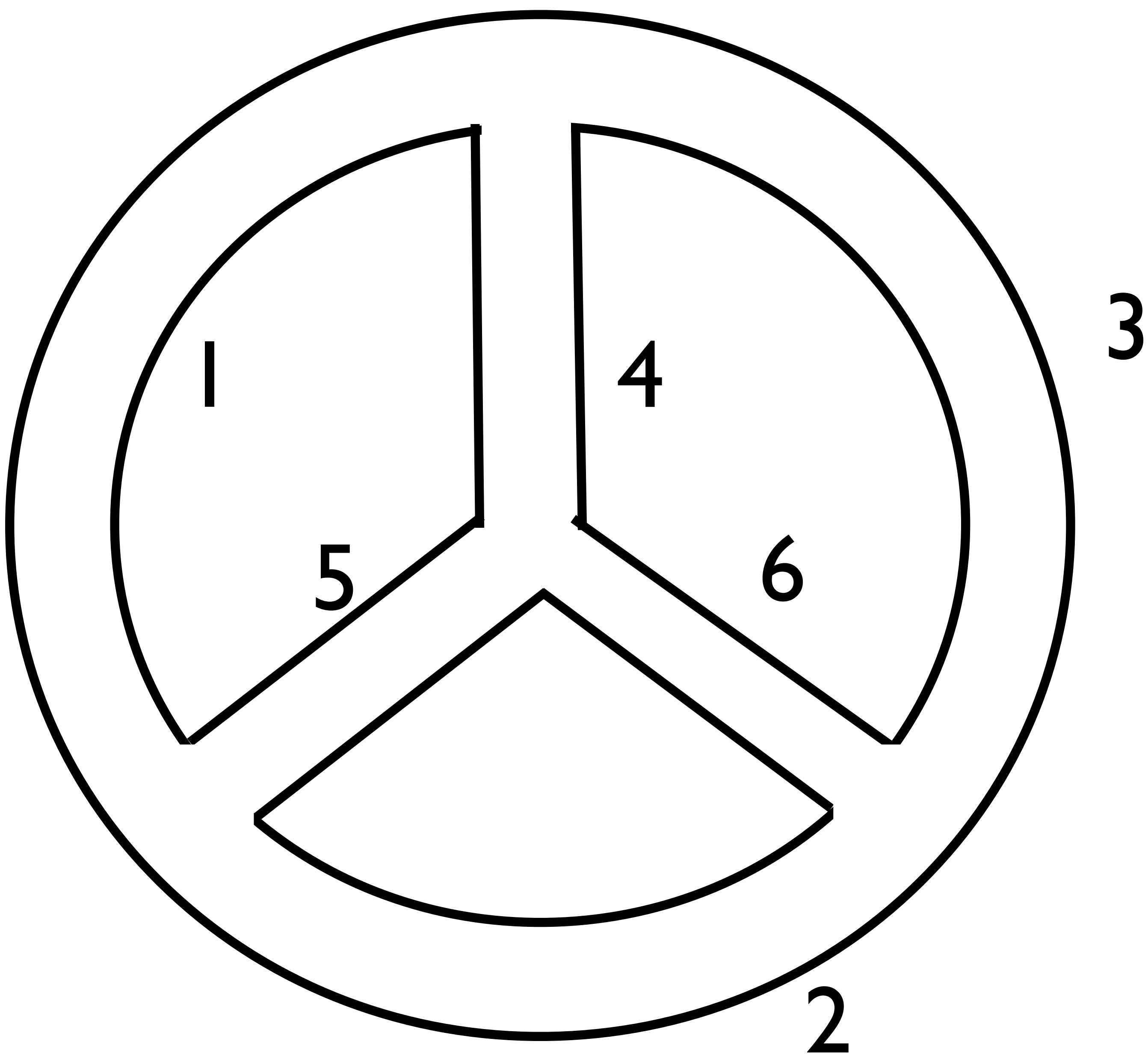
Summary & outlook

- At large-Nc, the sign problem can be avoided.(not all observables, but many “neutral” operators are calculable.)
- The large-Nc equivalence provides a unified view of the QCD-like theories with baryon/isospin chemical potentials.
- The equivalence of random matrix theories may be useful for various systems. Wigner-Dyson ensembles, Bogoliubov-de Gennes ensembles,...
- Lattice simulation outside the BEC/BCS cross over region. Does the QCD critical point exist?

Backup slides

Proof
(Bershadsky-Johansen '98)

- All planar diagrams agree when $g_{SU}^2 = g_{SO}^2$



Insert "projectors" to each propagator:

$$\mathcal{P}(A_\mu) = \frac{1}{4}\sum_{n=1}^{4} J_c^n A_\mu J_c^{-n}$$

$$= \frac{1}{2}\left(A_\mu + J_c A_\mu J_c^{-1}\right)$$

Following factor is multiplied to U(Nc) diagram:

$$\sum_{n_i=1,2}\left(\frac{1}{2}\right)^{N_P}\cdot Tr(J^{n_1}J^{-n_4}J^{n_5})\cdot Tr(J^{-n_2}J^{-n_5}J^{-n_6})\cdot Tr(J^{-n_3}J^{n_6}J^{n_4})$$
$$\times Tr(J^{-n_1}J^{n_2}J^{n_3})$$

Constraints due to "regularity" :

not independent

$$J^{n_1}J^{-n_4}J^{n_5}=\pm\mathbf{1}_2,\quad J^{-n_2}J^{-n_5}J^{-n_6}=\pm\mathbf{1}_2,\quad J^{-n_3}J^{n_6}J^{n_4}=\pm\mathbf{1}_2,\quad J^{-n_1}J^{n_2}J^{n_3}=\pm\mathbf{1}_2$$

$n_1,n_2,..$

N_P-N_L-1 constraints

$$(1/2)^{N_P}\cdot 2^{N_P-N_L+1}\cdot 2^{N_L}$$

From projectors

N_L traces

Can we go beyond $\mu = m_\pi/2$?

Deformation

- The equivalence fails once b-pion condenses.

$$S_{ab} = q_a^T C \gamma^5 q_b$$

- It may be avoided by deforming the parent while preserving the daughter untouched, i.e. by adding "b-pion mass" which is projected to zero.

$$\mathcal{L}_d = \frac{c^2}{\Lambda_{\rm QCD}} \left(S^{\dagger ab} S_{ab} + P^{\dagger ab} P_{ab} \right)$$

$$P_{ab} = q_a^T C q_b$$

- However we must be careful so that the positivity of the determinant is not lost!

Certain deformations keep sign free nature.

A sign-free way of introducing the auxiliary fields

$$\mathcal{L}_d = s^\dagger_{ab} s^{ab} + p^\dagger_{ab} p^{ab} + i(s^\dagger_{ab} S^{ab} + p^\dagger_{ab} P^{ab} + \mathrm{h.c.})$$

- Pfaffian rather than determinant appears.
- Pf > 0 holds in the chiral limit when auxiliary fields are constant.
- Inhomogeneous condensation may appear.

Better way?

- Introduce "tachyonic mass" for a heavy b-meson.

$$
\begin{aligned}
\mathcal{L}_d &= \frac{c^2}{\Lambda^2}(S^{\dagger ab}S_{ab} - P^{\dagger ab}P_{ab}) \\
&= \frac{c^2}{2\Lambda^2}((\bar{q}^i_a q^j_a)^2 + (\bar{q}^i_a \gamma_5 q^j_a)^2 + \frac{1}{2}(\bar{q}^i_a \gamma^{\mu\nu} q^j_a)^2). \\
&\to (f_{ij})^2/2 + (g_{ij})^2/2 + (h_{\mu\nu,ij})^2/2 + ic_1 f_{ij}\bar{q}^i_a q^j_a \\
&+ ic_2 g_{ij}\bar{q}^i_a \gamma^5 q^j_a + ic_3 h_{\mu\nu,ij}\bar{q}^i_a \gamma^{\mu\nu} q^j_a
\end{aligned}
$$

Sign-free
in the chiral limit

Inhomogeneous condensation is killed.
But not clear how large "mass" can be introduced without causing instability.