11th workshop on non-perturbative QCD
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From correlators to Wilson loops and super-amplitudes

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Introduction

AdS/CFT Conjecture [Maldacena]

$\mathcal{N} = 4$ super Yang-Mills (SYM)

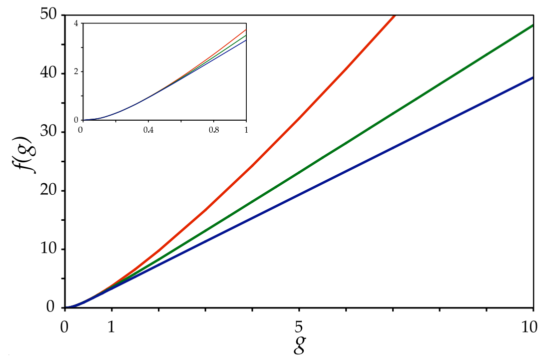
weak coupling **operators: dimensions**

$\Updownarrow$

strong coupling **states: energy levels**

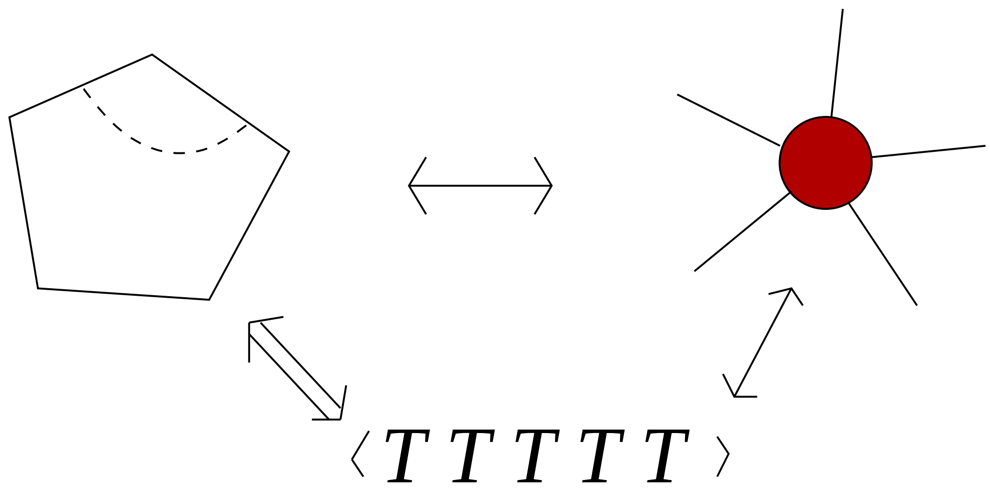
II B superstring theory on $AdS_5 \times S^5$

- **Spectrum** from an **integrable system** [Minahan, Zarembo], [Beisert, Staudacher], ...
- **Interpolating result** for $\mathbf{\Gamma}_{\text{cusp}}$ [Beisert, Eden, Staudacher] $\longrightarrow$



- **Scattering amplitudes** $\Leftrightarrow$ **Wilson loops**
  [Alday, Maldacena], [Drummond, Henn, Korchemsky, Sokatchev]
  [Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich]
- **Yangian symmetry** [Drummond, Henn, Plefka]
- **Parts of QCD** results can be obtained from $\mathcal{N} = 4$ SYM.

Weak coupling dualities:



- Polygonal **Wilson loops** with light-like edges are **dual** to on-shell **amplitudes**.
- **n-point functions** of the **stress tensor** $\mathcal{T}$ generate **both** in a **light-cone limit**.
- We **match** the recently proposed **all-loops integrand for amplitudes**.
  [Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka]
- **This talk: MHV** situation. **Paul Heslop** will address the **supersymmetric extension**.

Our work:

arXiv: 1007.3243, 1007.3246, 1009.2488, 1103.3714, 1103.4353 [hep-th]

Collaborators:

L. Alday, P. Heslop, G. Korchemsky, J. Maldacena, E. Sokatchev

$\mathcal{N} = 4$ SYM ...

... is the **maximally supersymmetric non-abelian gauge theory in four dimensions**.

$$\{\phi^{[AB]},\, \psi^{\alpha A},\, \bar\psi^{\dot\alpha}_A,\, A^\mu\}\,, \qquad A \in \{1,2,3,4\}\,, \qquad \phi^{AB} = \frac{1}{2}\,\epsilon^{ABCD}\,\bar\phi_{CD}\,.$$

On-shell supersymmetry:

$$\begin{aligned} Q^\alpha_A\,\phi^{BC} &= i\sqrt{2}\,(\delta^B_A\psi^{C\alpha} - \delta^C_A\psi^{B\alpha})\,, & Q^\alpha_A\,A_{\beta\dot\beta} &= -2\,i\,\delta^\alpha_\beta\,\bar\psi_{A\dot\beta}\,,\\ Q^\alpha_A\,\psi^B_\beta &= \delta^B_A F^\alpha_\beta + i\,g[\phi^{BC},\phi_{CA}]\,\delta^\alpha_\beta\,, & Q^\alpha_A\,\bar\psi^{\dot\beta}_B &= \sqrt{2}\,D^{\dot\beta\alpha}\phi_{AB} \end{aligned}$$

and the conjugate. Note: $\quad Q^\alpha_A\,F_{\beta\gamma} = i\,\delta^\alpha_\beta\,D_{\gamma\dot\alpha}\,\bar\psi^{\dot\alpha}_A + (\beta\leftrightarrow\gamma)\,.$

- There is **no quantum formalism** with manifest $\mathcal{N} = 4$ susy.

Field strength multiplet transforming in the **adjoint** of $\mathbf{SU(N_c)}$:

$$W^{[AB]} = \phi^{[AB]}(x) + \theta^{\alpha[A}\psi(x)^{B]}_\alpha + \theta^{[A}_{(\alpha}\theta^{B]}_{\beta)}F^{\alpha\beta} + O(\bar\theta)$$

$\mathcal{N} = 2$ **reduction**: $\qquad W_{\mathcal{N}=2} = W^{23}_{\mathcal{N}=4}|_{\theta_1=\theta_4=0}\,, \qquad q = \{\phi^{1i}, \psi^1, \bar\psi_4\}\,.$

Off-shell $\mathcal{N} = 2$ fields

Two quantum fields

$$q^+(x_A,\, \theta^+, \bar\theta^+, u)\,, \qquad \theta^+ = \theta^i\, u_i^+\,,\ \bar\theta^+ = \bar\theta_i\, u^{+i}\,, \quad x_A = x - 4\,i\,\theta^{(i}\bar\theta^{j)}\, u_i^+ u_j^-\,,$$
$$V^{++}(x_A,\, \theta^+, \bar\theta^+, u)\,, \qquad u = (u^+, u^-) \in SU(2)/U(1)$$

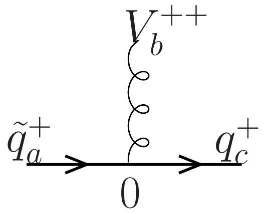
- Use $u \in SU(2)$ and keep track of $U(1)$ charge.

Action:

$$S_{\mathcal{N}=4\,\mathrm{SYM}} = S_{\mathrm{HM/SYM}} + S_{\mathcal{N}=2\,\mathrm{SYM}}$$

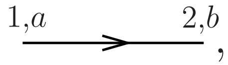
$$S_{\mathrm{HM/SYM}} = -2\int du\, d^4x_A\, d^2\theta^+\, d^2\bar\theta^+\ \mathrm{Tr}(\tilde q^+ D^{++} q^+ + i\sqrt{2}\,\tilde q^+[V^{++}, q^+])\,,$$
$$S_{\mathcal{N}=2\,\mathrm{SYM}} = -\frac{1}{4\,g^2}\int d^4x_L\, d^4\theta\ \mathrm{Tr}\ W^2 = -\frac{1}{4\,g^2}\int d^4x_R\, d^4\bar\theta\ \mathrm{Tr}\ \overline{W}^2\,, \qquad x_L^{\alpha\dot\alpha} = x^{\alpha\dot\alpha} - 2i\theta^{i\alpha}\bar\theta_i^{\dot\alpha}\,,$$
$$W = \frac{i}{4} u_i^+ u_j^+ \bar D^i_{\dot\alpha} \bar D^{j\dot\alpha} \sum_{r=1}^{\infty} \int du_1 \dots du_r\, \frac{(-i\sqrt{2})^r\, V^{++}(u_1)\dots V^{++}(u_r)}{(u^+u_1^+)(u_1^+u_2^+)\dots(u_r^+u^+)}\,.$$

Feynman rules:



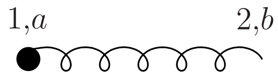
$$= \; i f_{bac} \int d^4x_0 \, du_0 \, d^4\theta_0^+$$

$$\langle \tilde{q}_a^+(1) q_b^+(2) \rangle \;=\; \frac{(12)}{4\pi^2 \, \hat{x}_{12}^2} \, \delta_{ab} \;=$$



$$, \qquad (12) = -(21) = u_1^{+i} \epsilon_{ij} u_2^{+j}$$

$$\langle W_a(1) V_b^{++}(2) \rangle \;=\; -\frac{g^2 \delta_{ab}}{4\pi^2 \, \tilde{x}_{12}^2} \, (\theta_{12})^2 =$$



with the supersymmetric coordinate differences

$$\hat{x}_{12}^\mu \;=\; x_{A1}^\mu - x_{A2}^\mu + \frac{2i}{(12)} [(1^-2)\theta_1^+ \sigma^\mu \bar{\theta}_1^+ + (2^-1)\theta_2^+ \sigma^\mu \bar{\theta}_2^+ + \theta_1^+ \sigma^\mu \bar{\theta}_2^+ + \theta_2^+ \sigma^\mu \bar{\theta}_1^+] \, ,$$

$$\tilde{x}_{12}^{\alpha\dot{\alpha}} \;=\; x_{L1}^{\alpha\dot{\alpha}} - x_{A2}^{\alpha\dot{\alpha}} - 4i u_{2i}^- \theta_1^{i\alpha} \, \bar{\theta}_2^{+\dot{\alpha}} \, , \qquad \theta_{12}^\alpha = u_{2i}^+ \theta_1^{i\alpha} - \theta_2^{+\alpha} \, .$$

Correlation functions of

$$O \;=\; \mathrm{Tr}(q^+ q^+) \, , \qquad \tilde{O} \;=\; \mathrm{Tr}(\tilde{q}^+ \tilde{q}^+) \, , \qquad \hat{O} \;=\; 2 \, \mathrm{Tr}(\tilde{q}^+ q^+) \, ,$$

i.e.

$$G_n = \int \mathcal{D}\Phi \; e^{i S_{\mathcal{N}=4 \, \mathrm{SYM}}} \; \tilde{O}(x_1) \, O(x_2) \, \tilde{O}(x_3) \ldots O(x_n) \, .$$

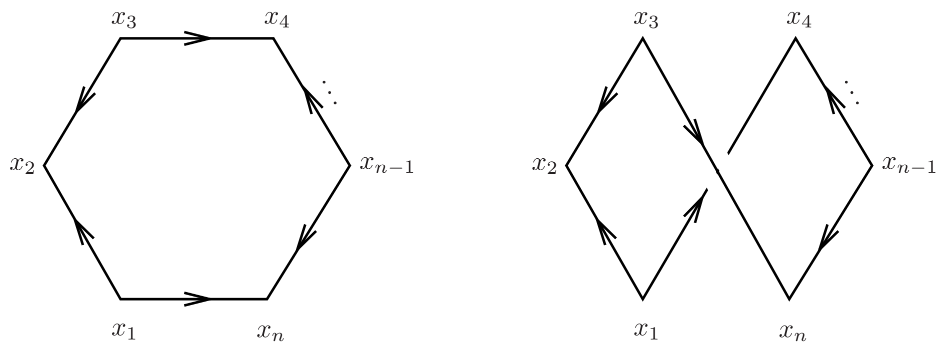
Wilson loops from correlators - an explanation

Consider only **scalar fields**. The **connected tree level** graphs yield

$$G_n^{(0)} = N_c^2 \sum_{\{i_1,\dots,i_n\}} S^{(0)}(x_{i_1,i_2}) S^{(0)}(x_{i_2,i_3}) \dots S^{(0)}(x_{i_n,i_1}) \,, \qquad S^{(0)}(x) = \frac{1}{4\pi^2\, x^2} \,.$$

Dominant term in the limit $x_{i,i+1}^2 \to 0$:

$$G_n^{(0)} \stackrel{x_{i,i+1}^2 \to 0}{\longrightarrow} N_c^2\, S^{(0)}(x_{12}) S^{(0)}(x_{23}) \dots S^{(0)}(x_{n1}) = \frac{(2\pi)^{-2n} N_c^2}{x_{12}^2 x_{23}^2 \dots x_{n1}^2}$$



(a) (b)

- Diagram (b) is suppressed by the factor $x_{34}^2 x_{1n}^2 / (x_{3n}^2 x_{14}^2)$.

• Scattering of a **fast colour charged particle** off a **slowly varying gauge field**

$$\lim_{x_{i,i+1}^2\to 0} G_n^{(0)} = \langle 0|\mathrm{Tr}[S(x_1,x_2;A)S(x_2,x_3;A)\dots S(x_n,x_1;A)]|0\rangle$$

Ansatz for the **propagator in the background gauge field**

$$S(x_i,x_{i+1};A) = S^{(0)}(x_{i,i+1})P\exp\left(ig\int_{x_i}^{x_{i+1}} dz\cdot\tilde{A}(z)\right)G(x_i,x_{i+1};A)$$

G should have an **OPE expansion** (Δ is the twist)

$$G(x,0;A) = \sum (x^2)^\Delta C_{\Delta,N}(x^2\mu^2)\; x_{\mu_1}\dots x_{\mu_N}\mathcal{O}_\Delta^{\mu_1\dots\mu_N}(0)$$

implying $\lim_{x^2\to 0} G = 1$.

It follows

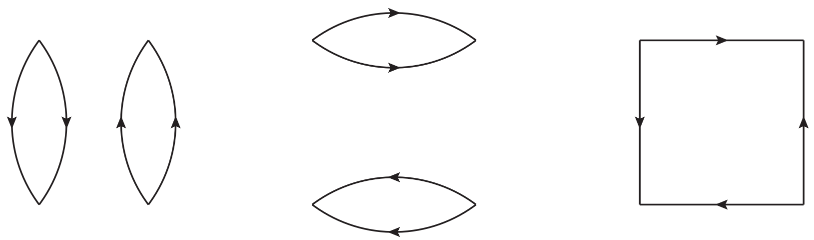
$$\boxed{\frac{G_n}{G_n^{(0)}} \to W^{\mathrm{adj}}[C_n] = \frac{1}{N_c^2-1}\langle 0|\;\mathrm{Tr}_A\left[P\exp\left(ig\int_{C_n} dz\cdot\tilde{A}(z)\right)\right]|0\rangle}$$

where

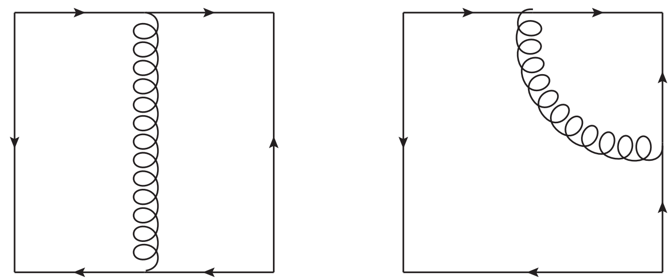
$$[\tilde{A}_\mu(z)]_{ab} = -if_{abc}A^c_\mu(z)\,, \qquad C_n = [x_1,x_2]\cup[x_2,x_3]\cup\dots\cup[x_n,x_1].$$

$\langle \tilde{\mathbf{O}}(\mathbf{1})\, \mathbf{O}(\mathbf{2})\, \tilde{\mathbf{O}}(\mathbf{3})\, \mathbf{O}(\mathbf{4}) \rangle$ [Eden, Howe, Schubert, Sokatchev, West (1998)]

Tree level:



One loop:



$$\langle O(1)\, \tilde{O}(2)\, O(3)\, \tilde{O}(4) \rangle_{\frac{g^2 N_c}{4\pi^2}} = \frac{1}{(4\pi^2)^4}\Big((14)^2 (23)^2 A_1 + (12)^2 (34)^2 A_2 + (12)(23)(34)(41) A_3 \Big)$$

A_1 and A_2 are simple:

$$A_1 = \frac{1}{x_{14}^2\, x_{23}^2}\, g(1,2,3,4)\,, \qquad A_2 = \frac{1}{x_{12}^2\, x_{34}^2}\, g(1,2,3,4)\,, \qquad g(1,2,3,4) = \frac{1}{4\pi^2} \int \frac{d^4 x_0}{x_{10}^2\, x_{20}^2\, x_{30}^2\, x_{40}^2}\,,$$

$$\begin{aligned} A_3 &= \frac{2\,\partial_2.\partial_4\, f(1,2,1,4)}{x_{23}^2\, x_{34}^2} + \frac{2\,\partial_1.\partial_3\, f(2,1,2,3)}{x_{14}^2\, x_{34}^2} + \frac{2\,\partial_2.\partial_4\, f(3,2,3,4)}{x_{12}^2\, x_{14}^2} + \frac{2\,\partial_1.\partial_3\, f(4,1,4,3)}{x_{12}^2\, x_{23}^2} \\ &+ \frac{(\partial_2+\partial_3)^2 f(1,2,3,4)}{x_{14}^2 x_{23}^2} + \frac{(\partial_1+\partial_2)^2 f(1,4,2,3)}{x_{12}^2 x_{34}^2}\,, \qquad f(1,2;3,4) = \frac{1}{(4\pi^2)^2} \int \frac{d^4 x_0\, d^4 x_{0'}}{x_{10}^2\, x_{20}^2\, x_{00'}^2\, x_{30'}^2\, x_{40'}^2}\,. \end{aligned}$$

Limit $x_{12}^2 = x_{23}^2 = x_{34}^2 = x_{41}^2 \to 0$ in **dimensional regularisation:** (μ suppressed)

$$g_\epsilon = \int \frac{d^{4-2\epsilon}x_0}{(x_{10}^2\, x_{20}^2\, x_{30}^2\, x_{40}^2)^{1-\epsilon}} = \frac{\Gamma(2-3\epsilon)}{(x_{24}^2)^{2-3\epsilon}} \int_0^1 \frac{dz_1 \dots dz_4\, \delta(1-\sum_i z_i)\,(z_1\dots z_4)^{-\epsilon}}{(z_2 z_4 + z_1 z_3\, X)^{2-3\epsilon}}\,, \qquad X = \frac{x_{13}^2}{x_{24}^2}$$

Mellin-Barnes (MB) method:

$$\frac{1}{(A+B)^\nu} = \frac{1}{2\pi i\,\Gamma(\nu)\,B^\nu}\int_{-i\infty}^{i\infty} dz\, A^z\, B^{-z}\,\Gamma(-z)\,\Gamma(\nu+z)\,, \qquad \int_0^1 \left(\Pi_i\, dz_i\, z_i^{q_i-1}\right)\delta(1-\sum_i z_i) = \frac{\Pi_i\,\Gamma(q_i)}{\Gamma(\sum_i q_i)}$$

It follows

$$g_\epsilon = \frac{1}{2\pi i\ (x_{24}^2)^{2-3\epsilon}\ X^{2-3\epsilon}}\int_{-i\infty}^{i\infty} dz\, X^{-z}\,\Gamma(-z)\,\Gamma(2-3\epsilon+z)\frac{\Gamma(1-\epsilon+z)^2\,\Gamma(-1+2\epsilon-z)^2}{\Gamma(2\epsilon)}$$

and finally:

$$\lim_{x_{i,i+1}^2 \to 0} \left(x_{13}^2\, x_{24}^2\right)^{1-\epsilon} g_\epsilon(1,2,3,4) = -2\left[-\frac{(x_{13}^2)^\epsilon}{\epsilon^2} - \frac{(x_{24}^2)^\epsilon}{\epsilon^2} + \log^2(X) + O(\epsilon)\right]$$

- **Wilson loop** at $O(g^2)$!

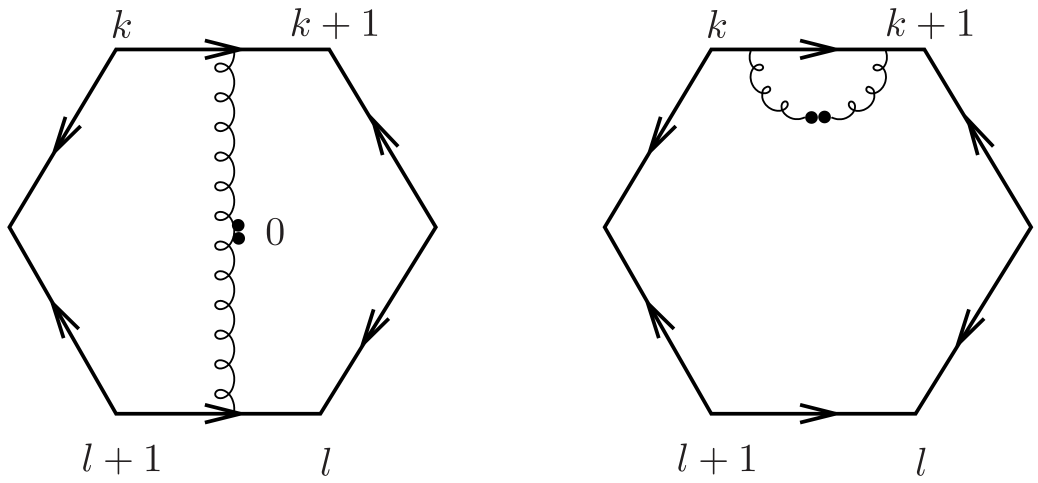
Lagrangian insertions

Differentiation w.r.t. g brings down $\mathbf{S}_{\mathcal{N}=\mathbf{2}\ \mathrm{SYM}}$:

$$g^2\frac{\partial}{\partial g^2}G_n = -\int d^Dx_0\ G^{(0)}_{n+1;1}(x_0;x_1,u_1,\ldots,x_n,u_n)\ +\ O(g^4)$$

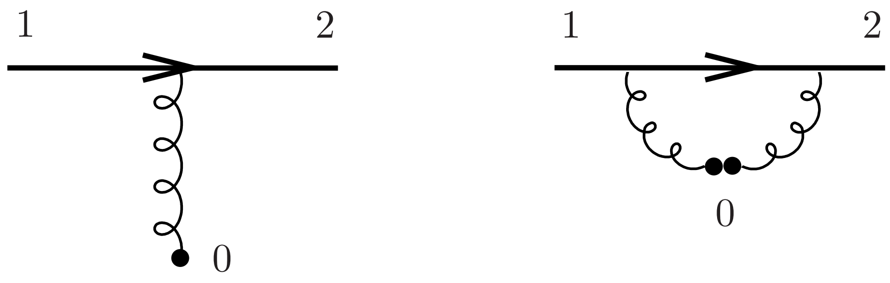
with the **Born-level** $(n+1)$-point correlator

$$G^{(0)}_{n+1;1}(x_0;x_1,u_1;\ldots;x_n,u_n) = \int d^4\theta_0\ \langle L_{\mathcal{N}=2\ SYM}(x_0,\theta_0)\, O(x_1,u_1)\ldots O(x_n,u_n)\rangle\ +\ O(g^4)\ .$$



(a) (b)

- **Rational** building blocks **T** and **TT**



(a) (b)

$$\langle \tilde{q}_a^+(1)\, W_b(0)\, q_c^+(2)\rangle = -\frac{2ig^2 f_{abc}}{(2\pi)^4}\,\frac{(12)}{x_{12}^2}\, i_{12}\,, \qquad \langle \tilde{q}_a^+(1)\, \mathrm{Tr}(W^2)(0)\, q_b^+(2)\rangle = \frac{g^4 N_c\, \delta_{ab}}{(2\pi)^6}\,\frac{(12)}{x_{12}^2}\, i_{12}^2$$

with

$$i_{12} \;=\; \frac{x_{12}^2}{(12)}\,\frac{\theta^+_{0/1}\cdot\theta^+_{0/2}}{x_{10}^2\, x_{20}^2} - \frac{\theta^+_{0/1}\cdot\theta^-_{0/1}}{x_{10}^2} + \frac{\theta^+_{0/2}\cdot\theta^-_{0/2}}{x_{20}^2} - \frac{\big(\theta^+_{0/1}[x_{10},x_{20}]\theta^+_{0/2}\big)}{(12)\, x_{10}^2\, x_{20}^2}\,.$$

Proportional to $(12)/x_{12}^2$, so **light cone limit simple** to take.

Summing up all graphs:

$$G^{(0)}_{n+1;1} \;=\; \frac{a}{4\pi^2}\, G^{(0)}_n \int d^4\theta_0 \Big(\sum_{k=1}^{n} i_{k,k+1}\Big)^2, \qquad\qquad a = \frac{g^2 N_c}{4\,\pi^2}$$

$$\to\; \frac{a}{8\pi^2}\, G^{(0)}_n \sum_{k,l} \frac{x^2_{k,l+1}x^2_{k+1,l} - x^2_{kl}x^2_{k+1,l+1} \;+\; 4\, i\, \epsilon_{\mu\nu\lambda\rho}\, x^\mu_{k,0}x^\nu_{k+1,0}x^\lambda_{l,0}x^\rho_{l+1,0}}{x^2_{k,0}x^2_{k+1,0}x^2_{l,0}x^2_{l+1,0}}$$

Amplitudes from correlators - no explanation yet

- **WL**: standard dimensional regularisation

$$\langle\bar\varphi(x_i)\,\varphi(x_{i+1})\rangle \;=\; \frac{1}{4\pi^2\left(x^2_{i,i+1}\right)^{1-\epsilon}}\,, \qquad\qquad \int d^{4-2\epsilon}x_0$$

- **Amplitudes**: algebra in four dimensions, modify only the measure at insertion points

$$\langle\bar\varphi(x_i)\,\varphi(x_{i+1})\rangle \;=\; \frac{1}{4\pi^2\,p_i^2}\,, \quad p_i \;=\; x_i - x_{i+1}\,, \qquad\qquad \int d^{4-2\epsilon}l\,, \quad l \;=\; x_{0k}$$

- The p_i are **dual momenta**.

- In the light-like limit the $g(k,k+1,l,l+1)$ become **two-mass easy box integrals** and

$$\lim_{x^2_{i,i+1}\to 0}\frac{G_n^{(1)}}{G_n^{(0)}} \;=\; 2\,\frac{A^{(1)}_{\mathrm{MHV}n}}{A^{(0)}_{\mathrm{MHV}n}}\,.$$

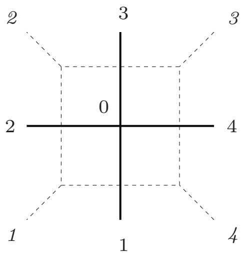
Four points at two loops

- **Correlator:** [Eden, Schubert, Sokatchev (2000)], [Bianchi, Kovacs, Rossi, Stanev (2000)]

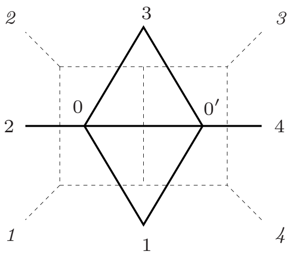
$$\lim_{x^2_{i,i+1}\to 0} G_4/G_4^{(0)} = 1+2\,a\,x^2_{13}x^2_{24}\,g(1,2,3,4) + a^2\Big[\big(x^2_{13}x^2_{24}\,g(1,2,3,4)\big)^2 + 2\,x^2_{13}x^2_{24}\Big(x^2_{13}\,h(1,2,3;1,3,4) + x^2_{24}\,h(1,2,4;2,3,4)\Big)\Big] + O(a^3)$$

- **Amplitude:** [Anastasiou, Bern, Dixon, Kosower (2003)]

$$A_4/A_4^{(0)} = 1 + a\,x^2_{13}x^2_{24}\,g(1,2,3,4) + a^2\,x^2_{13}x^2_{24}\Big[x^2_{13}\,h(1,2,3;1,3,4) + x^2_{24}\,h(1,2,4;2,3,4)\Big]$$



$g(1,2,3,4)$



$h(1,2,3;1,3,4)$

$$g(1,2,3,4) = \frac{1}{4\pi^2}\int \frac{d^Dx_0}{(x^2_{10}x^2_{20}x^2_{30}x^2_{40})}$$

$$h(1,2,3;1,2,4) = \frac{1}{(4\pi^2)^2}\int \frac{d^Dx_0 d^Dx_{0'}}{(x^2_{10}x^2_{20}x^2_{30})x^2_{00'}(x^2_{10'}x^2_{20'}x^2_{40'})}$$

$$\boxed{\lim_{x^2_{i,i+1}\to 0} G_4/G_4^{(0)} = \Big(A_4/A_4^{(0)}\Big)^2 + O(a^3)}$$

- **The same holds at five points.** This involves an identity between integrals.

Integrands [Eden, Korchemsky, Sokatchev (2010)]

l−loop correlators G_n by **l Lagrangian insertions**:

$$G_n^{(l)}(x_1, \dots, x_n) \propto \int \prod_{i=1}^{l} d^4 z_i \, d^4 \rho_i \, G^{(0)}_{n+l;l}(z_1, \dots, z_l; x_1, \dots, x_n)$$

Integrand:

$$G^{(0)}_{n+l;l} = \langle L(z_1) \dots L(z_l) \, O(x_1) \, O(x_2) \dots O(x_n) \rangle^{\text{tree}}$$

Apparently

$$\lim_{x^2_{i,i+1} \to 0} \ln(G_n / G_n^{(0)}) = 2 \ln\left(A_n / A_n^{(0)}\right) + O(1/N_c) + O(\epsilon) \, .$$

- **Parity odd sector:** Absent in G_n, $\log(A_n/A_n^{(0)})$. Present in $G_{n+l;l}$ due to $iF\tilde{F}$ in L.

$\widehat{\mathcal{A}}_{\mathbf{n+1}}$ = **integrand** of the $l-$loops $n-$point **MHV amplitude**, $\mathcal{I}_{n+l} = \lim_{x^2_{i,i+1} \to 0} (G^{(0)}_{n+l;l} / G^{(0)}_n) \, .$

Conjecture:

$$\boxed{1 + \sum_{l \geq 1} a^l \, \mathcal{I}_{n+l} = (1 + \sum_{l \geq 1} a^l \, \widehat{\mathcal{A}}_{n+l})^2}$$

All-loops integrands [Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka (2010)]

n **momentum twistors** Z^a define the outer points,
a pair $\{A, B\}, \{C, D\}, \ldots$ is needed for each integration point.

$$Z^a = (\lambda^\alpha, \mu^{\dot\alpha}), \qquad x_i^{\alpha\dot\alpha} = \frac{\lambda_i^\alpha \mu_{i+1}^{\dot\alpha} - \lambda_{i+1}^\alpha \mu_i^{\dot\alpha}}{\lambda_i^\beta \lambda_{i+1\,\beta}}$$

$x_0, x_{0'}$ depend on $\{A, B\}, \{C, D\}$.

We further need

$$\begin{aligned}
\langle AB \rangle &= \lambda_A^\beta \lambda_{B\,\beta}\,, \\
\langle ijkl \rangle &= \det(Z_i, Z_j, Z_k, Z_l)\,, \\
\langle AB\overline{ij} \rangle &= \langle A, i, i-1, i, i+1 \rangle \langle B, j, j-1, j, j+1 \rangle - (A \leftrightarrow B)\,.
\end{aligned}$$

- **Tests** by substituting four-vectors of **random complex integers** for the $Z_i = (\lambda_i, \mu_i)$.

According to [Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka (2010)]

$$\widehat{\mathcal{A}}_{5+1} = \frac{2}{5}\frac{\langle 1234\rangle\langle 2345\rangle\langle AB\rangle^4}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB45\rangle} + \frac{\langle AB\overline{25}\rangle\langle 2534\rangle\langle AB\rangle^4}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB45\rangle\langle AB51\rangle} + (\text{cyclic})\,,$$

$$\begin{aligned}\widehat{\mathcal{A}}_{5+2} &= \frac{1}{2}\frac{\langle 1234\rangle\langle 2345\rangle\langle 5123\rangle\langle AB\rangle^4\langle CD\rangle^4}{\langle AB12\rangle\langle AB23\rangle\langle AB51\rangle\langle ABCD\rangle\langle CD23\rangle\langle CD34\rangle\langle CD45\rangle} \\ &+ \frac{1}{2}\frac{\langle 1345\rangle\langle 3451\rangle\langle AB\overline{13}\rangle\langle AB\rangle^4\langle CD\rangle^4}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB51\rangle\langle ABCD\rangle\langle CD34\rangle\langle CD45\rangle\langle CD51\rangle} \\ &+ (AB \leftrightarrow CD) + (\text{cyclic})\,.\end{aligned}$$

Note

$$\langle i, i+1, j, j+1\rangle = \langle i, i+1\rangle\,\langle j, j+1\rangle\, x_{ij}^2$$

while the determinant is more complicated if the twistors belong to more than two points.

- **We confirm**

$$\boxed{\mathcal{I}_{5+1} = 2\,\widehat{\mathcal{A}}_{5+1}\,, \qquad \mathcal{I}_{5+2} = 2\,\widehat{\mathcal{A}}_{5+2} + \left(\widehat{\mathcal{A}}_{5+1}\right)^2 .}$$

- The same **holds at six points**.
- Correlators by $\mathcal{N} = 2$ **superfields** and **two insertions**, as in [Eden, Schubert, Sokatchev (2000)]

Conclusions ...

- In $\mathcal{N} = 4$ SYM, a light-like limit sends **n-point functions of** gauge invariant **composite operators** to **Wilson loops** or **scattering amplitudes**.
- Which **object** is obtained depends on the **regularisation**.
- The connection to Wilson loops is "manifest", while that to amplitudes needs to be understood.
- The **higher-loop amplitude integrands** of Arkani-Hamed et al. are **matched**.
- Our construction works for **MHV and non-MHV** cases.
- For the six-point one-loop NMHV amplitude, the usual **x variables** are just as **good** as the momentum twistors.

... and outlook

- Look for a proof of the correlator/amplitude duality, develop it into a proof of the Wilson loop/amplitude duality.
- Show the exact equality of the integrands.
- Analyse factorisation properties of the higher-loop correlators.
- [Mason, Skinner (2011)] suggest a proof of the correlator/Wilson loop duality for n-point functions of the Konishi operator. For what other correlators will this work?
- The supersymmetric Wilson loops are anomalous. [Belitsky, Korchemsky, Sokatchev (2011)] Can they be mended?
- BCFW recursion breaks amplitudes into three-point blocks. Link to the OPE for correlators?
- Reconstruct off-shell correlators from the amplitude integrands. Analyse invariants.
- Strong coupling?

n points $\to$ n cusps

In the cyclic sum above **only the last term in $\mathbf{i_{12}}$ contributes**. This was obtained from

$$I(0,1,2) \;=\; \frac{x_{12}^2}{4\pi^2(12)}\big(\theta^+_{0/1}[\partial_1,\partial_2]\theta^+_{0/2}\big)\int \frac{d^4x_{0'}}{x_{10'}^2x_{20'}^2x_{00'}^2}\,.$$

Dimensional regularisation:

$$\lim_{x_{12}^2\to 0}\; I_\epsilon(0,1,2) \;=\; -\,\frac{(1-\epsilon)}{(12)}\big(\theta^+_{0/1}[x_{10},x_{20}]\theta^+_{0/2}\big)\int_0^1 ds\big[-\big(x_{10}\,s + x_{20}\,(1-s)\big)\big]^{-(2-\epsilon)}$$

θ, u algebra as before, rewrite numerator:

$$\boxed{\lim_{x_{i,i+1}^2\to 0} G_n^{(1)}/G_n^{(0)} \;=\; \frac{1}{2}\,g^2N_c\sum_{k\neq l}\int_0^1 ds_k\, x^\mu_{k,k+1}\int_0^1 ds_l\, x^\nu_{l,l+1}\, D_{\mu\nu}(x_{ab}) \;=\; 2\,W^{(1)}[C_n]}$$

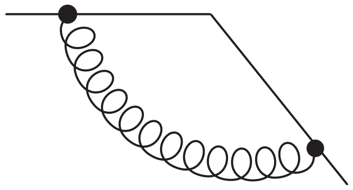
where

$$x_a = x_k s_k + x_{k+1}(1-s_k)\,, \qquad x_b = x_l s_l + x_{l+1}(1-s_l)\,.$$

Gluon propagator in **Landau gauge:**

$$D_{\mu\nu}(x) = \int \frac{d^{4-2\epsilon}k}{k^2}\, e^{ikx}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)$$

Mass regulator $x_{i,i+1}^2 = \delta$



WL in dimensional regularisation

$$W = e^{\text{Div}}\, W_{\text{finite}}\ , \qquad W_{\text{finite}} = e^{\Gamma_{\text{cusp}} A_{\text{BDS-like}}}\, W_{\text{conformal}}$$

$$\text{Div} = \sum_{i=1}^{n} \frac{\Gamma_{\text{cusp}}}{8} [\log(\mu^2 x_{i,i+2}^2)]^2 + g_{\text{col}} \log(\mu^2 x_{i,i+2}^2)$$

- $\mathbf{W}_{\text{finite}}$ [Bern et al.] obeys the **Ward identity** for anomalous special conformal symmetries. [Drummond, Henn, Korchemsky, Sokatchev (2007)]
- When $n \neq 4\,k$ there is a **unique solution $\mathbf{A}_{\text{BDS-like}}$** to the Ward identity involving only $\{\mathbf{x}^2_{\mathbf{j,j+2}}\}$. [Alday, Gaiotto, Maldacena (2009)]
- In the **correlator** remove it from the finite part and **combine it with the singularities**. Only **cross-ratios** can occur!
- When $n \neq 4\,k$ there is a **unique cross-ratio $\chi_{\mathbf{i,i+1}}$** built from $x_{i,i+1}^2, \{x_{j,j+2}^2\}$.

Conjecture at **one loop**: $\qquad a = g^2 N_c/(4\pi^2)$

$$\lim_{x_{i,i+1}^2 \to 0} \left(G_n^{(1)}/G_n^{(0)}\right) = -\frac{a}{2}\left[\sum_{i=1}^{n} \log(\chi_{i,i+1}) \log(\chi_{i+1,i+2}) - 2\,(A_{\text{BDS}} - A_{\text{BDS-like}}) + n * \frac{5}{2}\zeta(2)\right]$$

How to compute $\mathbf{G^{(0)}_{5+2;2}}$

By **R-weight**:

$$G^{(0)}_{5+2;2} = \langle L(0)\, L(0')\, O(1)\, \tilde{O}(2)\, O(3)\, \tilde{O}(4)\, \hat{O}(5)\rangle \propto \theta^8 + \theta^9\bar{\theta} + \ldots$$

Q, **S-supersymmetry**: The **leading term** must come out in terms of

$$\xi_{r\dot{\alpha}} = \rho_{r\dot{\alpha}} - \sigma_{r\dot{\alpha}}\,, \qquad r = 1, \ldots, 5\,,$$

$$\rho_{r\dot{\alpha}} = (\theta^+_r - \theta^i_0 u^+_{ri})^\alpha (x_{r0})_{\alpha\dot{\alpha}} x^{-2}_{r0}\,, \qquad \sigma_{r\dot{\alpha}} = (\theta^+_r - \theta^i_{0'} u^+_{ri})^\alpha (x_{r0'})_{\alpha\dot{\alpha}} x^{-2}_{r0'}\,.$$

U(1)-charge and **harmonic analyticity**:

$$\mathcal{G}^{(0)}_{5+2} = \xi^2_1\, \xi^2_2\, \xi^2_3\, \xi_{4\dot{\alpha}}\, \xi_{5\dot{\beta}}\, (45)\, f^{\dot{\alpha}\dot{\beta}}_{45}(x) + \text{(perms.)}$$

$\mathbf{f^{\dot{\alpha}\dot{\beta}}_{ij}}$ are defined by the **graphs**. Evaluate at $\boldsymbol{\theta_0 = \theta_{0'} = 0}$, then **reconstruct** by sending

$$\theta^{+\alpha}_r = \frac{x^2_{r0}\, x^2_{r0'}}{x^2_{00'}} \left(\frac{x^{\alpha\dot{\alpha}}_{r0}}{x^2_{r0}} - \frac{x^{\alpha\dot{\alpha}}_{r0'}}{x^2_{r0'}} \right) \xi_{r\dot{\alpha}}|_{\theta_0=\theta_{0'}=0} \longrightarrow \frac{x^2_{r0}\, x^2_{r0'}}{x^2_{00'}} \left(\frac{x^{\alpha\dot{\alpha}}_{r0}}{x^2_{r0}} - \frac{x^{\alpha\dot{\alpha}}_{r0'}}{x^2_{r0'}} \right) \xi_{r\dot{\alpha}}|_{\theta^+_r=0} = x^{\alpha\dot{\alpha}}_{r0}\, \lambda^{r+}_{0'\dot{\alpha}} - x^{\alpha\dot{\alpha}}_{r0'}\, \lambda^{r+}_{0\dot{\alpha}}\,,$$

$$\lambda^{r+}_{t\dot{\alpha}} = \frac{x_{00'\,\dot{\alpha}\alpha}}{x^2_{00'}}\, \theta^{\alpha i}_t u^+_{ri}\,, \qquad t \in \{0, 0'\}\,.$$

- To compute $\mathbf{f}_{\mathbf{45}}^{\dot{\alpha}\dot{\beta}}$ from the sum of graphs, put $\mathbf{u_1} = \mathbf{u_2} = \mathbf{u_3} = \mathbf{u_5}$ in all corresponding terms.
- Graphs with **more than one free line** are put to **zero.**
- We only need graphs with **four T-blocks.**
- The **light-like limit** selects the **pentagon 12345**.
- We find a large sum over traces

$$\mathrm{Tr}\,(\tilde{x}_{17}\,x_{18}\,\tilde{x}_{28}\,x_{27}\ldots)$$

which are **conformally covariant**.