FERMION MASSES FROM A CROSSOVER MECHANISM

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GROSS-NEVEU MODEL - LARGE N

$$S_{GN}\left[\Psi,\bar{\Psi}\right]=\int d^dx\left[\bar{\Psi}\left(x\right)(\partial\!\!\!/+m_B)\Psi\left(x\right)-\frac{G_B}{2}\left(\bar{\Psi}\cdot\Psi\right)^2\right] \tag{1}$$

Gap Equation (Large N, only Hartree terms)

$$m=m_B+G_B\int\frac{d^4p}{(2\pi)^4}\mathbf{tr}\left(\frac{1}{ip\!\!\!/+m}\right)\qquad\Rightarrow\qquad m-m_B=\frac{m\,G_B\Lambda^2}{4\pi^2}\left(1-\frac{m^2}{\Lambda^2}\ln\frac{\Lambda^2}{m^2}\right)$$

For $m_B=0$ **:**

$$\frac{4\pi^2}{G_B\Lambda^2}=1-\frac{m^2}{\Lambda^2}\ln\frac{\Lambda^2}{m^2}\qquad\Rightarrow\qquad G_B\geq G_c=\frac{4\pi^2}{\Lambda^2}\Rightarrow m\neq 0$$

Four point function (s **and** t **channels only contribute.** $q^2=s$ or t **) :**

$$\begin{aligned} D(q^2) &= \frac{G_B}{1-G_B\Pi\left(q^2\right)} \\ \Pi\left(q^2\right) &= \int\frac{d^4p}{(2\pi)^4}\mathbf{tr}\left(\frac{1}{ip\!\!\!/+m}\cdot\frac{1}{i\left(p\!\!\!/+q\!\!\!/\right)+m}\right) \end{aligned}$$

HUBBARD-STRATONOVICH TRANSFORMATION

Generating Functional $Z[J]$ for the Green's functions composite operator $\bar{\Psi}\Psi$:

$$\begin{aligned} Z[J] &= \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\, e^{\left[-\int \bar{\Psi}\partial\!\!\!/\Psi + \int \frac{G}{2}(\bar{\Psi}\Psi)^2 + \int J\bar{\Psi}\Psi\right]} \\ &= \mathcal{N} e^{-J^2/2} \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi \int \mathcal{D}\sigma\, e^{\left[-\int \bar{\Psi}(\partial\!\!\!/+\sigma)\Psi - \int \frac{1}{2G}\sigma^2 - \int \frac{J}{G}\sigma\right]} \\ &= \mathcal{N} e^{-J^2/2} \int \mathcal{D}\sigma\, e^{\left[-\int \frac{1}{2G}\sigma^2 + Tr\log(\partial\!\!\!/+\sigma) - \int \frac{J}{G}\sigma\right]} \end{aligned}$$

Gap Equation (Tadpole equation) :

$$\frac{1}{G}\sigma = N\,\frac{\mathbf{tr}\,\mathbb{I}}{(2\pi)^d}\sigma \int \frac{d^d p}{p^2+\sigma^2}$$

Inverse composite scalar propagator :

$$D^{-1}(q) = N\,\frac{\mathbf{tr}\,\mathbb{I}}{2}\left(q^2+4\sigma^2\right)\int \frac{d^d p}{(2\pi)^d}\frac{1}{(p^2+\sigma^2)\left[(p+q)^2+\sigma^2\right]}$$

CHIRAL SYMMETRY BREAKING : PHASE TRANSITION

Order parameter : $\langle \bar{\psi}\psi \rangle$

Critical exponents : $(\tau = 1/G - 1/G_c)$

Leading Order Large N

β **: scaling of** $\langle \bar{\psi}\psi \rangle$ **around** G_c**:** $\langle \bar{\psi}\psi \rangle \sim \tau^{\beta}$ — $\beta = \frac{1}{d-2}$

δ **: scaling of** $\langle \bar{\psi}\psi \rangle$ **with** m_B**:** $\langle \bar{\psi}\psi \rangle\big|_{\tau=0} \sim m_B^{\frac{1}{\delta}}$ — $\delta = d - 1$

γ **: scaling of** χ **near** $G = G_c$**:** $\chi \sim |\tau|^{-\gamma}$ — $\gamma = 1$

$$\left[\ \langle \bar{\psi}\psi(x)\,\bar{\psi}\psi(0)\rangle_{1PI} \quad \Rightarrow \quad D(q) \quad \Rightarrow \quad \chi \quad : \quad \chi^{-1} = GD(0)^{-1}\ \right]$$

η **: scaling of** $D(q)$ **for large momenta :** $D(q)|_{\tau=0} \sim \frac{1}{q^{2-\eta}}$ — $\eta = 4 - d$

ν **: scaling of the correlation length :** $\xi \sim \tau^{-\nu}$ — $\nu = \frac{1}{d-2}$

Correl. length ξ **:** $\xi = Z^{1/2}\chi^{-1/2}$ $\quad \left(Z = \frac{d^2 D^{-1}(q^2)}{dq^2}\bigg|_{\tau=0;q^2=0} = 1 - \frac{\eta}{2} \right)$

Remark : For $d = 4$ **: Mean Field Values (Hyperscaling violations)**

EQUIVALENCE GN MODEL - YUKAWA MODEL

Green's functions for the auxiliary field σ in the GN model versus Green's functions for the elementary field ϕ in the Yukawa model (N has been absorbed in G, λ, g^2).

GROSS - NEVEU MODEL | **YUKAWA MODEL**

$$0 = \frac{1}{G}\sigma - \frac{\mathbf{tr}\,\mathbb{I}}{(2\pi)^d}\sigma \int \frac{d^d p}{p^2+\sigma^2} \quad ; \quad 0 = \frac{M^2}{g^2}\phi + \frac{\lambda}{6g^4}\phi^3 - \frac{\mathbf{tr}\,\mathbb{I}}{(2\pi)^d}\phi \int \frac{d^d p}{p^2+\phi^2}$$

$$D^{-1}(q^2) = \frac{\mathbf{tr}\,\mathbb{I}}{2(2\pi)^d}(q^2+4\sigma^2)\int \frac{d^d p}{(p^2+\sigma^2)[(p+q)^2+\sigma^2]} \;;\; \Delta^{-1}(q^2) = \frac{q^2}{g^2} + \frac{\lambda\phi^2}{3g^4} + \frac{\mathbf{tr}\,\mathbb{I}}{2(2\pi)^d}(q^2+4\phi^2)\int \frac{d^d p}{(p^2+\phi^2)[(p+q)^2+\phi^2]}$$

In the scaling region ($p^2/\Lambda^2 << 1$ and $\sigma^2/\Lambda^2 << 1$), the GN and Yukawa Eqs. coincide (Zinn-Justin, 1991). In the Yukawa model, this region is nothing but the linearization region around the non-trivial (so called) Wilson-Fisher-Yukawa Fixed Point .

$\Rightarrow$ **GN model belongs to the same universality class of the Yukawa model**

WILSONIAN APPROACH

The wilsonian action at the scale k :

$$S_k\left[\Psi,\bar{\Psi}\right]=\int d^dx\left[\bar{\Psi}\left(x\right)\partial\!\!\!/\,\Psi\left(x\right)+U_k\left(\bar{\Psi}\cdot\Psi\right)\right] \tag{2}$$

The field $\Psi(x)$ contains Fourier modes up to k.

In order to define the Wilsonian action $S_{k-\delta k}$ at an infinitesimally lower scale $k-\delta k$, we split the modes:

$$\Psi(x)\equiv\psi(x)+\xi(x)=\sum_{|p|\leq k-\delta k}\frac{e^{-ip\cdot x}}{V}\psi_p+\sum_{k-\delta k\leq|p|\leq k}\frac{e^{-ip\cdot x}}{V}\xi_p \tag{3}$$

$S_{k-\delta k}\left[\psi,\bar{\psi}\right]$ **is defined from :**

$$e^{-S_{k-\delta k}\left[\psi,\bar{\psi}\right]} \quad = \quad \mathcal{N}_d\int\mathcal{D}\bar{\xi}\mathcal{D}\xi e^{-S_k\left[\psi+\xi,\bar{\psi}+\bar{\xi}\right]} \tag{4}$$

Local Potential Approximation : keeping for S_k the ansatz (2) at all scales k.

The integration in $D\xi$ (gaussian and exact!) is easily performed. For $U_{k-\delta k}$ we get:

WILSONIAN APPROACH cont'd

$$U_{k-\delta k} = U_k - \frac{1}{2}\int' \frac{d^dp}{(2\pi)^d}\,\mathbf{tr}\ln\left(\frac{\Delta}{k}\right) \tag{5}$$

where the integral is performed within the shell $[k-\delta k, k]$ and Δ is :

$$\Delta = \begin{pmatrix} \frac{\delta^2 U_k}{\delta\bar\psi\delta\bar\psi} & -i\not p + \frac{\delta^2 U_k}{\delta\psi\delta\bar\psi} \\ -i\not p^T + \frac{\delta^2 U_k}{\delta\bar\psi\delta\psi} & \frac{\delta^2 U_k}{\delta\psi\delta\psi} \end{pmatrix} \tag{6}$$

The above equation is immediately written as an RG differential equation:

$$k\frac{dU}{dk} = k^d C_d\left(N\mathbf{tr}\,\mathbb{I}\ln\left(\frac{k^2+U_\rho^2}{k^2}\right) - \ln\left(1+\frac{2\rho\,U_\rho U_{\rho\rho}}{k^2+U_\rho^2}\right)\right) \tag{7}$$

[$\rho = \bar\psi\psi$, $U_\rho = dU_k(\rho)/d\rho$, $\mathbf{tr}\,\mathbb{I}$ is the trace of the identity matrix in Dirac space ($\mathbf{tr}\,\mathbb{I} = 2^{d/2}$ for even values of d and $\mathbf{tr}\,\mathbb{I} = 2^{(d-1)/2}$ for odd values), $C_d = \frac{1}{(4\pi)^{d/2}\Gamma(d/2)}$]

Defining: $t = \ln(k_0/k)$ (k_0 arbitrary scale), $\sigma = k^{1-d}\rho$ and $V(\sigma,t) = k^d\,U_k(\rho)$, the dimensionless version of the above equation is :

$$\frac{\partial}{\partial t}V = d\,V - (d-1)\,\sigma V_\sigma - C_d N\mathbf{tr}\,\mathbb{I}\ln\left(1+V_\sigma^2\right) + C_d\ln\left(1+\frac{2\sigma V_{\sigma\sigma}}{1+V_\sigma^2}\right) \tag{8}$$

WILSONIAN APPROACH cont'd

$$k\frac{dU}{dk} = k^d C_d \left(N\mathbf{tr}\,\mathbb{I}\ln\left(\frac{k^2+U_\rho^2}{k^2}\right) - \ln\left(1+\frac{2\rho\,U_\rho U_{\rho\rho}}{k^2+U_\rho^2}\right)\right) \tag{9}$$

In the r.h.s. of this equation we recognize an Hartree and a Fock term (first and second term). In the large N limit, we neglect the Fock term :

$$k\frac{dU}{dk} = k^d C_d N\mathbf{tr}\,\mathbb{I}\ln\left(\frac{k^2+U_\rho^2}{k^2}\right) \tag{10}$$

This is the ladder approximation for our RG equation.

Dimless version :

$$\frac{\partial}{\partial t}V = d\,V - (d-1)\,\sigma V_\sigma - C_d N\mathbf{tr}\,\mathbb{I}\ln\left(1+V_\sigma^2\right) \tag{11}$$

We are now ready to study the GN model, or extended versions of the GN model, within the framework of the wilsonian RG approach.

GN MODEL à la WILSON

(Remember $\sigma = \frac{1}{k^{(d-1)}}\,\bar\psi\psi$ **,** $V(\sigma,t) = \frac{1}{k^d}\,U_k(\bar\psi\psi)$ **)**

$$V(\sigma,t) = -\frac{G(t)}{2}\sigma^2 \qquad \left(\text{dimful : } U_k(\bar\psi\psi) = -\frac{G_k}{2}\bar\psi\psi\bar\psi\psi\right) \tag{12}$$

The RG Eq. for $U_k(\bar\psi\psi)$ is just the RG Eq. for the Fermi constant:

$$\frac{dG}{dt} \;=\; (2-d)\,G + 2N\mathbf{tr}\,\mathbb{I}C_d G^2\,. \tag{13}$$

For $d = 2$, this is the equation found by Gross and Neveu. The β function vanishes at $G = 0$ and the theory is asymptotically free: $G = 0$ is an UV stable fixed point.

For $d > 2$, the beta function vanishes at

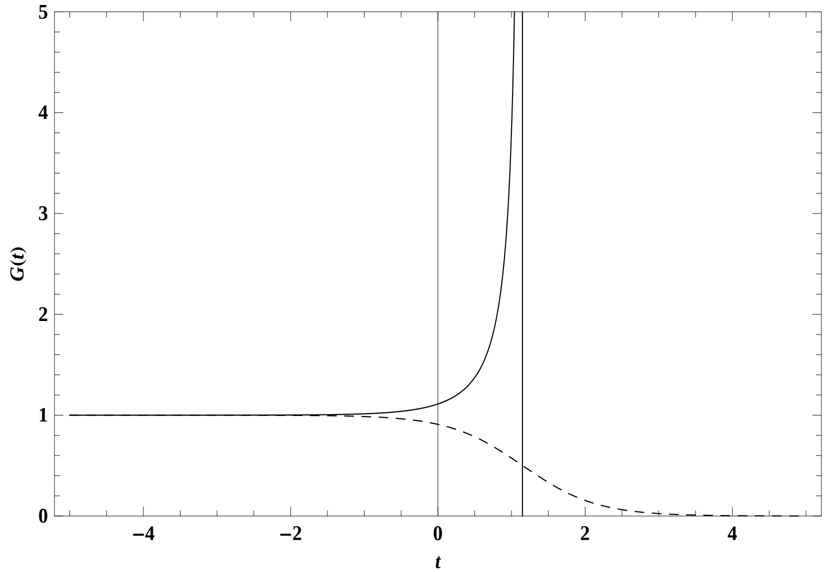
$$G = 0 \qquad\qquad \text{and} \qquad\qquad G = G_c = \frac{d-2}{2\,N\mathbf{tr}\,\mathbb{I}C_d} \tag{14}$$

and the solution to the RG equation for $G(t)$ is:

$$G\,(t) = \frac{G_c}{1 - Be^{(d-2)t}}\,, \tag{15}$$

B **is the arbitrary integration constant.**

GN MODEL à la WILSON cont'd

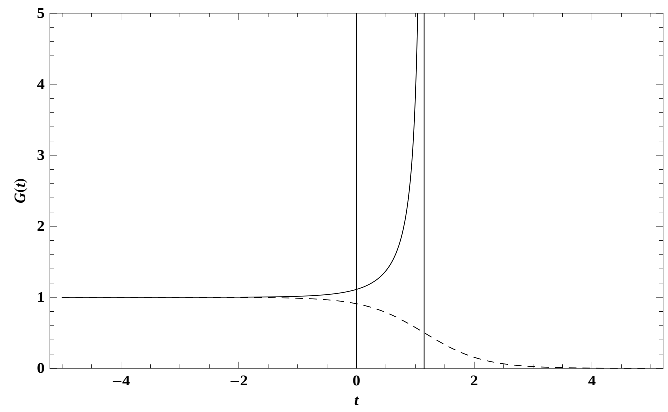


$$G(t) = \frac{G_c}{1 - Be^{(d-2)t}} \tag{16}$$

The $B < 0$ case **(dashed line). For $t \to -\infty$ $(k \to \infty) G(t)$ flows towards G_c. In the IR, $(t \to \infty$ or $k \to 0)$, $G(t)$ vanishes. $\Rightarrow G_c$ UV fixed point ; $G = 0$ IR fixed point.**

The $B > 0$ case **(solid line). In the UV, $G(t)$ again flows towards G_c . In its flow towards the IR, however, $G(t)$ diverges at a finite value of k, $k = k_c$ $(t = t_c)$.**

GN MODEL à la WILSON cont'd



As noted by Gross and Neveu, this is also what happens for the IR flow of $G(t)$ in the $d = 2$ case. By computing the effective potential in $d = 2$, they were able to relate this divergence of $G(t)$ with the existence of χSB and the consequent generation of a fermion mass.

Such a divergence in the RG flow is the precursor of the transition to the broken phase which occurs when the boundary of $G(t)$ at the UV scale Λ is such that: $G(t = 0) > G_c \quad \Rightarrow \quad G_c$ has to be identified with the critical point of the transition.

As we include fluctuations of longer and longer wavelength (i.e. we run the RG equation towards the IR), the system starts to develop an instability which eventually manifests itself in the divergence of the coupling constants.

CRITICAL EXPONENTS FROM WILSON RG EQUATIONS

Complete description of χSB transition : we need to add a mass term.

Analogy between χSB and ferromagnetic phase transition :

$$\tau = T - T_c \;\sim\; 1/G_B - 1/G_c \qquad h \;\sim\; m_B$$

Therefore :

$$V(\sigma,t) = m(t)\,\sigma - \frac{G(t)}{2}\sigma^2 \qquad \left(\text{dimful :} \quad U_k(\bar\psi\psi) = m_k\,\bar\psi\psi - \frac{G_k}{2}\,\bar\psi\psi\bar\psi\psi\right) \tag{17}$$

RG equation for $V(\sigma,t)$ $\Rightarrow$ **RG equation for $m(t)$ and $G(t)$:**

$$\frac{dm}{dt} = m + 2N\,\mathbf{tr}\,\mathbb{I}\,C_d\frac{Gm}{1+m^2} \tag{18}$$

$$\frac{dG}{dt} = (2-d)\,G + 2N\,\mathbf{tr}\,\mathbb{I}\,C_d G^2\frac{1-m^2}{(1+m^2)^2} \tag{19}$$

Gaussian fixed point

$$m = 0$$

$$G = 0$$

Non-Gaussian fixed point

$$m = 0 \tag{20}$$

$$G = \frac{d-2}{2N\mathbf{tr}\,\mathbb{I}C_d}\,. \tag{21}$$

CRITICAL EXP'S FROM WILSON RG EQUATIONS cont'd

Linearizing around the Gaussian FP :

$$\frac{dm}{dt} = m \quad \Rightarrow \quad m(t) \sim e^{t} \quad \Rightarrow \quad \lambda_m = 1 \tag{22}$$

$$\frac{dG}{dt} = (2-d)\,G \quad \Rightarrow \quad G(t) \sim e^{(2-d)t} \quad \Rightarrow \quad \lambda_G = 2-d \tag{23}$$

Linearizing around the Non-Gaussian FP :

$$\frac{dm}{dt} = (d-1)m \quad \Rightarrow \quad m(t) \sim e^{(d-1)t} \quad \Rightarrow \quad \lambda_m = d-1 \tag{24}$$

$$\frac{dG}{dt} = (d-2)\,G \quad \Rightarrow \quad G(t) \sim e^{(d-2)t} \quad \Rightarrow \quad \lambda_G = d-2 \tag{25}$$

Around the Non-Gaussian FP, for $d > 2$, *both m* and G are relevant couplings.

Technically : the (G, m) plane is an "UV critical surface" for the NGFP: m and G always reach this point in the UV. The theory is said : "asymptotically safe".

Canonical and Anomalous dimensions : λ_m and λ_G around the GFP are the *canonical dimensions* **of m and G. By comparing with λ_m and λ_G around the NGFP, we find the** *anomalous dimensions* **γ_m and γ_G of m and G are:**

$$\gamma_m = d-2 \qquad \gamma_G = 2(d-2) \tag{26}$$

CRITICAL EXP'S FROM WILSON RG EQUATIONS cont'd

Hyperscaling relations : obtained under very general conditions : the scale invariance of $1PI$ vertex functions, existence of a fixed point, linearization of RG equations around the fixed point. Under these conditions : critical exponents given in terms of the eigenvalues related to the relevant parameters (m and G in our case, the magnetic field and the temperature in the ferromagnetic case).

For the χSB transition we get :

$$\begin{aligned} \beta &= \frac{d-\lambda_m}{\lambda_G} \qquad \delta = \frac{\lambda_m}{d-\lambda_m} \\ \gamma &= \frac{2\lambda_m - d}{\lambda_G} \qquad \nu = \frac{1}{\lambda_G} \\ \eta &= 2 + d - 2\lambda_m \end{aligned} \tag{27}$$

Replacing in these Eqs. λ_m and λ_G :

$$\beta = \frac{1}{d-2} \tag{28}$$

$$\delta = d-1 \tag{29}$$

$$\gamma = 1 \tag{30}$$

$$\nu = \frac{1}{d-2} \tag{31}$$

$$\eta = 4-d\,. \tag{32}$$

These values coincide with those obtained at the leading order of the $1/N$ expansion.

CRITICAL EXP'S FROM WILSON RG EQUATIONS cont'd

Warning :

$d = 4$ **is the upper critical dimension: for** $d > 4$**, the theory becomes free and the mean field results are exact. At** $d = 4$**, the correlation functions get weak logarithmic corrections to power law behavior. The LPA does not allow us to detect the corresponding hyperscaling violations, which are due to non local contributions (absent in the LPA). This point has to be further investigated. The occurrence of the transition, however, is observed already at this stage of the approximation.**

We have seen how the RG equations for $m(t)$ **and** $G(t)$ **allow to compute the critical exponents which give the behavior of the correlators for values of** G **close to the critical point. To this end, it was sufficient to study the linearized equations in the neighborhood of the non trivial fixed point (**$G = G_c, m = 0$**).**

But we can do more - We can do better

We can extend the previous analysis to the study of the G **and** m **flows in the whole (**G, m**) plane, rather than limiting ourselves to the linearization region around the non-trivial fixed point. These flows have something to tell us in connection with the problem of the generation of fermion masses.**

Crossover mechanism for generating fermion masses

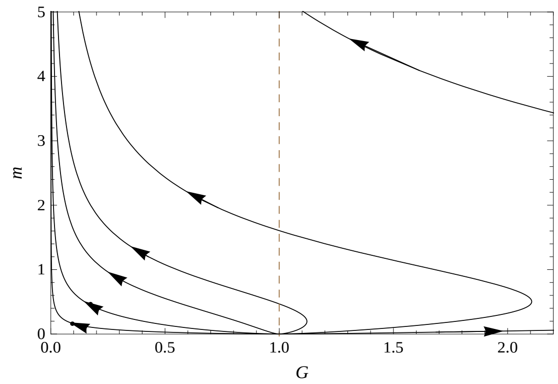
THE RG EQUATIONS FOR $m(t)$ and $G(t)$

$$\frac{dm}{dt} = m + 2N\,\mathbf{tr}\,\mathbb{I}\,C_d \frac{Gm}{1+m^2} \quad (33)$$

$$\frac{dG}{dt} = (2-d)\,G + 2N\,\mathbf{tr}\,\mathbb{I}\,C_d G^2 \frac{1-m^2}{\left(1+m^2\right)^2} \quad (34)$$

From now on, let us specify to the $d=4$ case ... so that we can draw pictures..

FLOW IN THE (G,m) PLANE



Four different types of trajectories. Two of them lie on the G axis ($m = 0$, already studied) :

(a) UV boundaries $m = 0$ and $G < G_c$: the trajectory flows along the G axis (right $\rightarrow$ left) towards the gaussian fixed point (Symmetric phase).

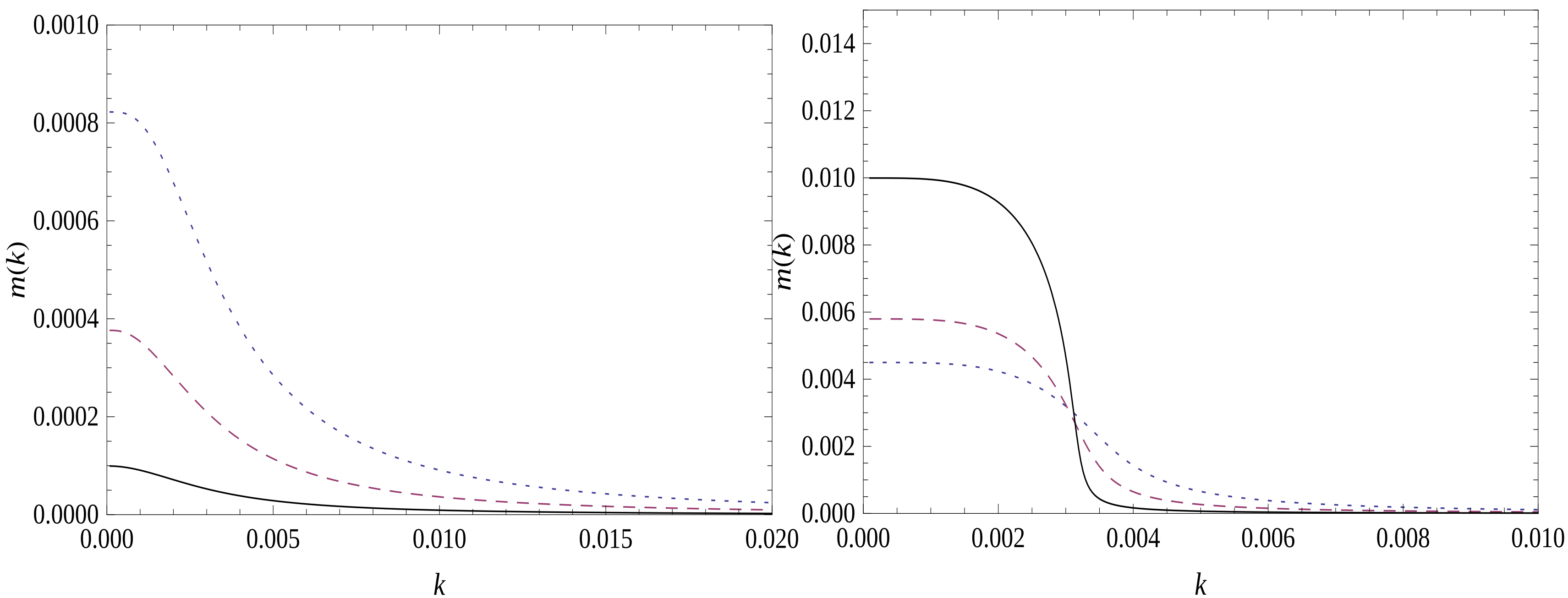
(b) UV boundaries $m = 0$ and $G > G_c$, the trajectory flows along the G axis (left $\rightarrow$ right) and reaches $G = \infty$ at a finite value of t (finite value of k) (Broken phase).

UV boundary $m \neq 0$: two types of trajectories :

(c) UV boundary $G < G_c$: the trajectory flows from right to left asymptotically converging to the m axis.

(d) UV boundary $G > G_c$: the trajectory initially flows from left to right, moving away from the $G = G_c$ axis. Then, after creating a more or less pronounced horizontal bell, it turns back, crosses the $G = G_c$ axis and eventually converges asymptotically to the m axis as in the previous case.

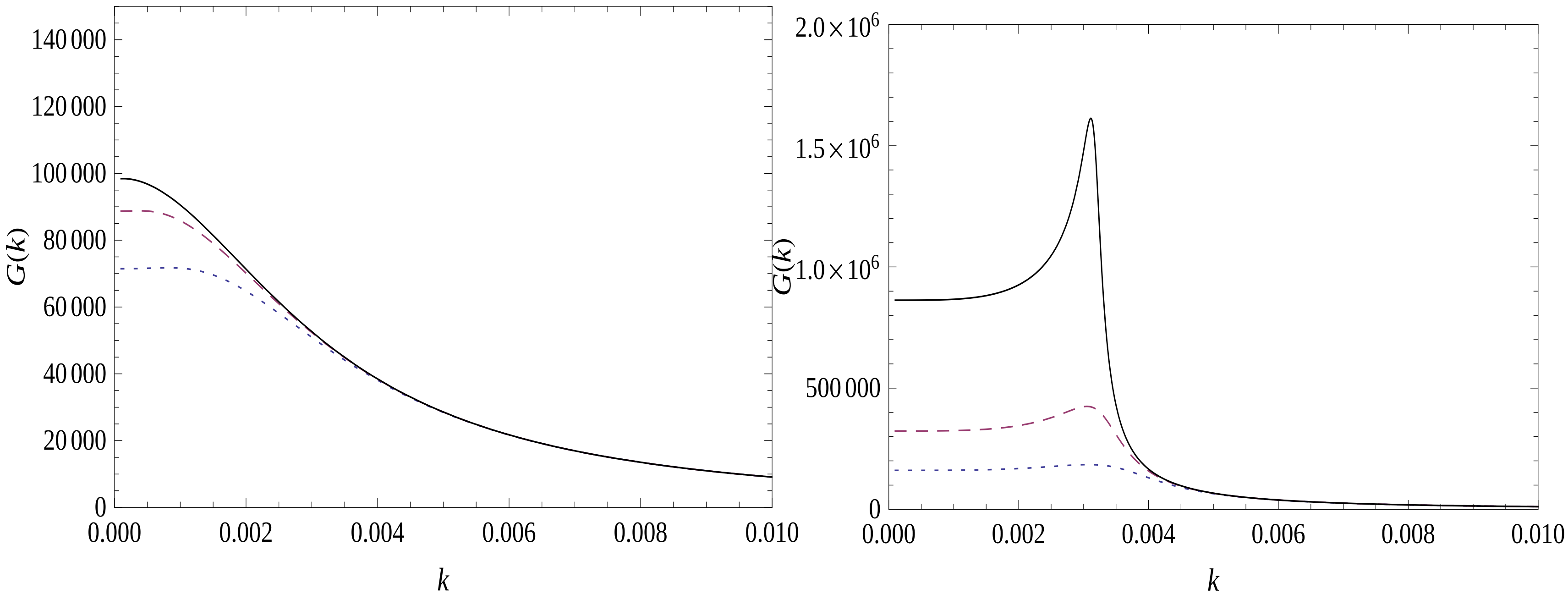
FLOW OF THE DIMENSIONFUL MASS $m(k)$



(a) **(b)**

Figura 1: (a): The running dimensionful mass obtained by setting, at $k = \Lambda = 1$, $G_B = G_c - 10^{-5}$ and $m_B = 10^{-8}, 4 * 10^{-9}, 10^{-9}$ (dotted, dashed and solid lines, respectively). (b): The running dimensionful mass obtained by setting, at $k = \Lambda = 1$, $G_B = G_c + 10^{-5}$ and $m_B = 10^{-8}, 4 * 10^{-9}, 10^{-9}$ (dotted, dashed and solid lines, respectively). All quantities are expressed in cut-off units.

FLOW OF THE DIMENSIONFUL FERMI CONSTANT $G(k)$



(a) **(b)**

Figura 2: (a): The running of the dimensionful coupling $G(k)$ obtained by setting, at $k = \Lambda = 1$, $G_B = G_c - 10^{-5}$ and $m_B = 10^{-8}, 4 * 10^{-9}, 10^{-9}$ (dotted, dashed and solid lines, respectively). (b): The running of the dimensionful coupling $G(k)$ obtained by setting, at $k = \Lambda = 1$, $G_B = G_c + 10^{-5}$ and $m_B = 10^{-8}, 4 * 10^{-9}, 10^{-9}$ (dotted, dashed and solid lines, respectively). All quantities are expressed in cut-off units.

IR and UV ASYMPTOTICS OF $m(k)$ and $G(k)$

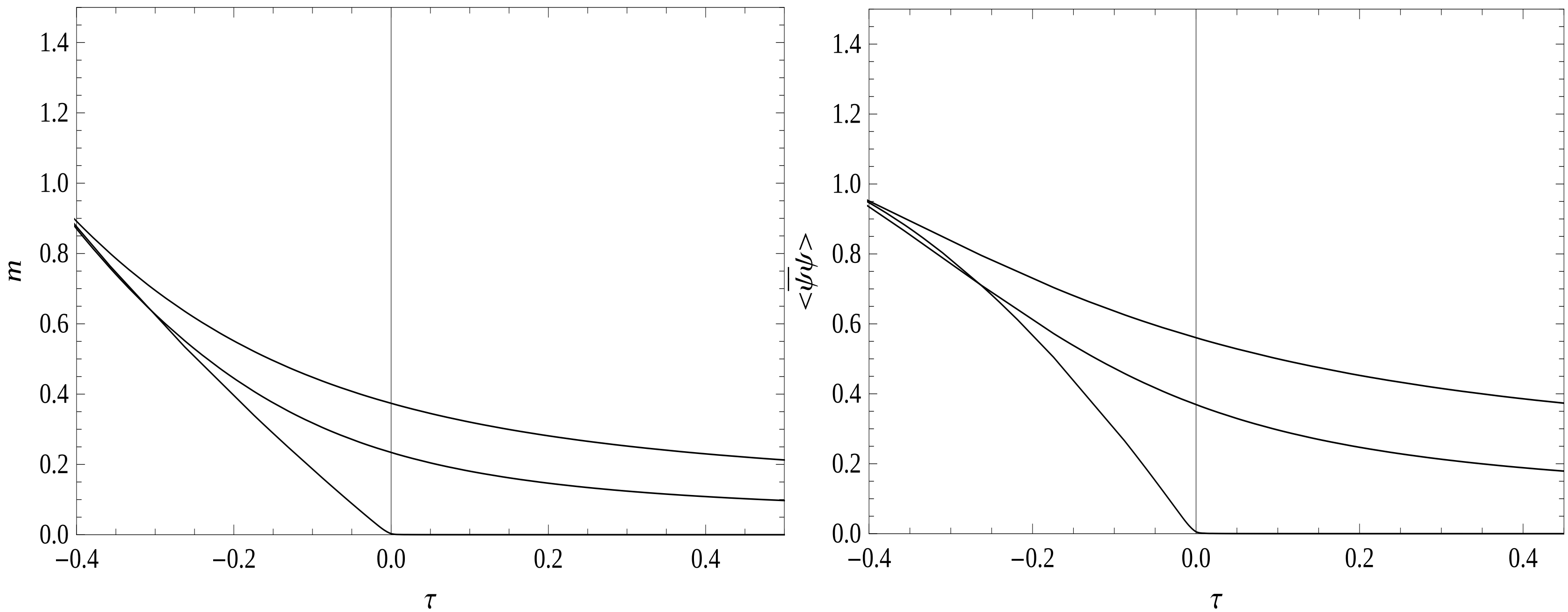
From the GR equations for $m(k)$ and $G(k)$ it is easy to get their asymptotic IR ($k \to 0$) and UV ($k \to \infty$) analytical behavior for $m(k)$ and $G(k)$:

$$k \;\to\; 0 \;:\quad m(k) \sim \overline{m} - \frac{\overline{G}}{8\pi^2\,\overline{m}}\,k^4 \quad ; \qquad G(k) \sim \overline{G} + \frac{\overline{G}^2}{8\pi^2\overline{m}^2}k^4 \tag{35}$$

$$k \;\to\; \infty \;:\quad m(k) \sim \frac{M^3}{k^2} \quad ; \qquad G(k) \sim \frac{4\pi^2}{k^2} \tag{36}$$

This behavior is easily checked against the numerical solutions of the previous slides.

CROSSOVER and χSB



(a) **(b)**

Figura 3: Different phase diagrams showing (a) the mass and (b) $\langle \bar{\psi}\psi \rangle$ as functions of $\tau = 1/G_B - 1/G_c$. The different lines correspond (from top to bottom) to the choices: $m_B = 10^{-1}, 5 * 10^{-2}, 10^{-4}$. All quantities are expressed in cut-off units.

Analytical approximations

NJL approximation

Specifying to the $d = 4$ case (NG is redefined as G):

$$\frac{dm}{dk} = -\frac{k^3}{2\pi^2}\frac{Gm}{k^2+m^2} \tag{37}$$

Approx. (defining the dimensionless Fermi constant $\tilde{G}_\Lambda = G\,\Lambda^2$) :

$$\frac{dm(k)}{dk} = -\frac{k^3}{2\pi^2}\,\overline{m}\,\frac{\tilde{G}_\Lambda}{\Lambda^2}\,\frac{1}{k^2+\overline{m}^2} \tag{38}$$

Integrating this equation between $k = 0$ and $k = \Lambda$:

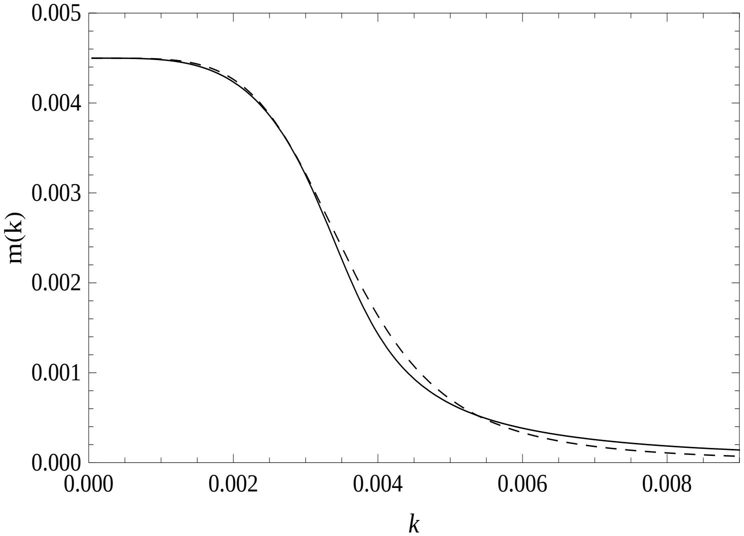
$$m\,(\Lambda) - m\,(0) = -\frac{m\tilde{G}_\Lambda}{4\pi^2}\left[\Lambda^2 - m^2 \ln\left(1+\frac{\Lambda^2}{m^2}\right)\right] \tag{39}$$

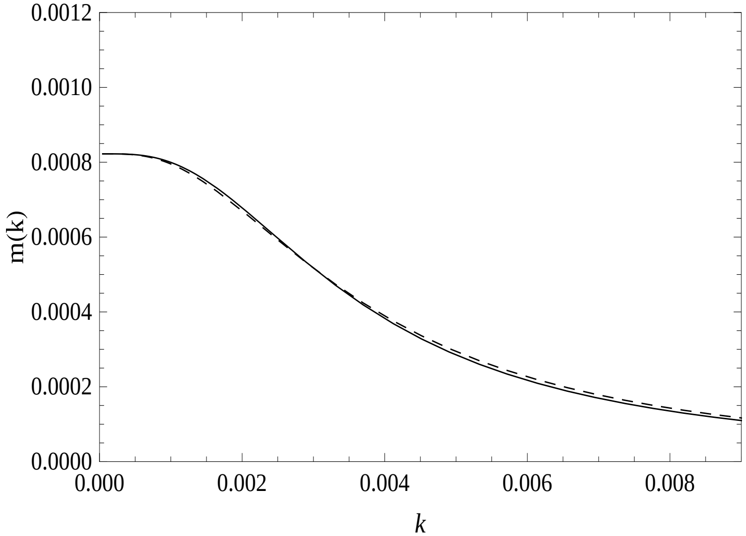
For vanishing bare mass $m(\Lambda) = m_B = 0$,

$$\frac{4\pi^2}{\tilde{G}_\Lambda} = 1 - \frac{\overline{m}^2}{\Lambda^2}\ln\frac{\Lambda^2}{\overline{m}^2}\,, \tag{40}$$

which is the NJL result. For values of $\tilde{G}_\Lambda$ greater than $G_c = 4\pi^2$; non vanishing solution for the fermion mass $\overline{m}$, while for values of $\tilde{G}_\Lambda$ smaller than $G_c = 4\pi^2$, the only solution for $\overline{m}$ is $\overline{m} = 0$.

Analytical approximation beyond NJL





$$\begin{aligned} m(k) &= \overline{m}\exp\Bigg(-\frac{1}{a^4+2a^2\left(\overline{m}^2+M_3^2\right)+\left(\overline{m}^2-M_3^2\right)^2}\Big(\overline{m}^2\left(-a^2+\overline{m}^2-M_3^2\right)\ln\left(\frac{k^2}{\overline{m}^2}+1\right) \\ &\quad+\left(a^4+a^2\left(3\overline{m}^2+2M_3^2\right)-\overline{m}^2M_3^2+M_3^4\right)\ln\left(\frac{k^2-2ak}{a^2+M_3^2}+1\right) \\ &\quad+\frac{2a\left(a^4+a^2\left(\overline{m}^2+2M_3^2\right)-3\overline{m}^2M_3^2+M_3^4\right)}{M_3}\left(\tan^{-1}\left(\frac{k-a}{M_3}\right)+\tan^{-1}\left(\frac{a}{M_3}\right)\right) \\ &\quad+4a\overline{m}^3\tan^{-1}\left(\frac{k}{\overline{m}}\right)\Big)\Bigg) \end{aligned} \tag{41}$$

Adding Higher Powers of $\bar{\psi}\psi$

Adding Higher Powers of $\bar{\psi}\psi$

Let us see what happens in the presence of higher powers of $\bar{\psi}\psi$:

$$V(\sigma,t) = m(t)\sigma - \frac{G(t)}{2}\sigma^2 + \frac{g_3(t)}{3!}\sigma^3 + \frac{g_4(t)}{4!}\sigma^4$$

$$\begin{aligned}
\frac{dm}{dt} &= m + \frac{2C_d G m N \,\mathbf{tr}\,\mathbb{I}}{m^2+1} \\
\frac{dG}{dt} &= (2-d)G + \frac{2C_d g_3 m N \,\mathbf{tr}\,\mathbb{I}}{m^2+1} - \frac{2C_d G^2 \left(m^2-1\right) N \,\mathbf{tr}\,\mathbb{I}}{\left(m^2+1\right)^2} \\
\frac{dg_3}{dt} &= (3-2d)g_3 + \frac{6C_d g_3 G \left(1-m^2\right) N \,\mathbf{tr}\,\mathbb{I}}{\left(m^2+1\right)^2} - \frac{2C_d g_4 m N \,\mathbf{tr}\,\mathbb{I}}{m^2+1} + \frac{4C_d G^3 m \left(m^2-3\right) N \,\mathbf{tr}\,\mathbb{I}}{\left(m^2+1\right)^3} \\
\frac{dg_4}{dt} &= (4-3d)g_4 + \frac{12C_d G^4 \left(m^4-6m^2+1\right) N \,\mathbf{tr}\,\mathbb{I}}{\left(m^2+1\right)^4} - \frac{8C_d g_4 G \left(m^2-1\right) N \,\mathbf{tr}\,\mathbb{I}}{\left(m^2+1\right)^2} \\
&- \frac{24C_d g_3 G^2 m \left(m^2-3\right) N \,\mathbf{tr}\,\mathbb{I}}{\left(m^2+1\right)^3} + \frac{6C_d g_3^2 \left(m^2-1\right) N \,\mathbf{tr}\,\mathbb{I}}{\left(m^2+1\right)^2}
\end{aligned}$$

FIXED POINTS

Gaussian fixed point

$$m = 0$$

$$G = 0$$

$$g_3 = 0$$

$$g_4 = 0$$

Non-Gaussian fixed point

$$m = 0 \tag{42}$$

$$G = \frac{d-2}{2N\,\mathbf{tr}\,\mathbb{I}C_d}\,. \tag{43}$$

$$g_3 = 0 \quad (\text{if } d \neq 3) \tag{44}$$

$$g_4 = \frac{3(d-2)^4}{4(4-d)N^3\,\mathbf{tr}\,\mathbb{I}^3 C_d^3} \tag{45}$$

LINEARIZATION AROUND THE FIXED POINTS

Define the vector $\underline{\mathbf{v}} \equiv (m, G, g_3, g_4)$**. The fixed point equations reads:**

$$\beta_i\left(\{v_j\}\right) = 0 \tag{46}$$

where β_i **is the** β **function of the i-th component of** $\underline{\mathbf{v}}$**. Solutions (GFP and NGFP) :**

$$\begin{aligned} \underline{\mathbf{v}}^*_{gau} &= (0,0,0,0)\,, \\ \underline{\mathbf{v}}^*_{ng} &= (0,a,0,b) \end{aligned} \tag{47}$$

Expand the β_i**'s around the** $\underline{\mathbf{v}}^*$**'s and linearize** $\Rightarrow$ **Jacobian matrix:** $J_{ij} = \left.\frac{\partial \beta_i}{\partial v_j}\right|_{\underline{\mathbf{v}}=\underline{\mathbf{v}}^*}$

Around each of the $\underline{\mathbf{v}}^*$**'s the system is now:**

$$\frac{d\underline{\mathbf{v}}}{dt} = J\left(\underline{\mathbf{v}} - \underline{\mathbf{v}}^*\right), \tag{48}$$

Solutions of this equation have the general form :

$$\underline{\mathbf{v}} = \underline{\mathbf{v}}^* + Ae^{\lambda_1 t}\underline{\mathbf{v}}_1 + Be^{\lambda_2 t}\underline{\mathbf{v}}_2 + Ce^{\lambda_3 t}\underline{\mathbf{v}}_3 + De^{\lambda_4 t}\underline{\mathbf{v}}_4 \tag{49}$$

Gaussian Fixed Point

GFP : $\underline{\mathbf{v}}^*_{gau} = (0,0,0,0)$**.**

Jacobian Matrix :

$$J\left(\underline{\mathbf{v}}^*_{gau}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2-d & 0 & 0 \\ 0 & 0 & 3-2d & 0 \\ 0 & 0 & 0 & 4-3d \end{pmatrix} \tag{50}$$

$\Rightarrow$ **Around** $\underline{\mathbf{v}}^*_{gau} = (0,0,0,0)$ **trivial canonical scaling :**

$$\begin{pmatrix} m(t) \\ G(t) \\ g_3(t) \\ g_4(t) \end{pmatrix} = Ae^{t} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + Be^{(2-d)t} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + Ce^{(3-2d)t} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + De^{(4-3d)t} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{51}$$

Non Gaussian Fixed Point

NGFP : $\underline{\mathbf{v}}^*_{ng} = \left(0, \frac{d-2}{2\mathcal{N}_d}, 0, \frac{3(d-2)^4}{4(d-4)\mathcal{N}_d^3}\right) \qquad (\mathcal{N}_d = N\,\mathbf{tr}\,\mathbb{I}C_d)$

Jacobian Matrix :

$$J\left(\underline{\mathbf{v}}^*_{ng}\right) = \begin{pmatrix} d-1 & 0 & 0 & 0 \\ 0 & d-2 & 0 & 0 \\ \frac{3(d-2)^3}{(d-4)\mathcal{N}_d^2} & 0 & d-3 & 0 \\ 0 & -\frac{12(d-2)^3}{(d-4)\mathcal{N}_d^2} & 0 & d-4 \end{pmatrix}$$

Has to be diagonalized. The running around $\underline{\mathbf{v}}^*_{ng}$ Non-Trivial :

$$\begin{pmatrix} m(t) \\ G(t) \\ g_3(t) \\ g_4(t) \end{pmatrix} = Ae^{(d-1)t}\begin{pmatrix} \frac{2(d-4)\mathcal{N}_d^2}{3(d-2)^3} \\ 0 \\ 1 \\ 0 \end{pmatrix} + Be^{(d-2)t}\begin{pmatrix} 0 \\ -\frac{(d-4)\mathcal{N}_d^2}{6(d-2)^3} \\ 0 \\ 1 \end{pmatrix} + Ce^{(d-3)t}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + De^{(d-4)t}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad \lambda_1 = d-1\ ,\ \lambda_2 = d-2\ ,\ \lambda_3 = d-3\ ,\ \lambda_4 = d-4$$

Note the marginality of O_3 in $d=3$ and of O_4 in $d=4$

Extended GN model

Only even powers : G and g_4

$$U(\bar{\psi}\psi) = -\tfrac{G}{2}(\bar{\psi}\psi)^2 + g_4(\bar{\psi}\psi)^4$$

$$\frac{dG}{dt} = (2-d)G + 2G^2 \quad (52)$$

$$\frac{dg_4}{dt} = (4-3d)g_4 + 12G^4 + 8Gg_4. \quad (53)$$

Extended GN model : $U(\bar\psi\psi) = -\frac{G}{2}(\bar\psi\psi)^2 + g_4(\bar\psi\psi)^4$ for $d = 3$

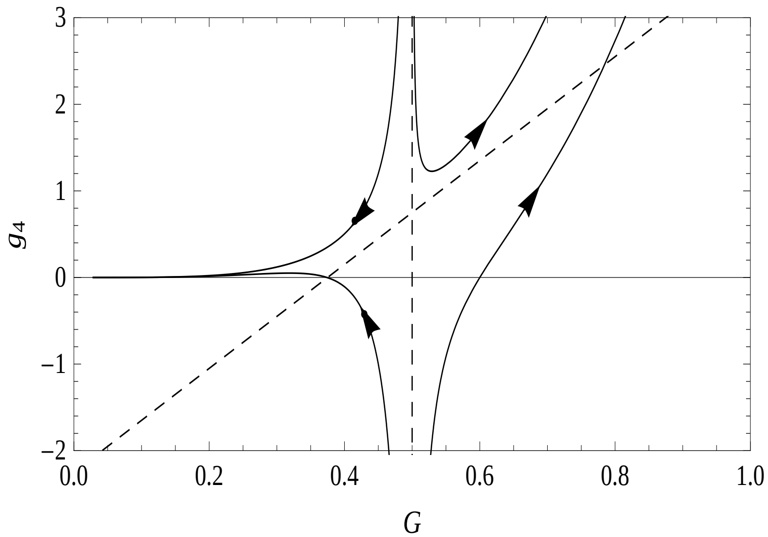
$$\frac{dG}{dt} = -G + 2G^2 \tag{54}$$

$$\frac{dg_4}{dt} = -5g_4 + 12G^4 + 8Gg_4. \tag{55}$$

Extended GN : $U(\bar\psi\psi) = -\frac{G}{2}(\bar\psi\psi)^2 + g_4(\bar\psi\psi)^4$

OUR FIXED POINT

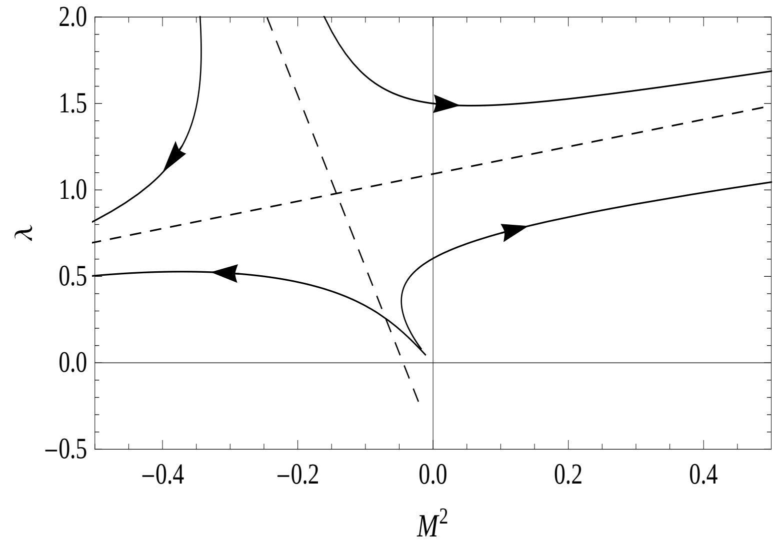
SYMM. **BROKEN**



Scalar theory (N=1) : $U(\phi) = \frac{1}{2}M^2\phi^2 + \frac{\lambda}{4}\phi^4$

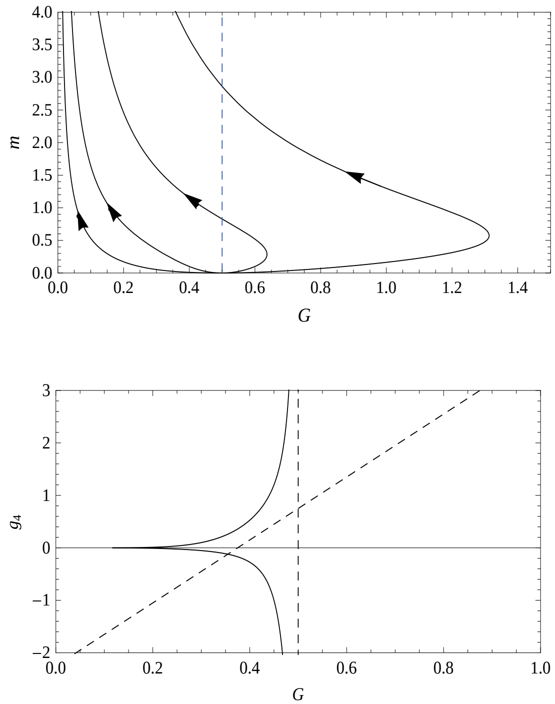
WILSON FISHER FIXED POINT

BROKEN **SYMM.**

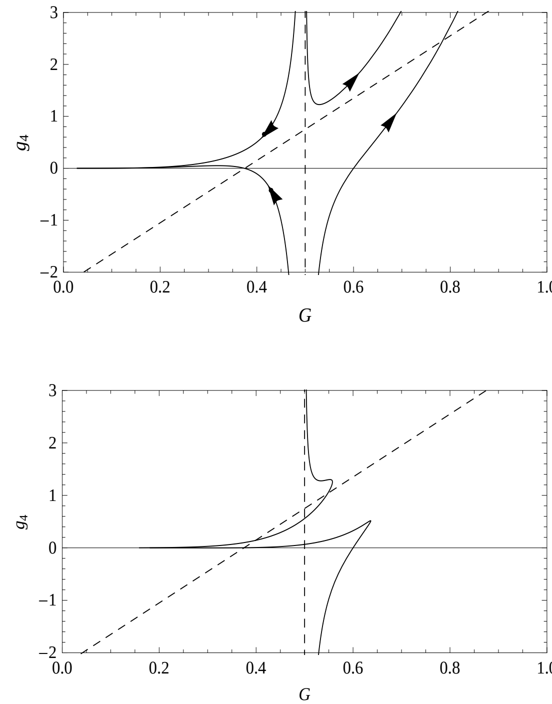


Crossover picture in $d = 3$

Adding the mass : $U(\bar{\psi}\psi)^2 = m\bar{\psi}\psi - \frac{G}{2}(\bar{\psi}\psi)^2 + g_4(\bar{\psi}\psi)^4$



(a)



(b)

(a) "unbroken phase". (b) "broken phase". Compare with the strictly unbroken and broken phases abov e (right panel).

Extended GN model $U(\bar\psi\psi) = -\frac{G}{2}(\bar\psi\psi)^2 + g_4(\bar\psi\psi)^4$ in $d = 4$

The solution is :

$$G = \frac{1}{1 - Be^{2t}} \quad (56)$$

$$g_4 = \frac{1}{(1 - Be^{2t})^4}\,[12t + C] \quad (57)$$

Remember :

1. The marginality found before ($\lambda_4 = 0$).

2. When we looked for the fixed point solution, we found that $g_4^* \to \infty$ goes to infinity as $d \to 4$.

All this is better understood if we consider again the case of generic d and perform the change $g_4 \to x = g_4^{-1}$. The RG system of equations is then:

$$\frac{dG}{dt} = (2 - d)G + 2G^2 \quad (58)$$

$$\frac{dx}{dt} = (3d - 4)x - 12G^4x^2 - 8Gx. \quad (59)$$

Fixed point (relevant to our discussion) :

$$G = (d - 2)/2 \quad (60)$$

$$x = \frac{4}{3}\frac{(4 - d)}{(d - 2)^4}. \quad (61)$$

Note : for $d = 4$: $x = 0$

Extended GN model $U(\bar{\psi}\psi) = -\frac{G}{2}(\bar{\psi}\psi)^2 + g_4(\bar{\psi}\psi)^4$ in $d = 4$ cont'nd

Solution of the systen for $d = 4$:

$$G = \frac{1}{1 - Be^{2t}} \tag{62}$$

$$x = (1 - Be^{2t})^4 \left(\frac{x_0}{1 + 12x_0 t} \right) \tag{63}$$

On the critical surface, $G = G_c$ (B=0), x scales as λ in a $\lambda\phi^4$ theory

$$x = \frac{x_0}{1 + 12x_0 t} \tag{64}$$

BEYOND large N - generic d

Up to now, we have considered the "Hartree Approximation". Let us keep now the Fock term.

$$k\frac{dU}{dk} = k^d C_d \left(N\mathbf{tr}\,\mathbb{I} \ln\left(\frac{k^2+U_\rho^2}{k^2}\right) - \ln\left(1+\frac{2\rho U_\rho U_{\rho\rho}}{k^2+U_\rho^2}\right)\right) \tag{65}$$

As before we have the Gussian Fixed Point and the Non-Gaussian FP :

$$\begin{aligned} m &= 0 \\ G &= \frac{d-2}{2C_d(N\mathbf{tr1}-2)} \\ g_3 &= 0 \\ g_4 &= -\frac{3(d-2)^4(N\mathbf{tr1}-8)}{4\mathbf{Cd}^3(N\mathbf{tr1}-2)^3((d-4)N\mathbf{tr1}-10d+24)} \end{aligned}$$

Remark : the $N \to \infty$ limit does not commute with the $d \to 4$ limit.

Performing first the $N \to \infty$ limit, we get the result of the previous transparency : $g_4 \to \infty$.
However, if we first take $d = 4$ ......

BEYOND large N - $d = 4$

The Non-Gaussian Fixed Point :

$$
\begin{aligned}
m &= 0 \\
G &= \frac{8\pi^2}{2N-1} \\
g_3 &= 0 \\
g_4 &= \frac{1536\pi^6 (N-2)}{(2N-1)^3}
\end{aligned}
$$

For $d = 4$, g_4 does not diverge !!!

BEYOND large N - $d = 4$

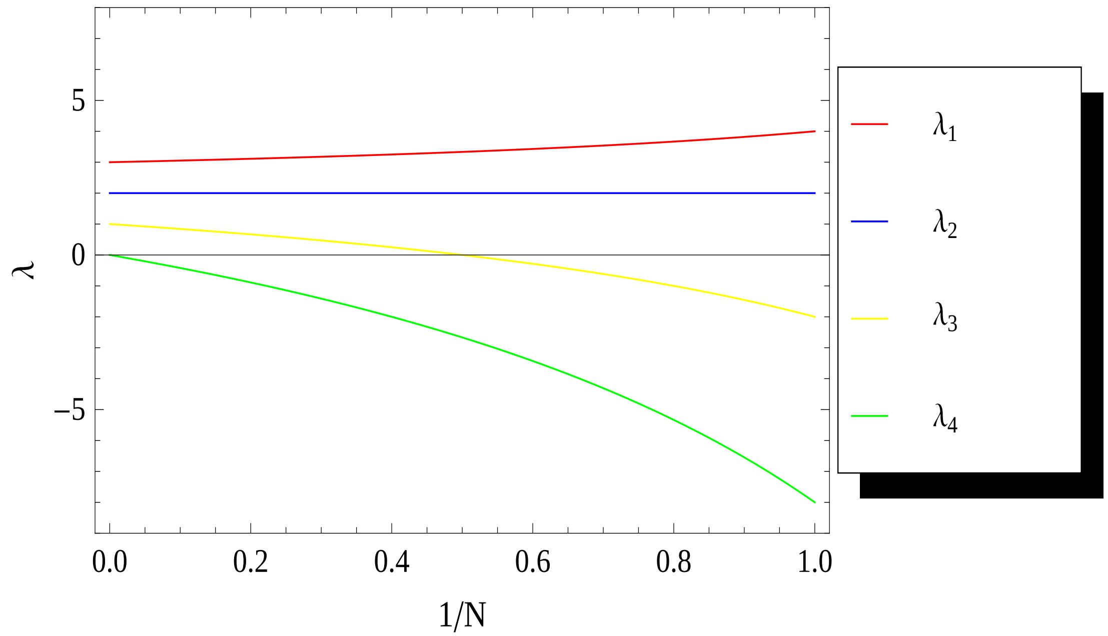
Eigenvalues of the Eigenoperators for the Non-Gaussian Fixed Point :

$$\begin{aligned}
\lambda_1 &= \frac{4N-1}{2N-1} + 1 \\
\lambda_2 &= 2\frac{4N-2}{2N-1} - 2 = 2 \\
\lambda_3 &= 3\frac{4N-3}{2N-1} - 5 \\
\lambda_4 &= 4\frac{4N-4}{2N-1} - 8 \\
\lambda_5 &= 5\frac{4N-5}{2N-1} - 11 \\
\lambda_6 &= 6\frac{4N-6}{2N-1} - 14
\end{aligned}$$

In the r.h.s.: the second term gives the gaussian scaling, the first term gives the anomalous scaling (linearization around the non trivial fixed point).

Taking the large N limit of these λ_i we get the same results as in the Hartree approximation.

BEYOND large N - $d = 4$: Scaling and Critical exponents



Transition of λ_3 from relevance to irrelevance : at $N = 2$.

Critical exponents :

N	β	γ	δ	η
∞	$1/2$	1	3	0
3	$2/5$	$6/5$	4	$-2/5$
2	$1/3$	$4/3$	5	$-2/3$
1	0	2	∞	-2

Conclusions

1. We established the wilsonian RG equation for the potential of a generalized N flavors GN model in $2 \leq d \leq 4$.

2. We found that this equation contains "Hartree" and "Fock" contributions.

3. Neglecting the latter, the leading order of the equation in the $1/N$ expansion was obtained.

4. By truncating the potential to the Fermi and mass terms, we derived the RG equations for the running mass $m(k)$ and Fermi constant $G(k)$ and found that the theory possesses a non-trivial fixed point.

5. By linearizing the RG equations around this fixed point, we were able to compute the critical exponents of the system and found that our results are in agreement with those previously obtained at the leading order of the more conventional large N expansion.

6. By studying the full RG equations for $G(k)$ and $m(k)$, we found that the chiral phase transition arises as a cross-over phenomenon triggered by the presence of an infinitesimal bare mass. When the bare value of the Fermi constant is greater than the fixed point value G_c, the running of $m(k)$ from the UV to the IR shows a "steepy" cross-over and a finite fermion mass is generated. Physically, what happens is that for values of G_B greater than G_c, the system is unstable against quantum fluctuations. The presence of an even infinitesimal bare mass provides an amplification mechanism which resolves the instability via the generation of a finite physical mass.

7. We have also found analytical approximations to $m(k)$ and $G(k)$:

7a. By considering a simple approximation for the RG equation of $m(k)$, we got the NJL result. However, although this approximation is able to grasp qualitatively the main features of the RG equations (which lead to the chiral phase transition due to quantum fluctuations), quantitatively more accurate analytical profiles for $m(k)$ and $G(k)$ require more sophisticated approximations.

7b. We found analytical approximations to $m(k)$ and $G(k)$ which very accurately reproduce the numerical results.

8. We have shown some interesting features related to the presence of higher powers of $\bar{\psi}\psi$ in the potential.

9. Finally, we have considered the RG equations beyond the large N limit expansion by keeping the "Fock" term in the RG equations.

To be done : Impact of higher derivative terms.

RG equations with Higher Powers of $\bar{\psi}\psi$ for d=3

$$\begin{aligned}
\frac{dm}{dt} &= m + \frac{GmN}{\pi^2\left(m^2+1\right)} \\
\frac{dG}{dt} &= -G + \frac{g_3 mN}{\pi^2\left(m^2+1\right)} - \frac{G^2\left(m^2-1\right)N}{\pi^2\left(m^2+1\right)^2} \\
\frac{dg_3}{dt} &= -3g_3 + \frac{3g_3 G\left(1-m^2\right)N}{\pi^2\left(m^2+1\right)^2} - \frac{g_4 mN}{\pi^2\left(m^2+1\right)} + \frac{2G^3 m\left(m^2-3\right)N}{\pi^2\left(m^2+1\right)^3} \\
\frac{dg_4}{dt} &= -5g_4 - \frac{12g_3 G^2 m\left(m^2-3\right)N}{\pi^2\left(m^2+1\right)^3} - \frac{4g_4 G\left(m^2-1\right)N}{\pi^2\left(m^2+1\right)^2} \\
&+ \frac{3g_3^2\left(m^2-1\right)N}{\pi^2\left(m^2+1\right)^2} + \frac{6G^4\left(m^4-6m^2+1\right)N}{\pi^2\left(m^2+1\right)^4} \qquad (66)
\end{aligned}$$

Non-Gaussian fixed point **Eigenvalues**

$$m = 0 \qquad \lambda_1 = 2 \qquad (67)$$

$$G = \frac{\pi^2}{N} \qquad \lambda_2 = 1 \qquad (68)$$

$$g_3 = \text{any value} \qquad \lambda_3 = 0 \qquad (69)$$

$$g_4 = \sqrt{\frac{6\pi^3}{N^3} - \frac{3g_3^2 N}{\pi}} \qquad \lambda_4 = -1 \qquad (70)$$

.... Tricritical point ...???......

RG equations with Higher Powers of $\bar{\psi}\psi$ for d=4

$$\begin{aligned}
\frac{dm}{dt} &= m + \frac{GmN}{2\pi^2 (m^2+1)} \\
\frac{dG}{dt} &= -2G + \frac{g_3 mN}{2\pi^2 (m^2+1)} - \frac{G^2 (m^2-1) N}{2\pi^2 (m^2+1)^2} \\
\frac{dg_3}{dt} &= -5g_3 + \frac{3g_3 G (1-m^2) N}{2\pi^2 (m^2+1)^2} - \frac{g_4 mN}{2\pi^2 (m^2+1)} + \frac{G^3 m (m^2-3) N}{\pi^2 (m^2+1)^3} \\
\frac{dg_4}{dt} &= -8g_4 - \frac{2g_4 G (m^2-1) N}{\pi^2 (m^2+1)^2} + \frac{3G^4 (m^4-6m^2+1) N}{\pi^2 (m^2+1)^4} \\
&+ \frac{3g_3^2 (m^2-1) N}{2\pi^2 (m^2+1)^2} - \frac{6g_3 G^2 m (m^2-3) N}{\pi^2 (m^2+1)^3} \qquad (71)
\end{aligned}$$

Non-Gaussian fixed point

$$\begin{aligned}
m &= 0 \\
G &= \frac{4\pi^2}{N} \\
g_3 &= 0 \\
g_4 &= \infty
\end{aligned}$$

Eigenvalues

$$\lambda_1 = 3 \qquad (72)$$

$$\lambda_2 = 2 \qquad (73)$$

$$\lambda_3 = 1 \qquad (74)$$

$$\lambda_4 = 0 \qquad (75)$$