Wilson loops and amplitudes in N=4 Super Yang-Mills

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N=4 Super Yang-Mills

- maximal supersymmetric theory (without gravity)
  conformally invariant, β fn. = 0
  - spin 1 gluon
    4 spin 1/2 gluinos
    6 spin 0 real scalars

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  - only planar diagrams
- AdS/CFT duality   Maldacena 97
  - large-λ limit of 4dim CFT $\leftrightarrow$ weakly-coupled string theory

    (aka weak-strong duality)

AdS/CFT duality, amplitudes & Wilson loops

- planar scattering amplitude at strong coupling — Alday Maldacena 07

$$\mathcal{M} \sim \exp\left[i \frac{\sqrt{\lambda}}{2\pi} (Area)_{cl} \right]$$

area of string world-sheet $\left(\begin{array}{l}\text{classical solution} \\ \text{neglect } O(1/\sqrt{\lambda}) \text{ corrections}\end{array}\right)$

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- amplitude has same form as ansatz for MHV amplitudes at weak coupling

$$M_n = M_n^{(0)} \exp\left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon)\, m_n^{(1)}(l\epsilon) + Const^{(l)} + E_n^{(l)}(\epsilon) \right) \right]$$

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- computation ``formally the same as ... the expectation value of a Wilson loop given by a sequence of light-like segments”

MHV amplitudes ⇔ Wilson loops

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- no amplitudes are known beyond the 6-point 2-loop amplitude

MHV amplitudes in planar N=4 SYM

- at any order in the coupling, colour-ordered MHV amplitude in N=4 SYM can be written as tree-level amplitude times helicity-free loop coefficient $M_n^{(L)} = M_n^{(0)} m_n^{(L)}$

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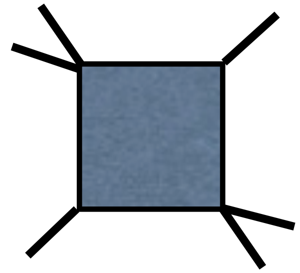
- at 1 loop

$$m_n^{(1)} = \sum_{pq} F^{2\mathrm{me}}(p, q, P, Q) \qquad n \geq 6$$



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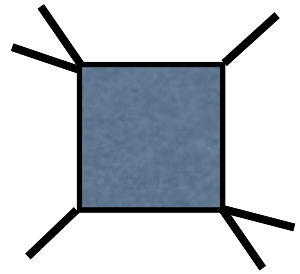

- at 2 loops, iteration formula for the *n*-pt amplitude

$$m_n^{(2)}(\epsilon) = \frac{1}{2}\left[m_n^{(1)}(\epsilon)\right]^2 + f^{(2)}(\epsilon)\, m_n^{(1)}(2\epsilon) + Const^{(2)} + R$$

Anastasiou Bern Dixon Kosower 03

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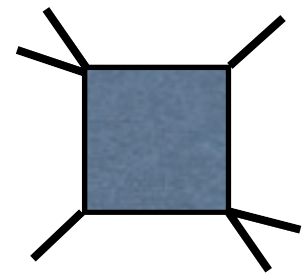

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- at all loops, ansatz for a resummed exponent

$$m_n^{(L)} = \exp\left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon)\, m_n^{(1)}(l\epsilon) + Const^{(l)} + E_n^{(l)}(\epsilon)\right)\right] + R$$

Bern Dixon Smirnov 05

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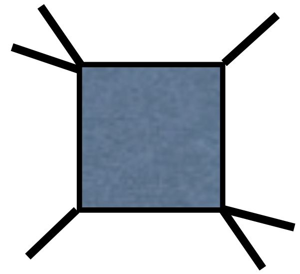

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remainder function

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Anastasiou Bern Dixon Kosower 03

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ansatz for MHV amplitudes in planar N=4 SYM

Bern Dixon Smirnov 05

$$
\begin{aligned}
M_n &= M_n^{(0)}\left[1+\sum_{L=1}^{\infty} a^L m_n^{(L)}(\epsilon)\right] \\
&= M_n^{(0)} \exp\left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) m_n^{(1)}(l\epsilon) + Const^{(l)} + E_n^{(l)}(\epsilon)\right)\right]
\end{aligned}
$$

coupling $a = \frac{\lambda}{8\pi^2}(4\pi e^{-\gamma})^\epsilon$ $\qquad \lambda = g^2 N$ 't Hooft parameter

$$f^{(l)}(\epsilon) = \frac{\hat{\gamma}_K^{(l)}}{4} + \epsilon \frac{l}{2} \hat{G}^{(l)} + \epsilon^2 f_2^{(l)} \qquad E_n^{(l)}(\epsilon) = O(\epsilon)$$

$\hat{\gamma}_K^{(l)}$ cusp anomalous dimension, known to all orders of *a*

Korchemsky Radyuskin 86
Beisert Eden Staudacher 06

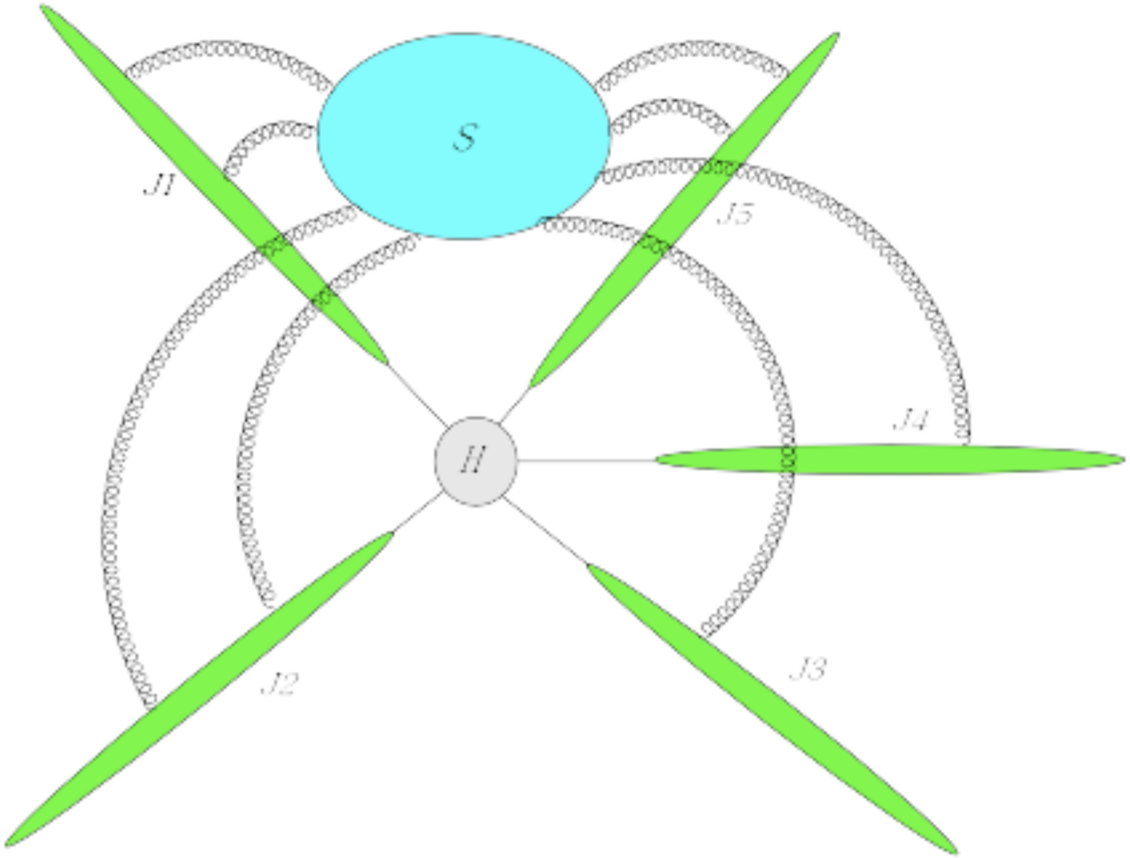
$\hat{G}^{(l)}$ collinear anomalous dimension, known through O(a^4)

Bern Dixon Smirnov 05
Cachazo Spradlin Volovich 07

ansatz generalises the iteration formula for the 2-loop *n*-pt amplitude $m_n^{(2)}$

$$m_n^{(2)}(\epsilon) = \frac{1}{2}\left[m_n^{(1)}(\epsilon)\right]^2 + f^{(2)}(\epsilon)\, m_n^{(1)}(2\epsilon) + Const^{(2)} + \mathcal{O}(\epsilon)$$

Factorisation of a multi-leg amplitude in QCD



Mueller 1981
Sen 1983
Botts Sterman 1987
Kidonakis Oderda Sterman 1998
Catani 1998
Tejeda-Yeomans Sterman 2002
Kosower 2003
Aybat Dixon Sterman 2006
Becher Neubert 2009
Gardi Magnea 2009

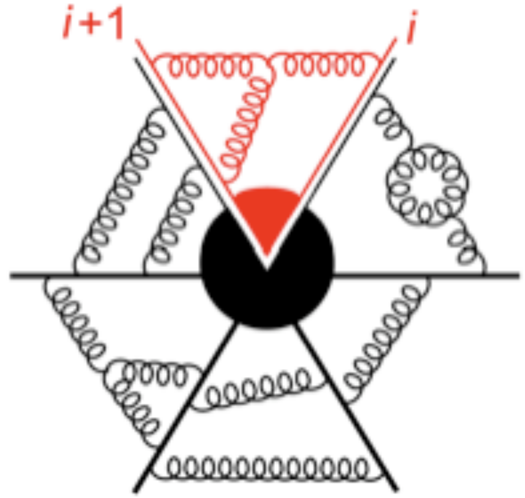
$$\mathcal{M}_N(p_i/\mu,\epsilon) = \sum_L \mathcal{S}_{NL}(\beta_i\cdot\beta_j,\epsilon)\, H_L\left(\frac{2p_i\cdot p_j}{\mu^2},\frac{(2p_i\cdot n_i)^2}{n_i^2\mu^2}\right)\prod_i \frac{J_i\left(\frac{(2p_i\cdot n_i)^2}{n_i^2\mu^2},\epsilon\right)}{\mathcal{J}_i\left(\frac{2(\beta_i\cdot n_i)^2}{n_i^2},\epsilon\right)}$$

$p_i = \beta_i Q_0/\sqrt{2}$ value of Q_0 is immaterial in *S, J*

to avoid double counting of soft-collinear region (IR double poles),
$\mathcal{J}_i$ removes eikonal part from J_i, which is already in S
$J_i/\mathcal{J}_i$ contains only single collinear poles

$N = 4$ SYM in the planar limit

- colour-wise, the planar limit is trivial: can absorb S into J_i



- each slice is square root of Sudakov form factor

$$\mathcal{M}_n = \prod_{i=1}^{n} \left[\mathcal{M}^{[gg\to 1]} \left(\frac{s_{i,i+1}}{\mu^2}, \alpha_s, \epsilon \right) \right]^{1/2} h_n(\{p_i\}, \mu^2, \alpha_s, \epsilon)$$

- β fn = 0 ⇒ coupling runs only through dimension $\bar{\alpha}_s(\mu^2)\mu^{2\epsilon} = \bar{\alpha}_s(\lambda^2)\lambda^{2\epsilon}$

Sudakov form factor has simple solution

$$\ln \left[\Gamma \left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) \right] = -\frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{\alpha_s(\mu^2)}{\pi} \right)^n \left(\frac{-Q^2}{\mu^2} \right)^{-n\epsilon} \left[\frac{\gamma_K^{(n)}}{2n^2\epsilon^2} + \frac{G^{(n)}(\epsilon)}{n\epsilon} \right]$$

⇒ IR structure of $N = 4$ SUSY amplitudes

Magnea Sterman 90
Bern Dixon Smirnov 05

Brief history of the ansatz

the ansatz checked for the 3-loop 4-pt amplitude — Bern Dixon Smirnov 05

2-loop 5-pt amplitude — Cachazo Spradlin Volovich 06; Bern Czakon Kosower Roiban Smirnov 06

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at 2 loops, the remainder function characterises the deviation from the ansatz

$$R_n^{(2)} = m_n^{(2)}(\epsilon) - \frac{1}{2}\left[m_n^{(1)}(\epsilon)\right]^2 - f^{(2)}(\epsilon)\, m_n^{(1)}(2\epsilon) - Const^{(2)}$$

$R_6^{(2)}$ known numerically — Bern Dixon Kosower Roiban Spradlin Vergu Volovich 08; Drummond Henn Korchemsky Sokatchev 08; Anastasiou Brandhuber Heslop Khoze Spence Travaglini 09

analitically — Duhr Smirnov VDD 09

Wilson loops & Ward identities

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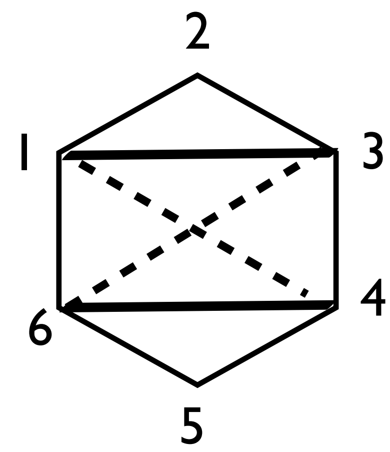
- for n = 6, the conformally invariant cross ratios are

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} \qquad u_2 = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2} \qquad u_3 = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2}$$



x_i are variables in a dual space s.t. $p_i = x_i - x_{i+1}$

thus $x_{k,k+r}^2 = (p_k + \ldots + p_{k+r-1})^2$

Wilson loops

- $$W[\mathcal{C}_n] = \mathrm{Tr}\ \mathcal{P}\ \exp\left[ig \oint \mathrm{d}\tau \dot{x}^\mu(\tau) A_\mu(x(\tau)) \right]$$

closed contour $\mathcal{C}_n$ made by light-like external momenta

$p_i = x_i - x_{i+1}$

Alday Maldacena 07

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  Alday Maldacena 07

- non-Abelian exponentiation theorem: vev of Wilson loop as an exponential, allows us to compute the log of W

  Gatheral 83
  Frenkel Taylor 84

$$\langle W[\mathcal{C}_n]\rangle = 1 + \sum_{L=1}^{\infty} a^L W_n^{(L)} = \exp \sum_{L=1}^{\infty} a^L w_n^{(L)}$$

through 2 loops $\quad w_n^{(1)} = W_n^{(1)} \qquad w_n^{(2)} = W_n^{(2)} - \frac{1}{2}\left(W_n^{(1)}\right)^2$

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- relation between 1 loop amplitudes & Wilson loops

  $$w_n^{(1)} = \frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} m_n^{(1)} = m_n^{(1)} - n\frac{\zeta_2}{2} + \mathcal{O}(\epsilon)$$

  Brandhuber Heslop Travaglini 07

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  - at 2 loops

$$w_n^{(2)}(\epsilon) = f_{WL}^{(2)}(\epsilon)\, w_n^{(1)}(2\epsilon) + C_{WL}^{(2)} + R_{n,WL}^{(2)} + \mathcal{O}(\epsilon)$$

with $f_{WL}^{(2)}(\epsilon) = -\zeta_2 + 7\zeta_3\epsilon - 5\zeta_4\epsilon^2$

(to be compared with $f^{(2)}(\epsilon) = -\zeta_2 - \zeta_3\epsilon - \zeta_4\epsilon^2$ for the amplitudes)

$$R_{4,WL} = R_{5,WL} = 0$$

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- $R_{n,WL}^{(2)}$ arbitrary function of conformally invariant cross ratios

$$u_{ij} = \frac{x_{ij+1}^2 x_{i+1j}^2}{x_{ij}^2 x_{i+1j+1}^2} \qquad \text{with} \qquad x_{k,k+r}^2 = (p_k + \ldots + p_{k+r-1})^2$$

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- duality Wilson loop ⇔ MHV amplitude is expressed by

$$R_{n,WL}^{(2)} = R_n^{(2)}$$

Collinear limits of Wilson loops

collinear limit $a||b$

Anastasiou Brandhuber Heslop Khoze Spence Travaglini 09

$R_6 \to 0$ $\quad$ $R_7 \to R_6$ $\quad$ $R_n \to R_{n-1}$

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triple collinear limit $a||b||c$

$R_6 \rightarrow R_6$ $\qquad$ $R_7 \rightarrow R_6$ $\qquad$ $R_8 \rightarrow R_6 + R_6$ $\qquad$ $R_n \rightarrow R_{n-2} + R_6$

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quadruple collinear limit $a||b||c||d$

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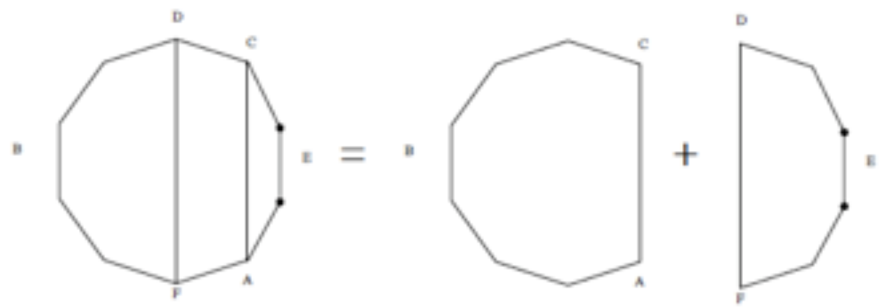
(k+1)-ple collinear limit $i_1||i_2||\cdots||i_{k+1}$

$R_n \rightarrow R_{n-k} + R_{k+4}$



(n-4)-ple collinear limit $i_1||i_2||\cdots||i_{n-4}$

$R_{n-1} \rightarrow R_{n-1}$ $\quad R_n \rightarrow R_{n-1}$

(n-3)-ple collinear limit $i_1||i_2||\cdots||i_{n-3}$

$R_n \rightarrow R_n$

Collinear limits of Wilson loops

Anastasiou Brandhuber Heslop Khoze Spence Travaglini 09

collinear limit $a||b$

$R_6 \to 0$ $\quad R_7 \to R_6$ $\quad R_n \to R_{n\text{-}1}$

triple collinear limit $a||b||c$

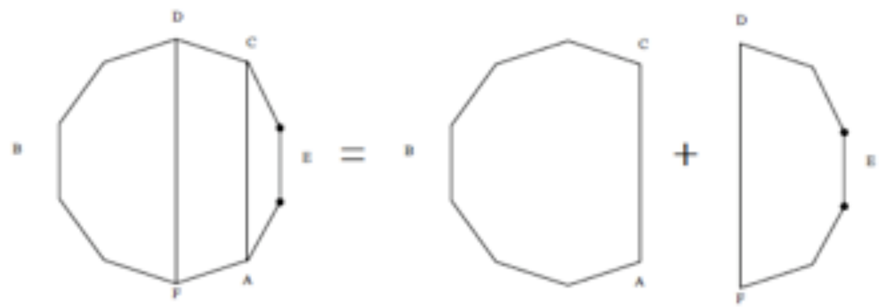
$R_6 \to R_6$ $\quad R_7 \to R_6$ $\quad R_8 \to R_6 + R_6$ $\quad R_n \to R_{n\text{-}2} + R_6$

quadruple collinear limit $a||b||c||d$

$R_7 \to R_7$ $\quad R_8 \to R_7$ $\quad R_9 \to R_6 + R_7$ $\quad R_n \to R_{n\text{-}3} + R_7$

(k+1)-ple collinear limit $i_1||i_2||\cdots||i_{k+1}$

$R_n \to R_{n\text{-}k} + R_{k+4}$



(n-4)-ple collinear limit $i_1||i_2||\cdots||i_{n-4}$

$R_{n\text{-}1} \to R_{n\text{-}1}$ $\quad R_n \to R_{n\text{-}1}$

(n-3)-ple collinear limit $i_1||i_2||\cdots||i_{n-3}$

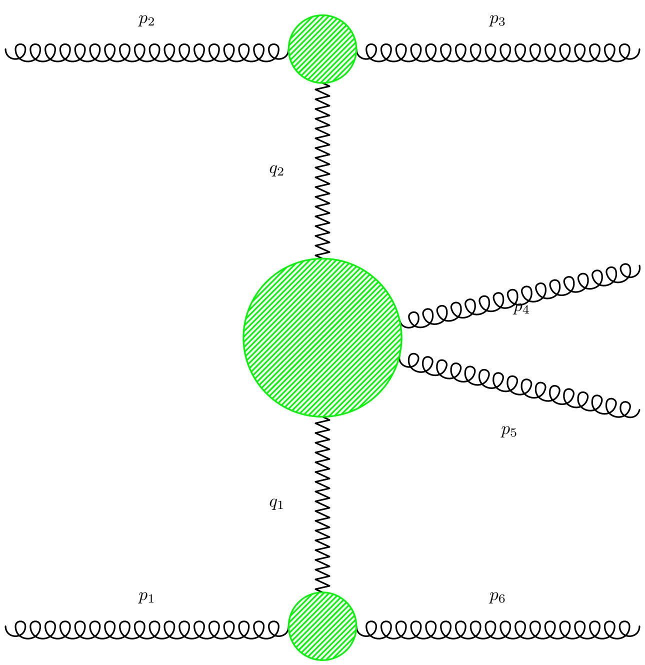
$R_n \to R_n$

- thus R_n is fixed by the (n-3)-ple collinear limit

Quasi-multi-Regge limit of hexagon Wilson loop

- 6-pt amplitude in the qmR limit of a pair along the ladder

$$y_3 \gg y_4 \simeq y_5 \gg y_6; \qquad |p_{3\perp}| \simeq |p_{4\perp}| \simeq |p_{5\perp}| \simeq |p_{6\perp}|$$



the conformally invariant cross ratios are

$$\begin{aligned} u_{36} &= \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12} s_{45}}{s_{123} s_{345}} \\ u_{14} &= \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2} = \frac{s_{23} s_{56}}{s_{234} s_{123}} \\ u_{25} &= \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2} = \frac{s_{34} s_{61}}{s_{234} s_{345}} \end{aligned}$$

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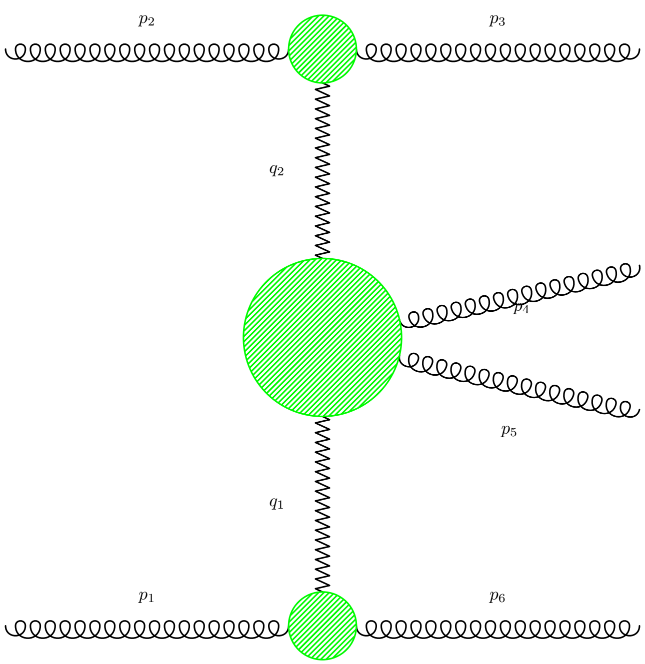

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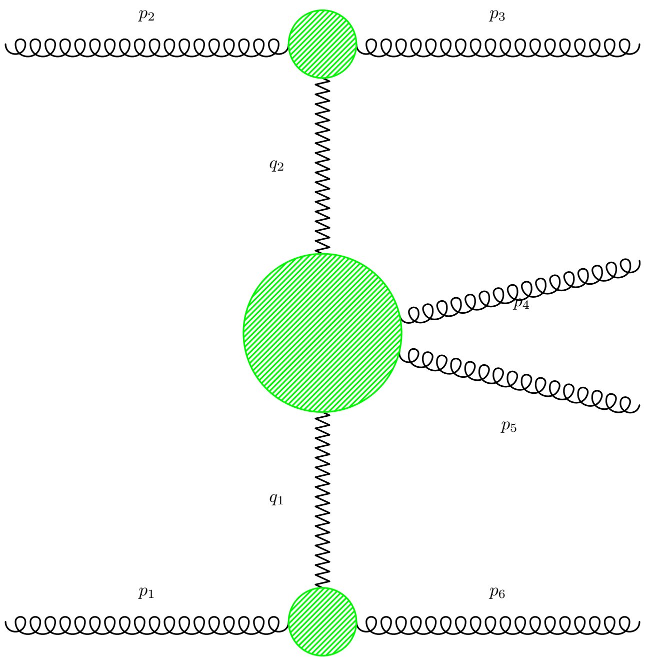

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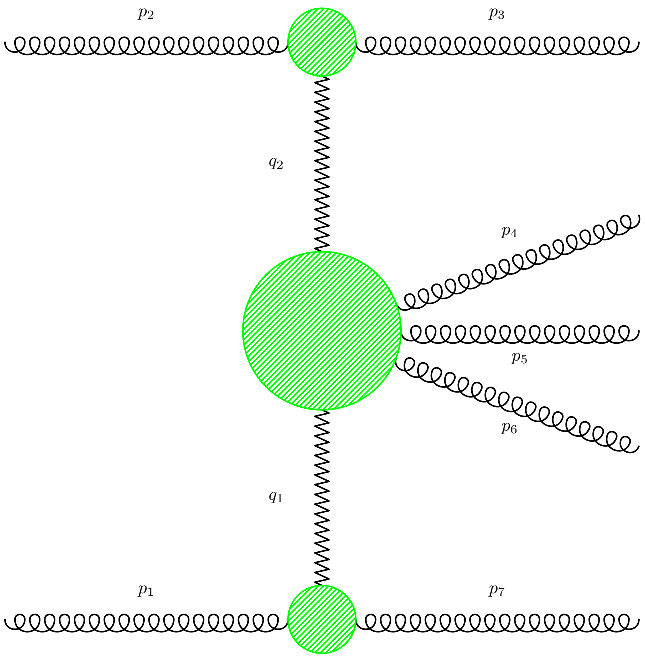
Duhr Glover Smirnov VDD 08

Quasi-multi-Regge limit of *n*-sided Wilson loop

- 7-pt amplitude in the qmR limit of a triple along the ladder

$$y_3 \gg y_4 \simeq y_5 \simeq y_6 \gg y_7; \qquad |p_{3\perp}| \simeq |p_{4\perp}| \simeq |p_{5\perp}| \simeq |p_{6\perp}| \simeq |p_{7\perp}|$$



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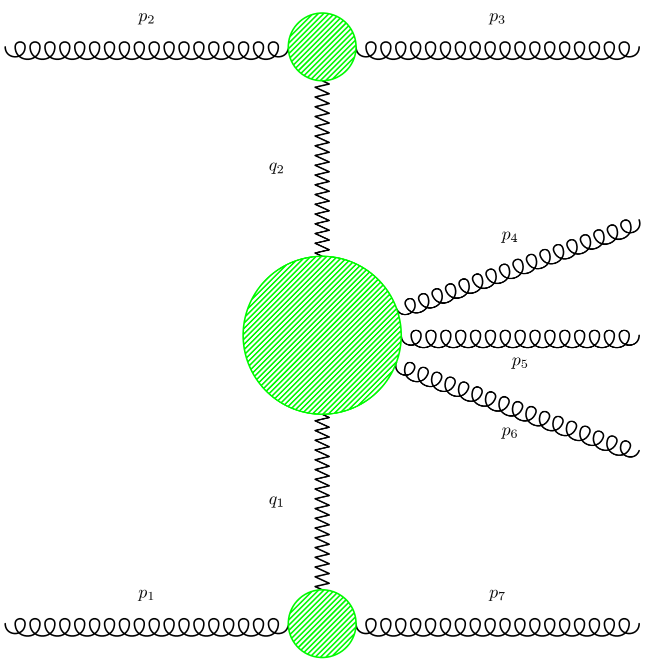

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- can be generalised to the *n*-pt amplitude in the qmR limit of a (*n*-4)-ple along the ladder

$$y_3 \gg y_4 \simeq \ldots \simeq y_{n-1} \gg y_n; \qquad |p_{3\perp}| \simeq \ldots \simeq |p_{n\perp}|$$

Duhr Smirnov VDD 09

Quasi-multi-Regge limit of Wilson loops

- L-loop Wilson loops are Regge exact

Drummond Korchemsky Sokatchev 07
Duhr Smirnov VDD 09

$$w_n^{(L)}(\epsilon) = f_{WL}^{(L)}(\epsilon)\, w_n^{(1)}(L\epsilon) + C_{WL}^{(L)} + R_{n,WL}^{(L)}(u_{ij}) + \mathcal{O}(\epsilon)$$

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- we may compute the Wilson loop in qmRk
  the result will be correct in general kinematics !!!

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$$\int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \frac{\mathrm{d}z_1}{2\pi i} \frac{\mathrm{d}z_2}{2\pi i} \frac{\mathrm{d}z_3}{2\pi i} (z_1\, z_2 + z_2\, z_3 + z_3\, z_1)\, u_1^{z_1}\, u_2^{z_2}\, u_3^{z_3}$$
$$\times \Gamma(-z_1)^2\, \Gamma(-z_2)^2\, \Gamma(-z_3)^2\, \Gamma(z_1+z_2)\, \Gamma(z_2+z_3)\, \Gamma(z_3+z_1)$$

the result is in terms of Goncharov polylogarithms

$$G(a, \vec{w}; z) = \int_0^z \frac{\mathrm{d}t}{t-a}\, G(\vec{w}; t)\,, \qquad G(a; z) = \ln\left(1 - \frac{z}{a}\right)$$

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- the remainder function $R_6^{(2)}$ is given in terms of $O(10^3)$ Goncharov polylogarithms $G(u_1, u_2, u_3)$

Duhr Smirnov VDD 09

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↓ finite answer, but in intermediate steps many divergences
output is punishingly long

our result has been simplified and given in terms of polylogarithms

Goncharov Spradlin Vergu Volovich 10

$$
\begin{aligned}
R^{(2)}_{6,WL}(u_1,u_2,u_3) &= \sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-) - \frac{1}{2}\mathrm{Li}_4(1-1/u_i)\right) \\
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where

$$x_i^\pm = u_i x^\pm \qquad x^\pm = \frac{u_1+u_2+u_3-1\pm\sqrt{\Delta}}{2u_1u_2u_3} \qquad \Delta = (u_1+u_2+u_3-1)^2-4u_1u_2u_3$$

$$L_4(x^+,x^-) = \sum_{m=0}^{3}\frac{(-1)^m}{(2m)!!}\log(x^+x^-)^m(\ell_{4-m}(x^+)+\ell_{4-m}(x^-))+\frac{1}{8!!}\log(x^+x^-)^4$$

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↑ answer is short and simple
introduces the *theory of motives* in TH physics

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Fn. F of $deg(F) = n$:   fn. with log cuts, s.t. $Disc = 2\pi i \times f$, with $w(f) = n\text{-}1$

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R_i rational functions

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the symbol is $\mathrm{Sym}[T_k] = R_1 \otimes \cdots \otimes R_k$

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$\Rightarrow$ a symbol determines a polynomial of uniform degree up to a constant

One-loop amplitude squared

the 2-loop *n*-pt amplitude is

$$m_n^{(2)}(\epsilon) = \frac{1}{2}\left[m_n^{(1)}(\epsilon)\right]^2 + f^{(2)}(\epsilon)\, m_n^{(1)}(2\epsilon) + Const^{(2)} + R$$

what about that ?

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↓ not practical for phenomenology (where DR rules the waves)

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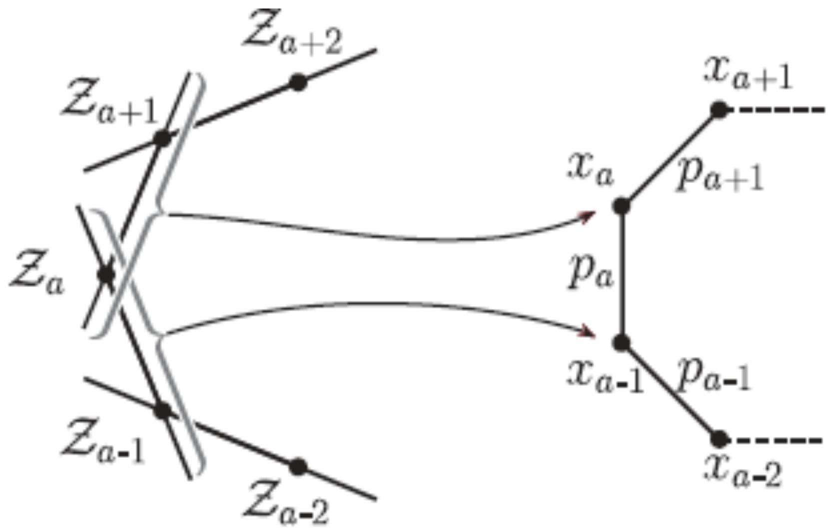

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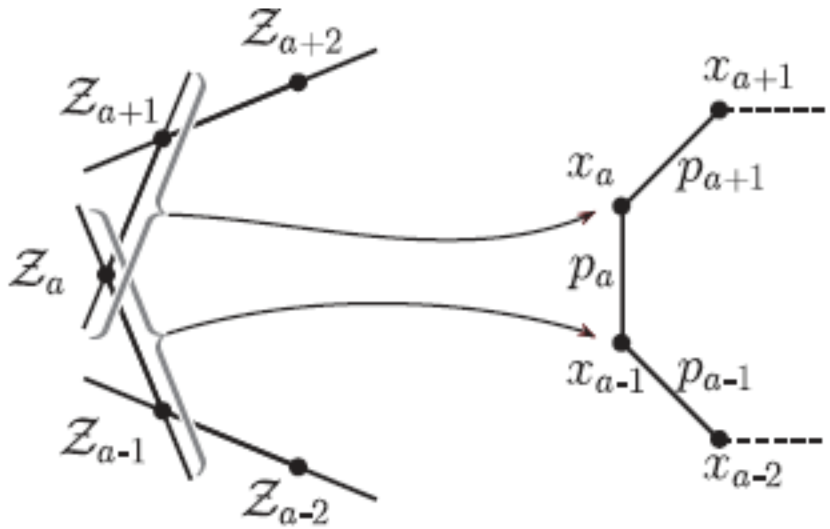
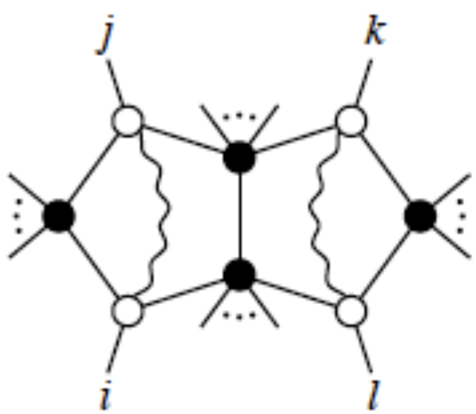


2-loop *n*-pt MHV amplitudes can be written as sum of pentaboxes in twistor space

$$m_n^{(2)} = \frac{1}{2} \sum_{i<j<k<l<i}$$



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Conclusions

- Planar N=4 SYM is a great lab where to test comparisons between strong and weak couplings
- features weak-strong duality and weak-weak duality
- Wilson loops are the ideal quantities to perform those comparisons
- first (and so far only) analytic computation of 2-loop hexagon Wilson loop
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- progress (symbols) recently to understand 2-loop n-side Wilson loops
- more is to come ... stay tuned!

In October 7-11,
we shall have a
School of Analytic Computing
in Atrani, Italy

lectures on amplitudes & Wilson loops by
Fernando Alday
Simon Caron-Huot
Claude Duhr
Johannes Henn
Henrik Johansson
Vladimir Smirnov