# Two-loop Remainder Functions in N=4 SYM

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# Introduction

- Planar N = 4 Super Yang-Mills is the ‘simplest gauge theory.’
- It is one of the rare theories where we can obtain explicit analytic results for multi-loop multi-leg processes.
- The AdS/CFT correspondence allows us to not only get perturbative answers, but also strong coupling results.
- Final aim: Solving the planar sector of N=4 Super Yang-Mills (Integrability).

# Introduction

- In the mean time: Use planar N=4 Super-Yang-Mills to explore the analytic structure of gauge theory amplitudes at higher loops.

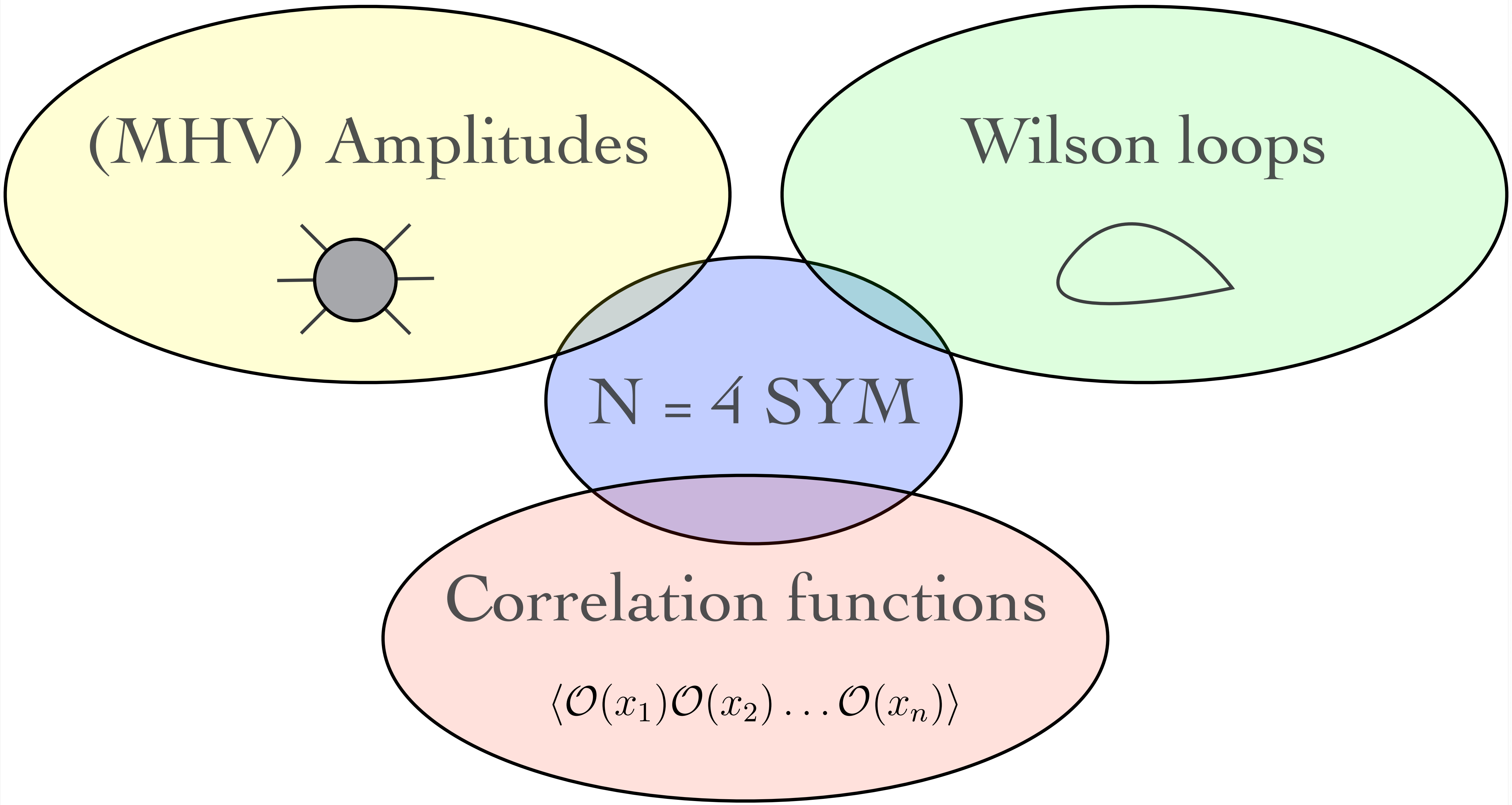
- Outline of the talk:

  ➡ The two-loop six-point remainder function.

  ➡ Towards higher-point remainders.

# A duality at work

[See talks by Eden and Heslop]



# A duality at work

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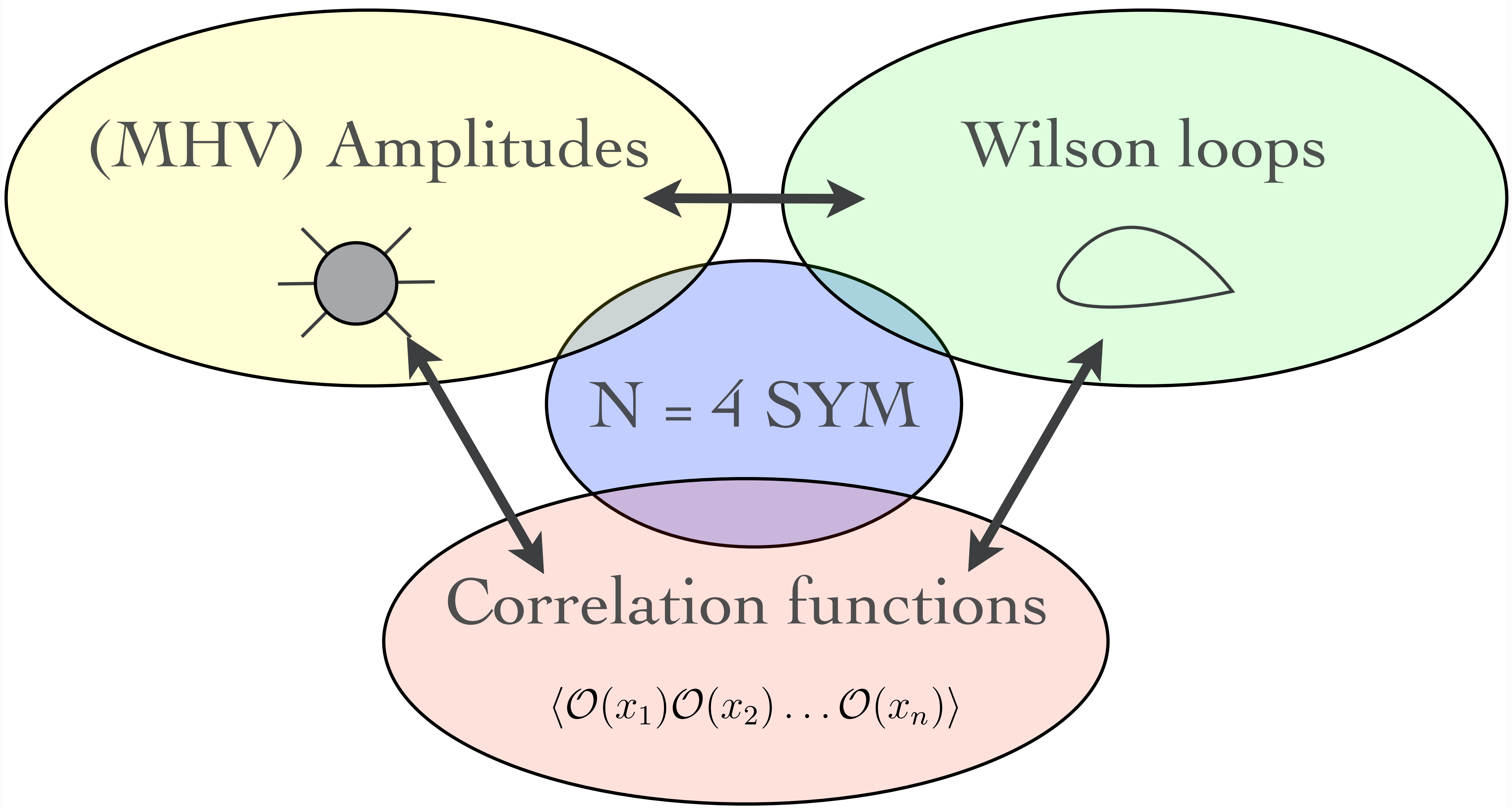
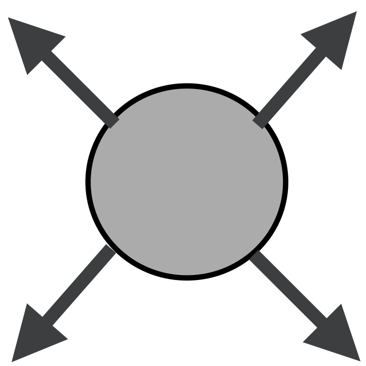
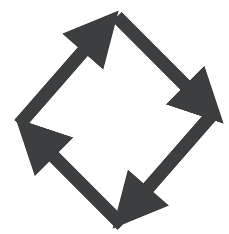


# Amplitude - Wilson loop duality

- MHV amplitudes
- Wilson loops



$$p_i = x_i - x_{i+1}$$



Superconformal symmetry

Dual superconformal symmetry

- Dual conformal invariance puts constraints in terms of an anomalous Ward identity.

# Anomalous Ward identity

- The solution to the Ward identities is, e.g., at two-loops,

[Drummond, Henn, Korchemsky, Sokatchev]

$$w_n^{(2)}(\epsilon) = f_{WL}^{(2)}(\epsilon)\, w_n^{(1)}(2\epsilon) + C_{WL}^{(2)} + \mathcal{O}(\epsilon)\,,$$

- This result is in agreement with an iteration for the amplitude conjectured at two loops (Anastasiou Bern, Dixon, Kosower) and beyond (Bern, Dixon, Smirnov).
- This conjecture was shown to fail for six points!

# Anomalous Ward identity

- The solution to the Ward identities is, e.g., at two-loops,

[Drummond, Henn, Korchemsky, Sokatchev]

$$w_n^{(2)}(\epsilon) = f_{WL}^{(2)}(\epsilon)\, w_n^{(1)}(2\epsilon) + C_{WL}^{(2)} + R_n^{(2)}(u_{ij}) + \mathcal{O}(\epsilon)\,,$$

- ... but we can always add an arbitrary function of conformal invariants and we still obtain a solution to the Ward identities!

$$u_{ij} = \frac{x_{ij+1}^2 x_{i+1j}^2}{x_{ij}^2 x_{i+1j+1}^2}$$

# The remainder function

- For on-shell amplitudes with $n$ = 4, 5, we do not have enough momenta to form non-trivial cross ratios
  - ➡ The full answer is given to all orders by the 'inhomogeneous' solution:

$$w_4^{(2)}(\epsilon) = f_{WL}^{(2)}(\epsilon)\, w_4^{(1)}(2\epsilon) + C_{WL}^{(2)} + \mathcal{O}(\epsilon)$$

$$w_5^{(2)}(\epsilon) = f_{WL}^{(2)}(\epsilon)\, w_5^{(1)}(2\epsilon) + C_{WL}^{(2)} + \mathcal{O}(\epsilon)$$

- For on-shell amplitudes with $n$ = 6 or more, we have non-trivial cross ratios:

$$w_6^{(2)}(\epsilon) = f_{WL}^{(2)}(\epsilon)\, w_6^{(1)}(2\epsilon) + C_{WL}^{(2)} + R_6^{(2)}(u_1, u_2, u_3) + \mathcal{O}(\epsilon)$$

$$u_1 = \frac{s_{12}\, s_{45}}{s_{123}\, s_{345}}, \qquad u_2 = \frac{s_{23}\, s_{56}}{s_{123}\, s_{234}}, \qquad u_3 = \frac{s_{34}\, s_{61}}{s_{234}\, s_{345}},$$

# The remainder function

- Dihedral symmetry of the amplitude implies symmetries for the remainder function.
  - ➡ For $n = 6$, the remainder function is completely symmetric.
- Multi-collinear limits:

$$\mathcal{R}_n \to \mathcal{R}_{n-k} + \mathcal{R}_{k+4}$$

[Brandhuber, Heslop, Khoze, Spence, Travaglini]

  - ➡ For $n = 6$, the remainder function vanishes in the two-particle collinear limits.
- It vanishes in the multi-Regge limit (in the Euclidean region).
- Depends on conformal ratios only, but functional form not fixed by symmetry.

# Strong coupling results

- Using a geometric setup allowed to obtain several special cases of remainder functions at strong coupling:
  - ➡ for six edges, in 3+1 dimensions when all cross ratios are equal

$$R(u,u,u) = -\frac{\pi}{6} + \frac{1}{3\pi}\phi^2 + \frac{3}{8}\left(\log^2 u + 2Li_2(1-u)\right)$$

[Alday, Gaiotto, Maldacena]

  - ➡ for eight edges, in 1+1 dimensions

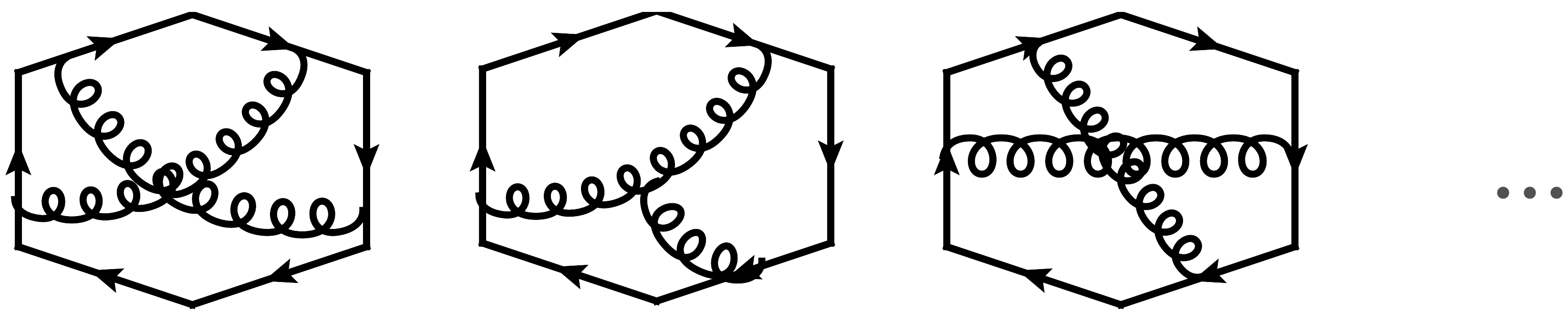
$$R^{\text{strong}}_{8,WL} = -\frac{1}{2}\ln\left(1+\chi^-\right)\ln\left(1+\frac{1}{\chi^+}\right) + \frac{7\pi}{6}$$

$$+ \int_{-\infty}^{+\infty} \mathrm{d}t \, \frac{|m| \, \sinh t}{\tanh(2t + 2i\phi)} \ln\left(1 + e^{-2\pi|m|\cosh t}\right)$$

[Alday, Maldacena]

# Weak coupling

- Anastasiou, Brandhuber, Heslop, Khoze, Spence and Travaglini worked out the two-loop Wilson loop diagrams:

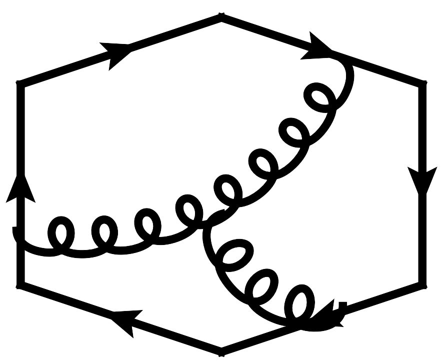


- Each of these diagrams is an integral, similar to a Feynman parameter integral.

- Allowed to perform a numerical study of the two-loop remainder functions.

# Weak coupling

- For $n$ = 6, many of the integrals can be computed explicitly, but one is particularly 'hard':



$$f_H(p_1,p_2,p_3;Q_1,Q_2,Q_3)$$
$$:= \frac{\Gamma(2-2\epsilon_{\rm UV})}{\Gamma(1-\epsilon_{\rm UV})^2}\int_0^1\Big(\prod_{i=1}^3 d\tau_i\Big)\int_0^1\Big(\prod_{i=1}^3 d\alpha_i\Big)\delta(1-\sum_{i=1}^3\alpha_i)\;(\alpha_1\alpha_2\alpha_3)^{-\epsilon_{\rm UV}}\frac{\mathcal{N}}{\mathcal{D}^{2-2\epsilon_{\rm UV}}}\,,$$

$$\begin{aligned}\mathcal{N} &= 2(p_1p_2)(p_1p_3)\Big[\alpha_1\alpha_2(1-\tau_1)+\alpha_3\alpha_1\tau_1\Big] + 2(p_1p_3)(p_2p_3)\Big[\alpha_3\alpha_1(1-\tau_3)+\alpha_2\alpha_3\tau_3\Big]\\ &+ 2(p_1p_2)(p_2p_3)\Big[\alpha_2\alpha_3(1-\tau_2)+\alpha_1\alpha_2\tau_2\Big] + 2\alpha_1\alpha_2\Big[2(p_1p_2)(p_3Q_3)-(p_2p_3)(p_1Q_3)-(p_3p_1)(p_2Q_3)\Big]\end{aligned}$$

$$+\ldots$$

- The integrals do not explicitly depend on conformal ratios.
- The integrals can however be computed numerically.

[Anastasiou, Brandhuber, Heslop, Khoze, Spence, Travaglini]

# An excursion to multi-Regge kinematics

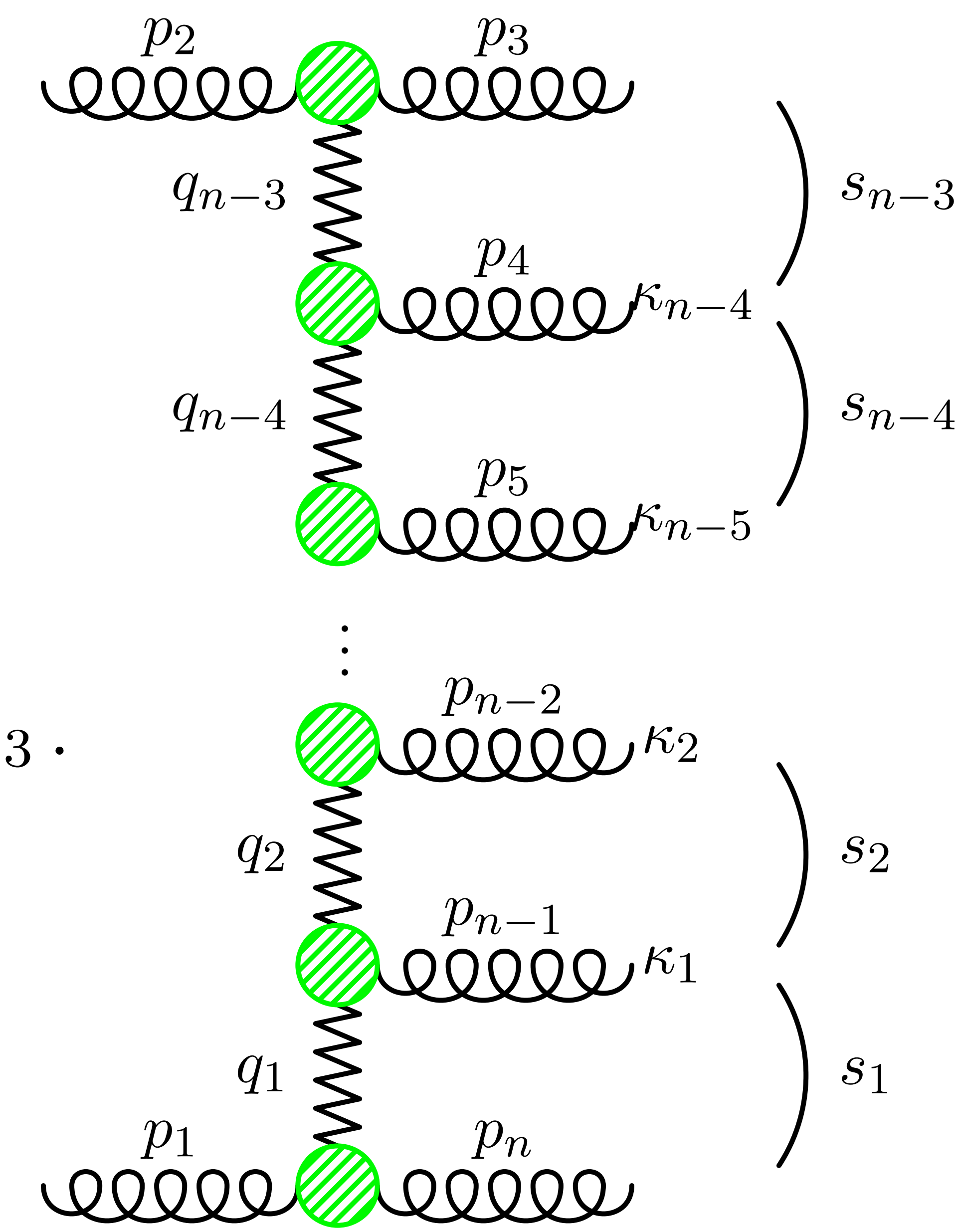
- Multi-Regge kinematics are defined by

$$y_3 \gg y_4 \gg \ldots \gg y_{n-1} \gg y_n$$

$$|p_{3\perp}| \simeq |p_{4\perp}| \simeq \ldots \simeq |p_{n-1\perp}| \simeq |p_{n\perp}|,$$

- This implies a hierarchy of scales:

$$s \gg s_1,\ s_2, \ldots, s_{n-3} \gg -t_1,\ -t_2, \ldots, -t_{n-3}\,.$$

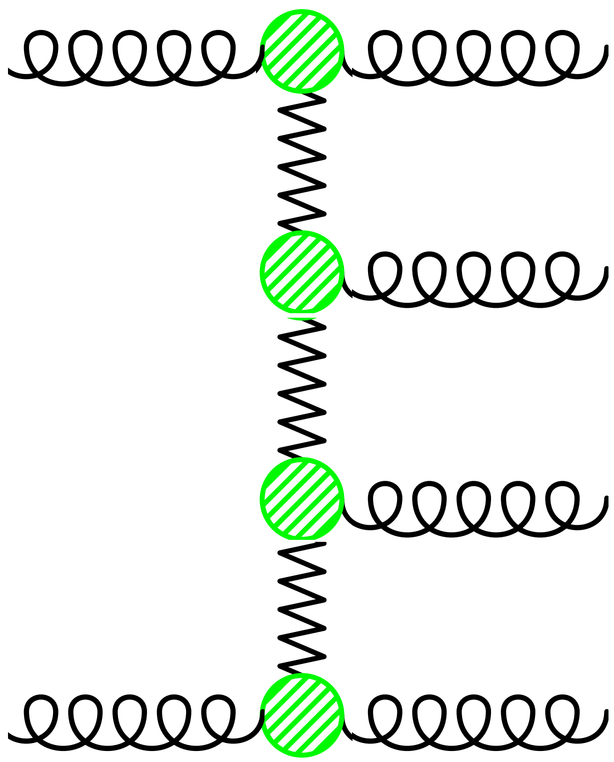


# Multi-Regge limits

- Multi-Regge kinematics

$$y_3 \gg y_4 \gg y_5 \gg y_6$$

$$|p_{3\perp}|^2 \simeq |p_{4\perp}|^2 \simeq |p_{5\perp}|^2 \simeq |p_{6\perp}|^2$$



- In the multi-Regge limit, the cross ratios become trivial:

$$u_1 = \frac{s_{12}\, s_{45}}{s_{345}\, s_{456}} \simeq 1$$

$$u_2 = \frac{s_{23}\, s_{56}}{s_{234}\, s_{456}} \simeq \mathcal{O}\left(\frac{t}{s}\right)$$

$$u_3 = \frac{s_{34}\, s_{61}}{s_{234}\, s_{345}} \simeq \mathcal{O}\left(\frac{t}{s}\right)$$

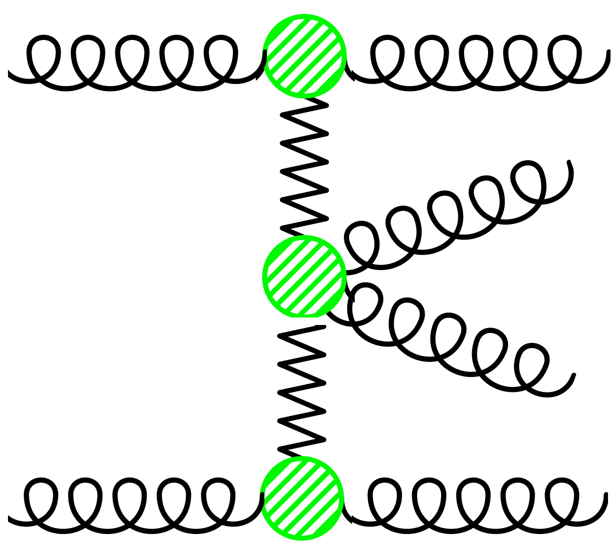
[Bartels, Lipatov, Vera; Brower, Nastase, Schnitzer; Del Duca, CD, Glover]

# Multi-Regge limits

- Quasi-multi-Regge kinematics

$$y_3 \gg y_4 \simeq y_5 \gg y_6$$

$$|p_{3\perp}|^2 \simeq |p_{4\perp}|^2 \simeq |p_{5\perp}|^2 \simeq |p_{6\perp}|^2$$



- In the quasi-multi-Regge limit, the cross ratios stay generic:

$$u_1^{\mathrm{QMRK}} = \frac{s_{45}}{(p_4^+ + p_5^+)(p_4^- + p_5^-)}$$

$$u_2^{\mathrm{QMRK}} = \frac{|p_{3\perp}|^2 p_5^+ p_6^-}{(|p_{3\perp} + p_{4\perp}|^2 + p_5^+ p_4^-)(p_4^+ + p_5^+) p_6^-}$$

$$u_3^{\mathrm{QMRK}} = \frac{|p_{6\perp}|^2 p_3^+ p_4^-}{p_3^+ (p_4^- + p_5^-)(|p_{3\perp} + p_{4\perp}|^2 + p_5^+ p_4^-)}$$

[Del Duca, CD, Glover]

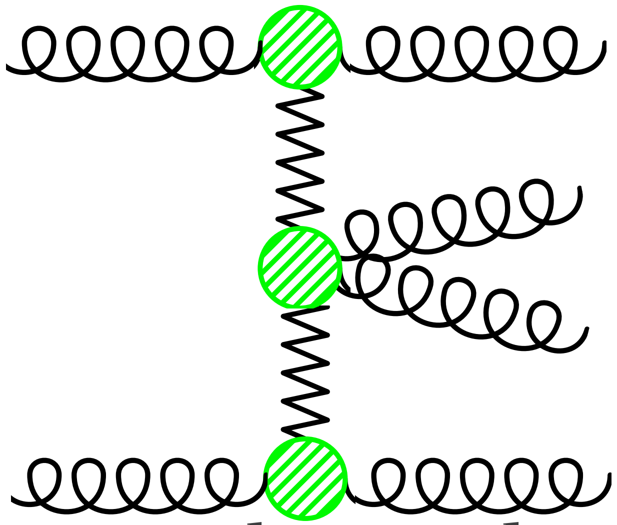
# Regge-exactness of Wilson loops

- The result is in fact even stronger:

  The (logarithm of the) Wilson-loop is **Regge-exact** in this limit, i.e., it is the same in this special kinematics and in arbitrary kinematics

$$y_3 \gg y_4 \simeq \ldots \simeq y_{n-1} \gg y_n$$

$$|p_{3\perp}|^2 \simeq \ldots \simeq |p_{n\perp}|^2$$



- This limit leaves the conformal cross ratios unchanged for an arbitrary number of edges.
- This result is in fact true for Wilson loops with an arbitrary number of edges and loops!

[Del Duca, CD, Smirnov]

# The six-point remainder function

- Due to Regge-exactness, it is enough to compute the remainder function in this restricted area of phase space.
- In the limit, all integrals are
  - ➡ at most three-fold.
  - ➡ dependent on conformal cross ratios only.
- The resulting integrals are much simpler and can be solved in a closed form, and we can extract the two-loop six-point remainder function,

$$w_n^{(2)}(\epsilon) = f_{WL}^{(2)}(\epsilon)\, w_n^{(1)}(2\epsilon) + C_{WL}^{(2)} + R_{n,WL}^{(2)} + \mathcal{O}(\epsilon)$$

[Del Duca, CD, Smirnov]

# The six-point remainder function

- The expression we obtained was considerably simplified by Goncharov, Spradlin, Vergu and Volovich.

$$R(u_1,u_2,u_3) = \sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-) - \frac{1}{2}\,\mathrm{Li}_4(1-1/u_i)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{J^4}{24} + \chi\frac{\pi^2}{12}\left(J^2+\zeta(2)\right)$$

$$x_i^\pm = u_i x^\pm,\quad x^\pm = \frac{u_1+u_2+u_3-1\pm\sqrt{\Delta}}{2u_1u_2u_3},\quad \Delta = (u_1+u_2+u_3-1)^2 - 4u_1u_2u_3.$$

- Arguments are cross ratios in momentum twistor space:

$$u_1 = \frac{\langle 1234\rangle\langle 4561\rangle}{\langle 1245\rangle\langle 3461\rangle},\qquad x_1^+ = -\frac{\langle 1456\rangle\langle 2356\rangle}{\langle 1256\rangle\langle 3456\rangle}$$

[Goncharov, Spradlin, Volovich, Vergu]

# Symbols

- The simplification of the hexagon remainder function went hand in hand with the introduction of a new mathematical tool: the symbol.
- Hand-waving idea: Associate a 'tensor calculus' to polylogarithms that incorporates the functional identities.

| Polylogarithm | Symbol |
|---|---|
| Function | Tensor |
| Functional equation | Algebraic identity |

# Remainders with more points

- The techniques we developed for the computation of the six-point remainder function can also be applied to Wilson loops with more edges.
- We focus on the 1+1 dimensional setup studied at strong coupling.

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# Remainders with more points

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- We focus on the 1+1 dimensional setup studied at strong coupling.
- The final answer involves 25.000 terms...

... but they all collapse to

$$R^{(2)}_{8,WL}(\chi^+,\chi^-) = -\frac{\pi^4}{18} - \frac{1}{2}\ln\left(1+\chi^+\right)\ln\left(1+\frac{1}{\chi^+}\right)\ln\left(1+\chi^-\right)\ln\left(1+\frac{1}{\chi^-}\right)$$

[Del Duca, CD, Smirnov]

# Remainders in 1+1 dimensions

- Interesting observation: R8 is the simplest function consistent with the cyclic symmetry and collinear limits.

$$R^{(2)}_{8,WL}(\chi^+,\chi^-) = -\frac{\pi^4}{18} - \frac{1}{2}\ln\left(1+\chi^+\right)\ln\left(1+\frac{1}{\chi^+}\right)\ln\left(1+\chi^-\right)\ln\left(1+\frac{1}{\chi^-}\right)$$

- Inspired by this simplicity, Heslop and Khoze have shown by a numerical analysis that this structure extends beyond eight points:

$$R_n = -\frac{1}{2}\Big(\sum_{\mathcal{S}} \log(u_{i_1 i_5})\log(u_{i_2 i_6})\log(u_{i_3 i_7})\log(u_{i_4 i_8})\Big) - \frac{\pi^4}{72}(n-4)$$

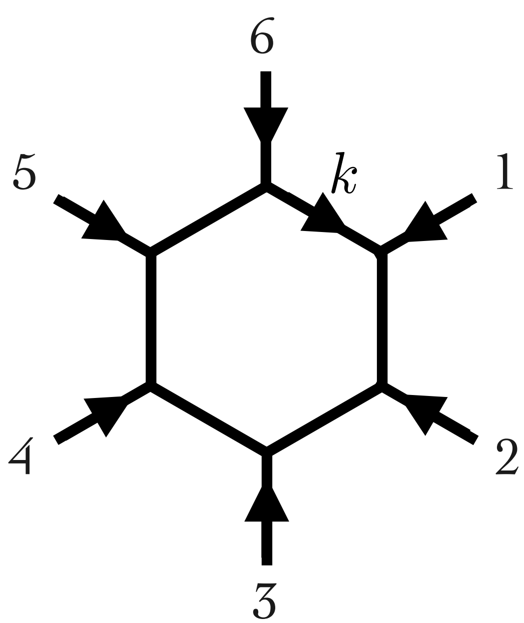
- This structure was recently confirmed by Gaiotto, Maldacena, Sever and Vieira using collinear OPE.

# Remainders with more points

- So far, no analytic results are known for remainder functions in general kinematics beyond six points.
- Recently, Caron-Huot computed the symbol of all two-loop remainder functions.
- Open question: Can we 'integrate' the symbol to a function?
  - ➡ Interesting point: The symbol already tells us that starting from $n = 7$ classical polylogarithms will no longer be enough.
- Insight might come from an unexpected front...

# One-loop Hexagons in 6 dimensions

- The massless scalar one-loop hexagon integral in D=6 dimensions
  - ➡ is finite,
  - ➡ dual conformally invariant,
  - ➡ a weight 3 function.




$$I_6^{D=6} = \int \frac{d^6 k}{i\pi^3} \prod_{i=0}^{5} \frac{1}{D_i} \,,$$

$$D_0 = k^2 \quad \text{and} \quad D_i = (k + p_i)^2, \quad \text{for} \quad i = 1, \ldots, 5\,.$$

# One-loop Hexagons in 6 dimensions

- The analytic form of the massless scalar hexagon in 6 dimensions looks very similar to the analytic expression for the two-loop remainder function!

[Dixon, Drummond, Henn; Del Duca, CD, Smirnov]

$$I_6^{D=6} = \frac{1}{x_{14}^2\, x_{25}^2\, x_{36}^2}\, \mathcal{I}_6(u_1, u_2, u_3)$$

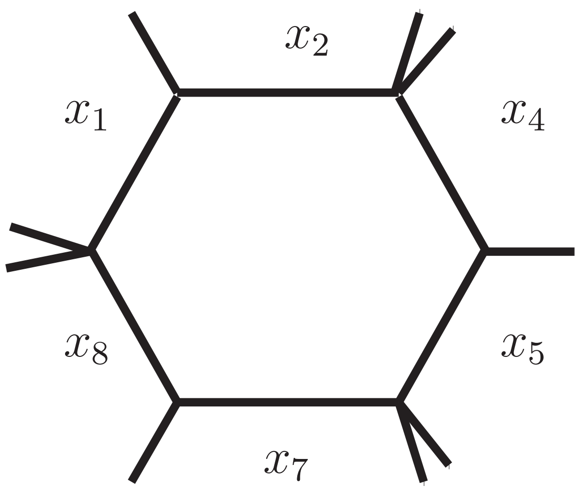
$$\mathcal{I}_6(u_1, u_2, u_3) = \frac{1}{\sqrt{\Delta}} \left[ -2 \sum_{i=1}^{3} L_3(x_{i+}, x_{i-}) + 2\zeta_2 J + \frac{1}{3} J^3 \right]$$

$$R(u_1, u_2, u_3) = \sum_{i=1}^{3} \left( L_4(x_i^+, x_i^-) - \frac{1}{2} \mathrm{Li}_4(1 - 1/u_i) \right)$$

$$- \frac{1}{8} \left( \sum_{i=1}^{3} \mathrm{Li}_2(1 - 1/u_i) \right)^2 + \frac{J^4}{24} + \chi \frac{\pi^2}{12} \left( J^2 + \zeta(2) \right)$$

# One-loop Hexagons in 6 dimensions

- This similarity motivated the study of more complicated hexagons:




$$\Phi_9(u_1,\ldots,u_6) = \frac{1}{\sqrt{\Delta_9}} \sum_{i=1}^{4} \sum_{g\in S_3} \sigma(g)\, \mathcal{L}_3(x^+_{i,g}, x^-_{i,g})$$

$$\mathcal{L}_3(x^+,x^-) := \frac{1}{18}\left(\ell_1(x^+) - \ell_1(x^-)\right)^3 + L_3(x^+,x^-)$$

$$\chi(i,j,k) := -\frac{\langle 4\bar{7}\rangle\langle X_i X_k\rangle\langle X_j 17\rangle}{\langle 1\bar{7}\rangle\langle X_j X_k\rangle\langle X_i 47\rangle}$$

$$x_1^+ := \chi(1,4,7)\,,$$
$$x_1^- := \bar{\chi}(1,4,7)\,,$$

etc.

$$\bar{\chi}(i,j,k) := -\frac{\langle \bar{4}7\rangle\langle X_i X_k\rangle\langle X_j \bar{1}\cap\bar{7}\rangle}{\langle \bar{1}7\rangle\langle X_j X_k\rangle\langle X_i \bar{4}\cap\bar{7}\rangle}$$

[Del Duca, Dixon, Drummond, CD, Henn, Smirnov]

# Conclusion & Outlook

- In the last 18 months, a lot of progress was made to compute multi-leg amplitudes/Wilson loops at strong and weak coupling:
  - ➡ Hexagon in 3+1 dimensions
  - ➡ Octagon in special kinematics (1+1 dimensions)
  - ➡ All even-sided polygons in 1+1 dimensions.
  - ➡ The symbols of all polygons in general kinematics.
- Next step: try to nail all two-loop MHV amplitudes.
- Together with all the other fascinating developments in the field, this might eventually allow to solve the planar sector of N=4 SYM.

# Back ups

# Symbols

- Simple example:

$$\mathrm{Li}_2(x) + \ln(1-x)\ln x = -\mathrm{Li}_2(1-x) - \frac{\pi^2}{6}$$

$$\mathrm{Symbol}(\mathrm{Li}_2(x)) = -(1-x)\otimes x$$

$$\mathrm{Symbol}(\ln(1-x)\ln x) = (1-x)\otimes x + x\otimes(1-x)$$

$$\mathrm{Symbol}(const) = 0$$

$$\mathrm{Symbol}(\mathrm{Li}_2(x) + \ln(1-x)\ln x) = x\otimes(1-x)$$

$$= -\mathrm{Symbol}(\mathrm{Li}_2(1-x))$$