Thermalization in QCD and AdS/CFT

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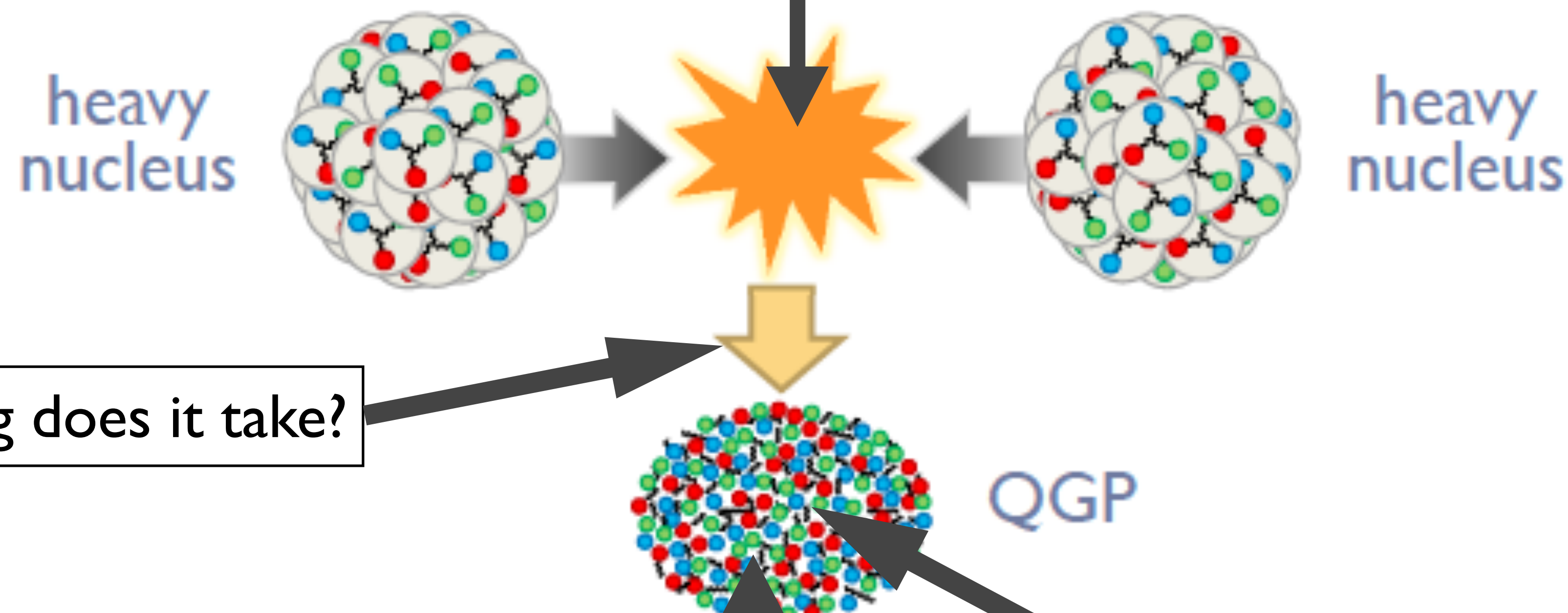
11th Workshop on Non-Perturbative QCD

Paris, 6−10 June 2011

The Thermalization Problem

Thermalization

Characteristic participant parton momentum scale: Q_s



How long does it take?

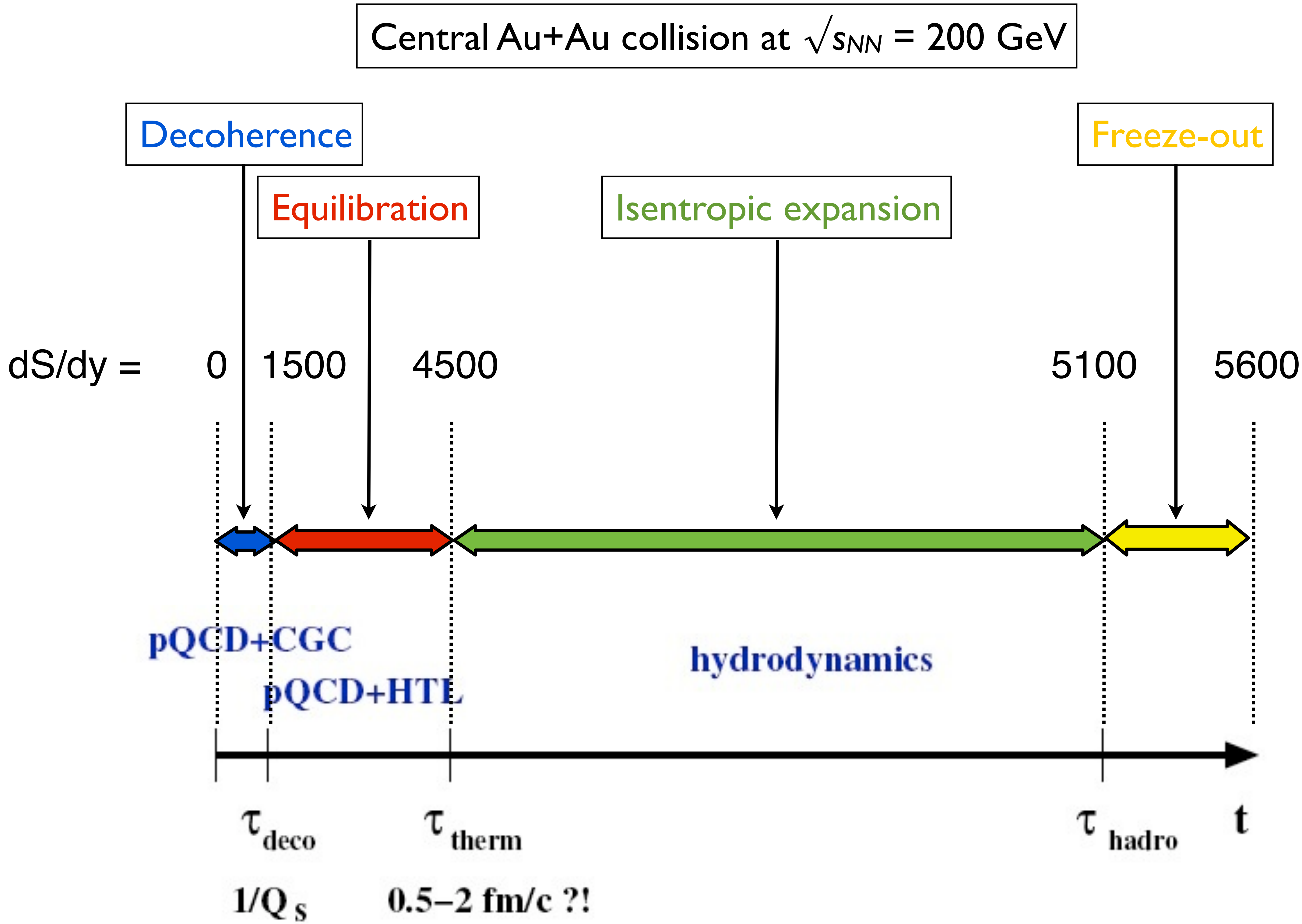
How thermal is it?

Characteristic parton momentum scale: $T << Q_s$
(at weak coupling)

How does the thermalization process work ?

"Bottom up", or "Top down" ?

Entropy evolution



The problem

The von Neumann entropy $S_{\mathrm{vN}} = -\mathrm{Tr}\left[\rho \ln \rho\right]$

is conserved for any closed quantum system described by a Hamiltonian.

Approach 1: For system *X* interacting with its environment *Y*, the reduced entropy

$$S_X = -\mathrm{Tr}_X\left[\rho_X \ln \rho_X\right] \quad \text{with} \quad \rho_X = \mathrm{Tr}_Y\left[\rho\right]$$

increases as a result of growing entanglement between *X* and *Y*. Consider, e.g., a rapidity interval [y,y+Δy] as “system” and the remainder as “environment”, which cannot effectively communicate due to causality.

Problem: How to split reaction volume unambiguously into *X* and *Y* ?

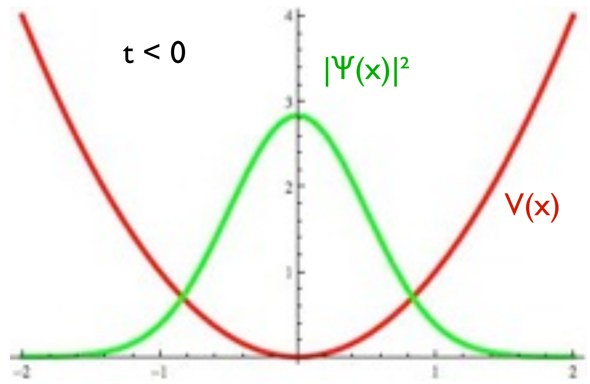
Approach 2: Consider the effective growth of the entropy due to the increasing intrinsic complexity of the quantum state after “coarse graining”.

Problem: How to coarse grain without assuming the answer ?

The Husimi Function

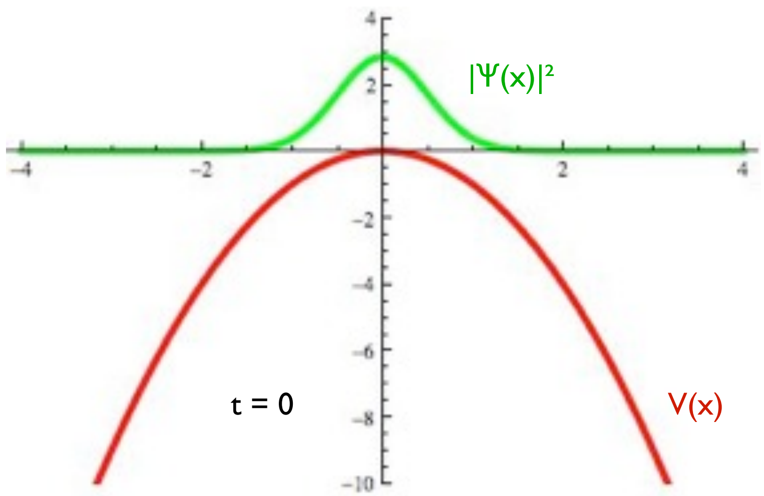
The “pencil on its tip”

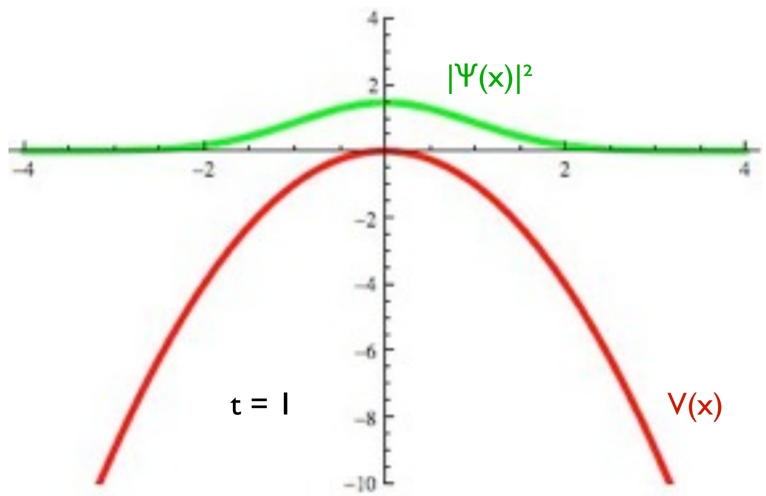
The decay of an unstable vacuum state is a common problem, e.g., in cosmology and in condensed matter physics. Paradigm case: inverted oscillator.

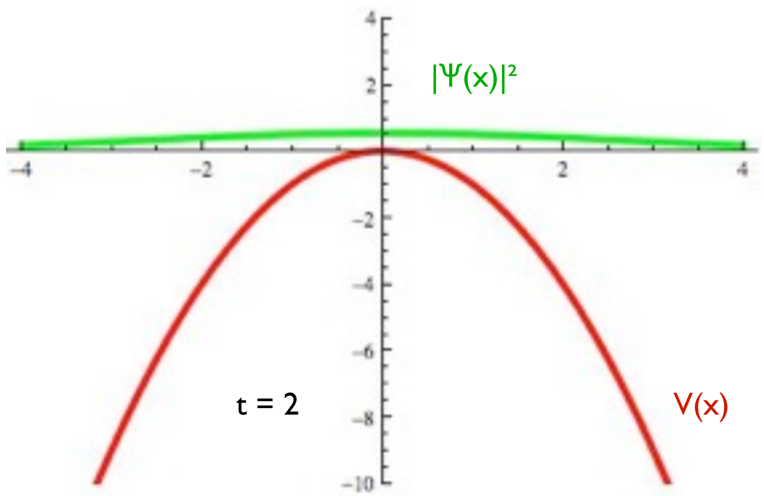


$$\hat{H}(t) = \frac{p^2}{2} + \frac{m(t)^2}{2} x^2$$

with $$m(t)^2 = \omega^2 \theta(-t) - \lambda^2 \theta(t)$$

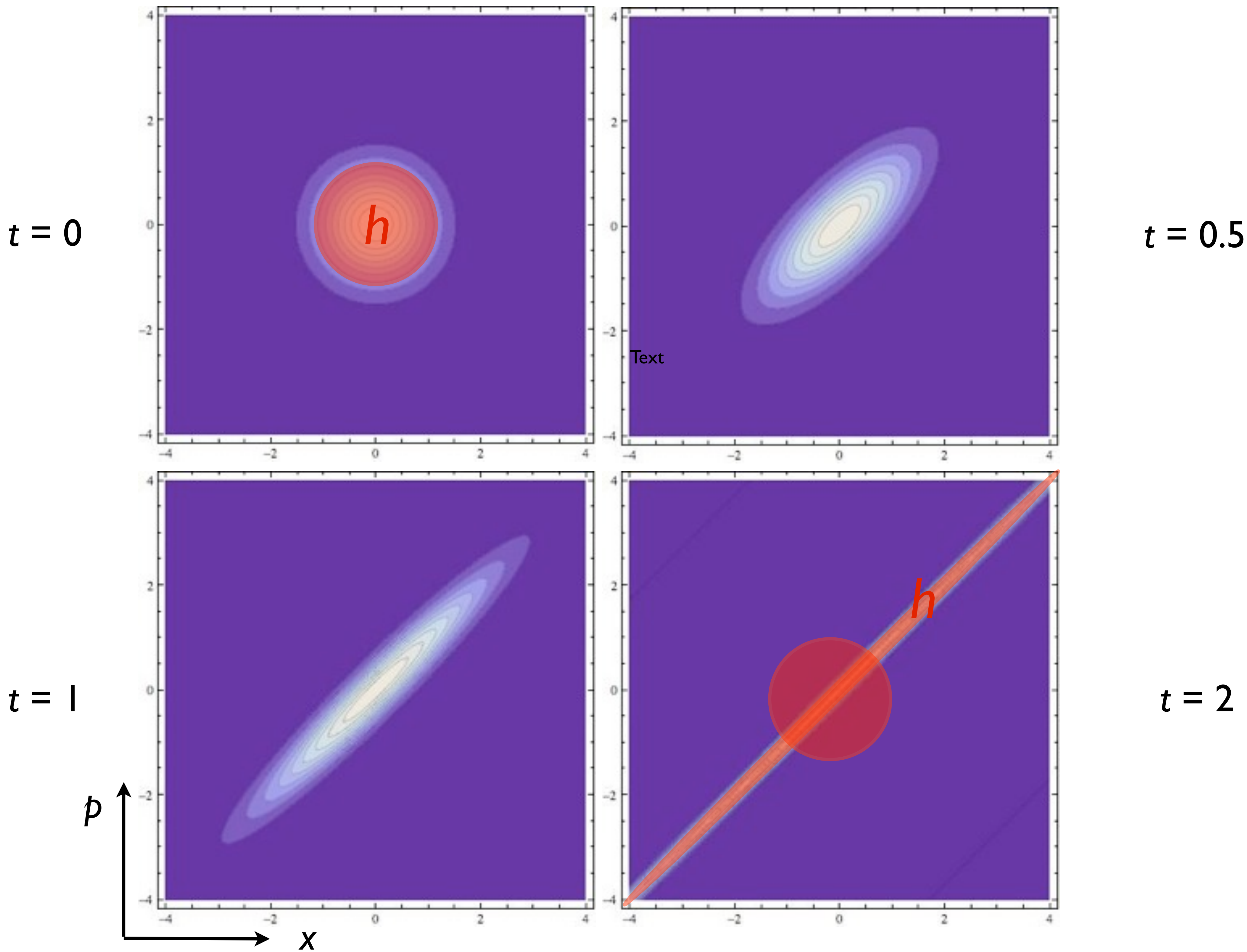






Wigner function: $$W(q,p;t) = \int du\, e^{-ipu} \langle q + \frac{1}{2}u|\ \hat{\rho}(t)\ |q - \frac{1}{2}u\rangle$$

Wigner function



Husimi transform

- Problem: Wigner function cannot be interpreted as a probability distribution, because $W(p,x)$ is not positive definite.
- Idea (*Husimi* - 1940): Smear the Wigner function with a Gaussian minimum uncertainty wave packet:

$$H_\Delta(p,x;t) \equiv \int \frac{dp'\,dx'}{\pi\hbar} \exp\left(-\frac{1}{\hbar\Delta}(p-p')^2 - \frac{\Delta}{\hbar}(x-x')^2\right) W(p',x';t)$$

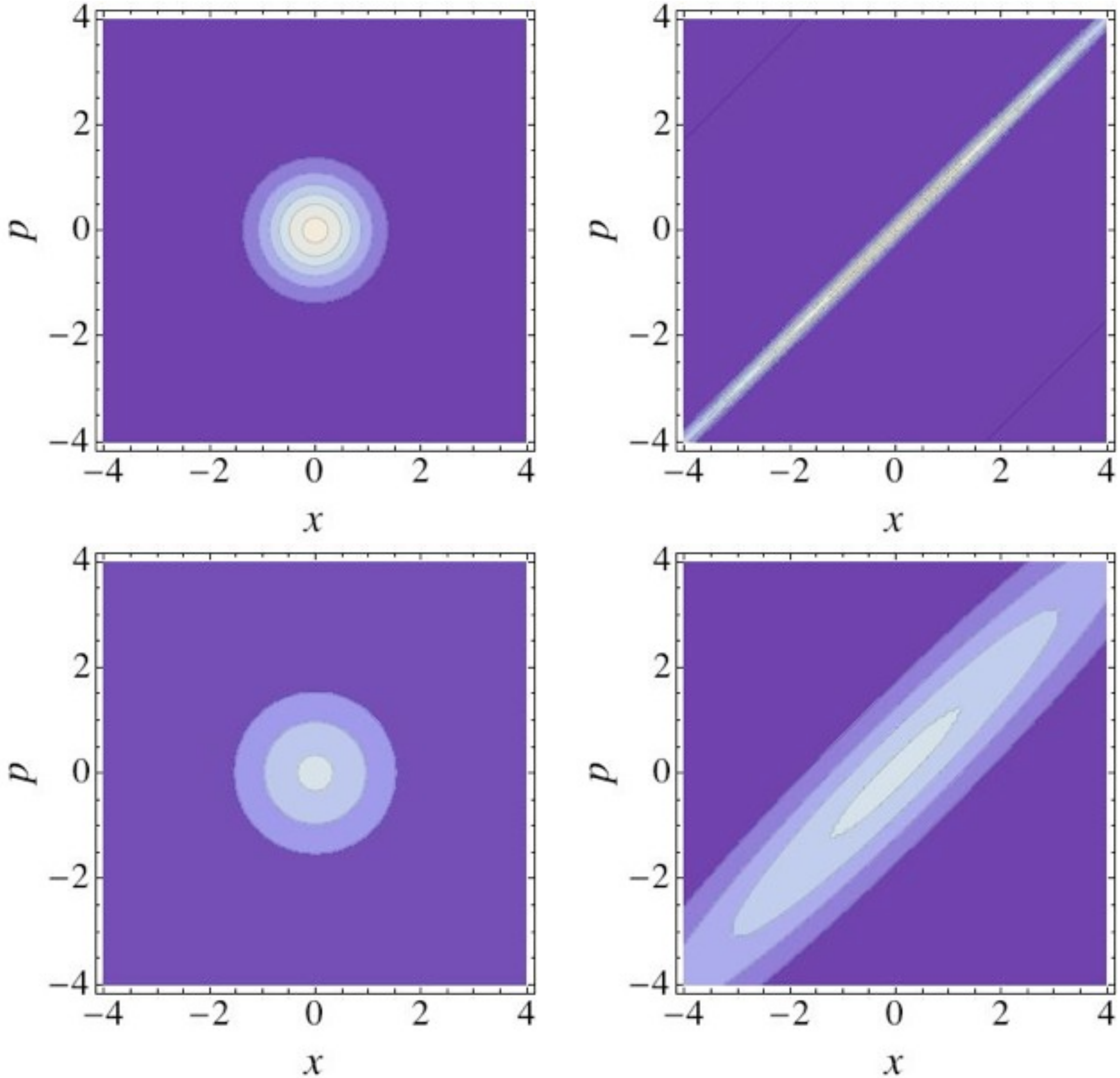
- $H(p,x)$ can be shown to be the expectation value of the density matrix in a coherent oscillator state $|x+ip\rangle$ and thus $H(p,x) \geq 0$ holds always.
- H(p,x) can be considered as a probability density, enabling the definition of a minimally coarse grained entropy (*Wehrl* - 1978):

$$S_{\mathrm{H},\Delta}(t) = -\int \frac{dp\,dx}{2\pi\hbar} H_\Delta(p,x;t) \ln H_\Delta(p,x;t)$$

Wigner vs. Husimi

Wigner function

$t = 0$



$t = 2$

Husimi function

$t = 0$

$t = 2$

S_H entropy growth

$$\frac{dS_H}{dt} = \frac{\lambda\sigma\rho\sinh 2\lambda t}{\sigma\rho\cosh 2\lambda t + 1 + \delta'\delta} \xrightarrow{t\to\infty} \lambda$$ with $\rho,\sigma,\delta,\delta'$ constants dep. on ω,λ

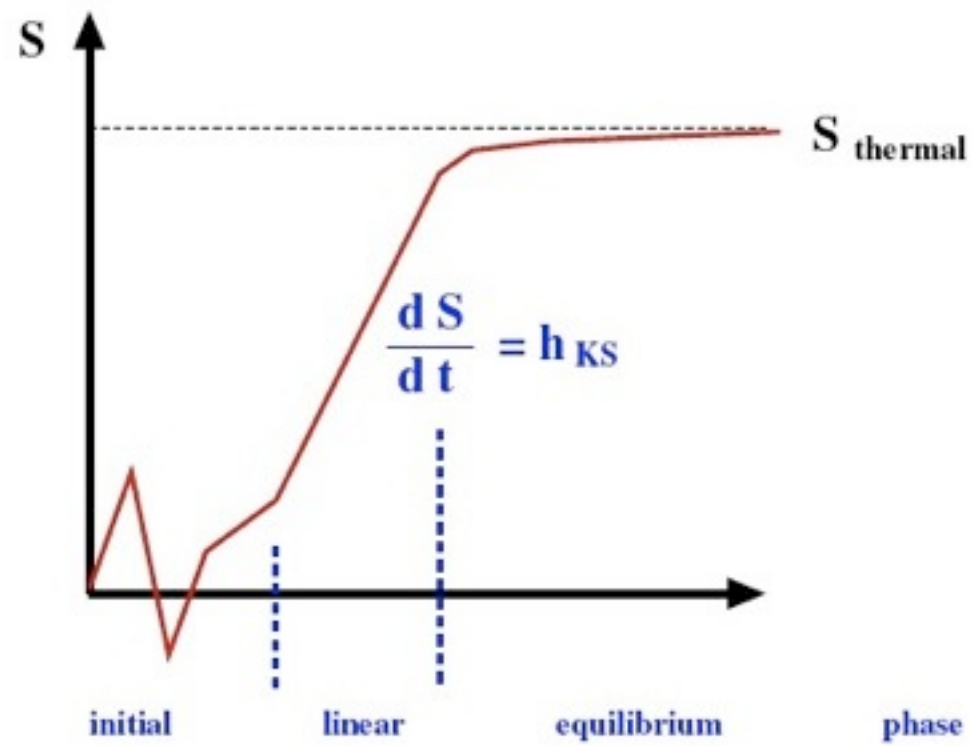
but independent of Δ and $\hbar$!!!

Many modes: $$\frac{dS_H}{dt} \xrightarrow{t\to\infty} \sum_k \lambda_k \theta(\lambda_k)$$

Kunihiro, BM, Ohnishi & Schäfer,
Prog. Theor. Phys. 121 (2009) 555

Kolmogorov-Sinai (KS) entropy growth rate h_{KS} of classical dynamical system theory.

KS-entropy growth rate describes the growth rate of the entropy for a coarse grained phase space density.
[Latora & Baranger, PRL 82 (1999) 520]



Quantum quench

$$\hat{H}(t) = \int_0^\infty \frac{dp}{2\pi}\left(\hat{\Pi}^\dagger(p)\hat{\Pi}(p) + (m^2(t)+p^2)\,\hat{\Phi}^\dagger(p)\hat{\Phi}(p)\right)$$ with $m^2(t) = m^2\,\theta(-t) - \mu^2\,\theta(t)$

"Quantum quench"

Split problem into stable ($p^2 > \mu^2$) and unstable ($p^2 < \mu^2$) modes.

Wigner functional:

$$W[\Pi,\Phi;t] = C\, e^{-\int \frac{dp}{2\pi}\left(\frac{|\Pi_p|^2}{E_p} + E_p|\Phi_p|^2\right)}$$ with $E_p = \sqrt{p^2+m^2}$

Each mode of W evolves along a classical trajectory:

$|p| < \mu$

$$\begin{aligned}\Phi_p^0 &= \Phi_p(t)\cosh\lambda_p t - \frac{\Pi_p(t)}{\lambda_p}\sinh\lambda_p t\\ \Pi_p^0 &= \Pi_p(t)\cosh\lambda_p t - \lambda_p\,\Phi_p(t)\sinh\lambda_p t\end{aligned}$$

$$\lambda_p = \sqrt{\mu^2 - p^2}$$

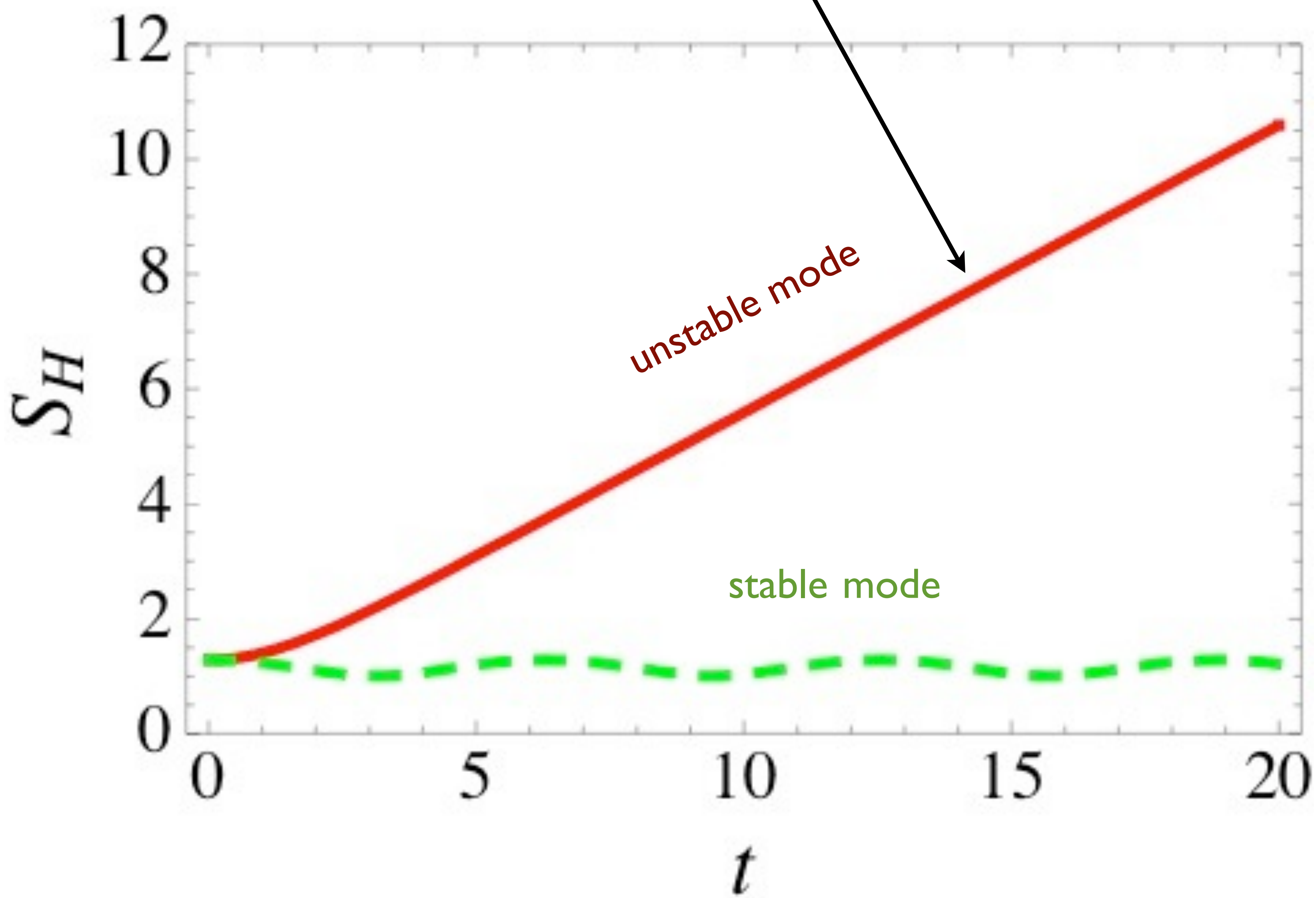
$|p| > \mu$

$$\begin{aligned}\Phi_p^0 &= \Phi_p(t)\cos\omega_p t - \frac{\Pi_p(t)}{\omega_p}\sin\omega_p t\\ \Pi_p^0 &= \Pi_p(t)\cos\omega_p t + \omega_p\,\Phi_p(t)\sin\omega_p t\end{aligned}$$

$$\omega_p = \sqrt{p^2 - \mu^2}$$

Instability begets entropy

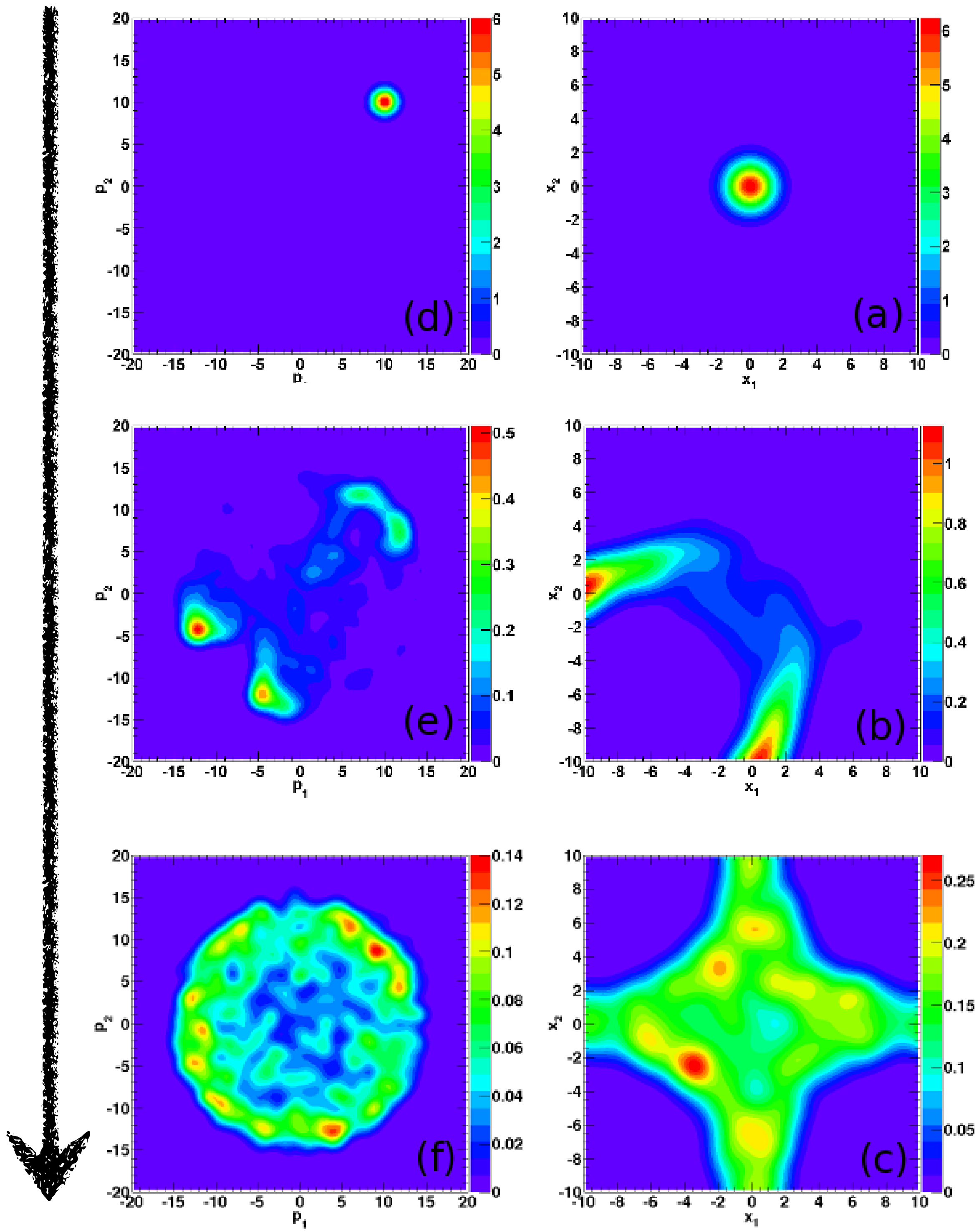
Only S_H of unstable modes grows !

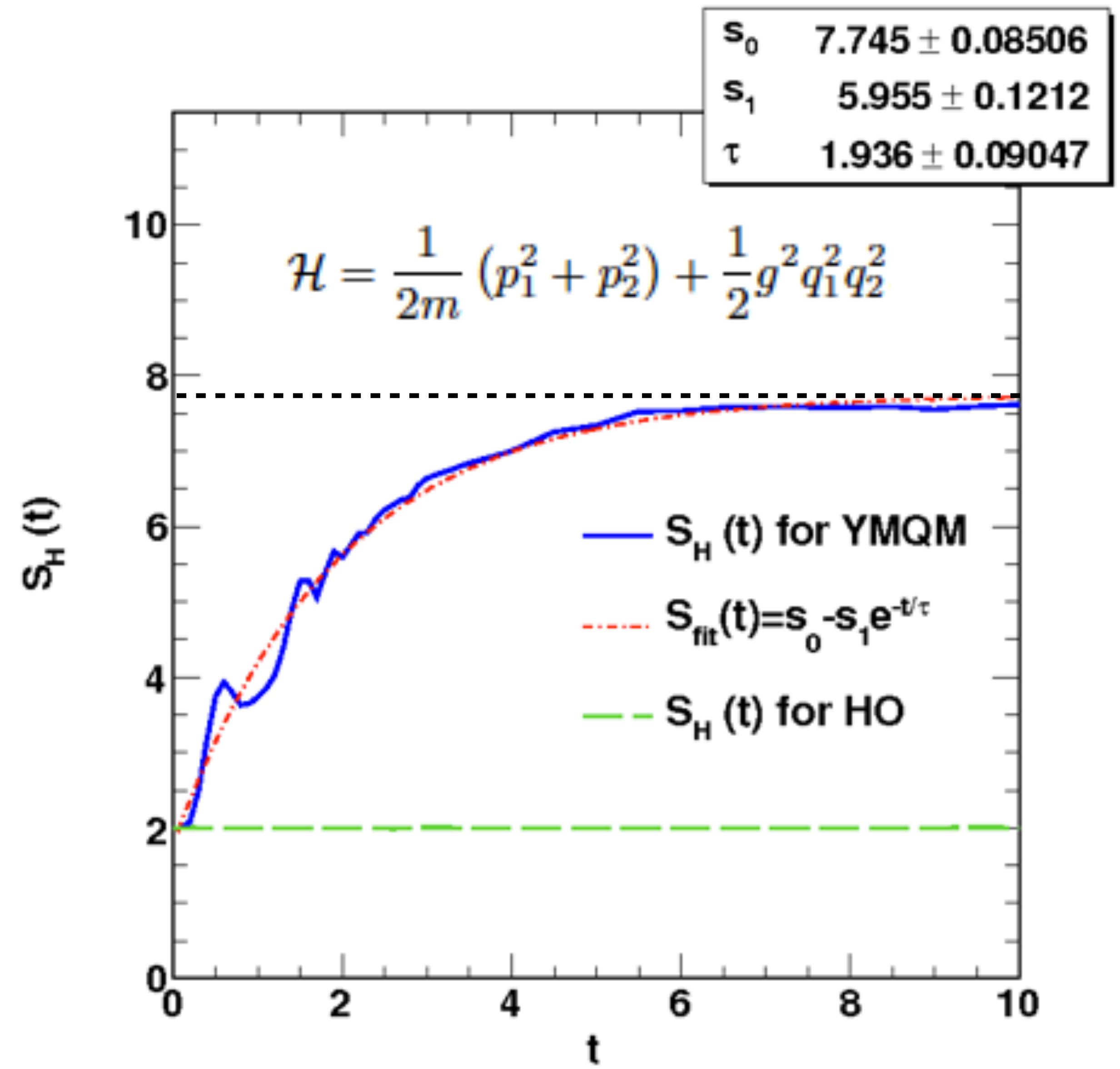


Yang-Mills theory

YM model system

HM Tsai & BM, arXiv: *1011.3508*





Yang-Mills theory

$$H = \frac{1}{2g^2}\int d^3x \left(\sum_{a,i} E_i^a(x)^2 + \frac{1}{2}\sum_{a,i,j} F_{ij}^a(x)^2\right) \Longrightarrow H = \frac{1}{2}\sum_{x,a,i} E_i^a(x)^2 + \frac{1}{4}\sum_{x,a,i,j} F_{ij}^a(x)^2$$

continuum lattice

classical EOM's:

$$\dot{A}_i^a(x) = E_i^a(x) \ ,$$
$$\dot{E}_i^a(x) = \sum_j \partial_j F_{ji}^a(x) + \sum_{b,c,j} f^{abc} A_j^b(x) F_{ji}^c(x)$$

infinitesimal fluctuations (Hessian):

$$\mathcal{H} = \begin{pmatrix} H_{EA} & H_{EE} \\ -H_{AA} & -H_{AE} \end{pmatrix}$$

$$H_{EE} = \delta^{ab}\delta_{ij}\delta_{x,y} \ ,$$
$$H_{EA} = H_{AE} = 0 \ ,$$
$$H_{AA} = \frac{1}{4}\delta^{ab}P + \frac{1}{2}\sum_c f^{abc}Q^c + \sum_{cde} f^{acd} f^{bce} R^{de},$$

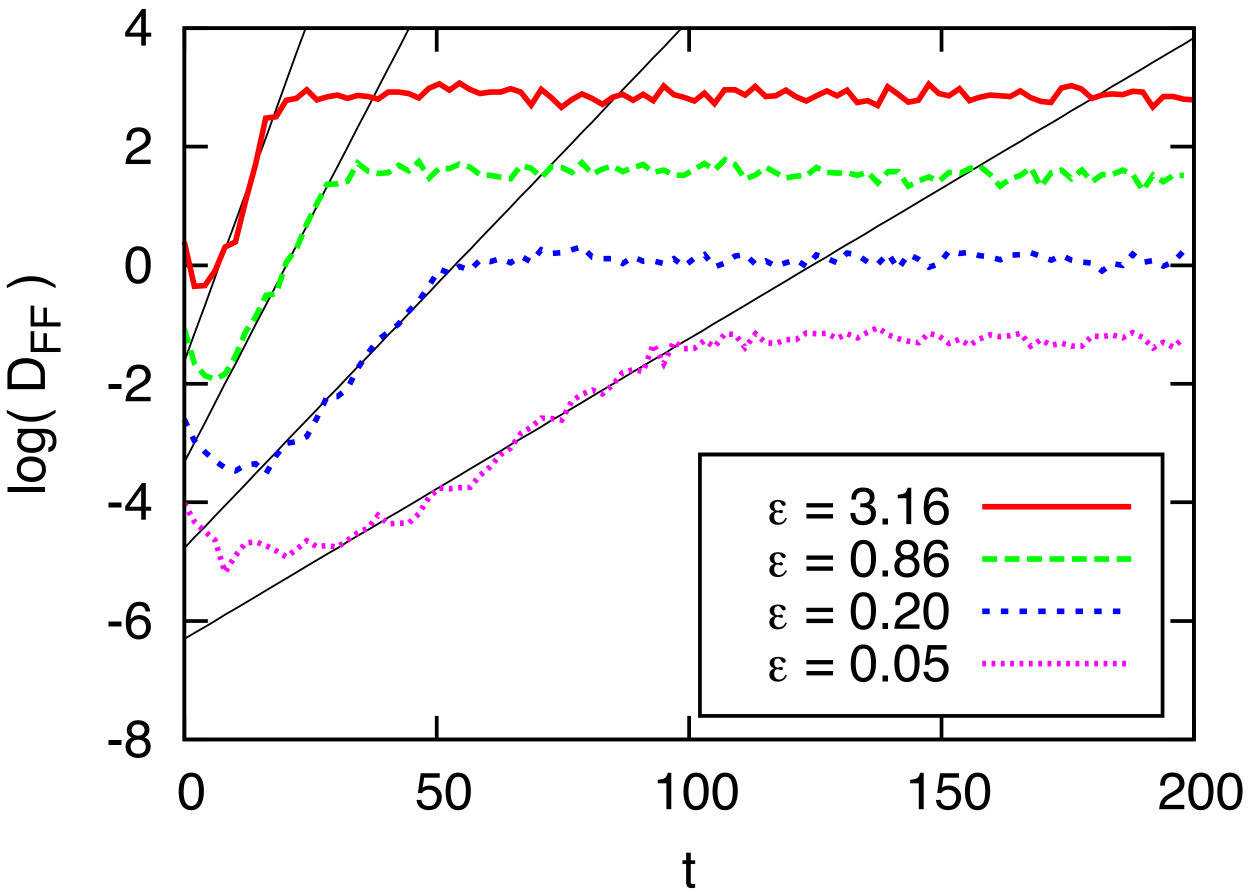
$$P = -\left(\delta_{x+\hat{\imath},y+\hat{\jmath}} - \delta_{x+\hat{\imath},y-\hat{\jmath}} - \delta_{x-\hat{\imath},y+\hat{\jmath}} + \delta_{x-\hat{\imath},y-\hat{\jmath}}\right) + \delta_{ij}\sum_k \left(2\delta_{x,y} - \delta_{x+\hat{k},y-\hat{k}} - \delta_{x-\hat{k},y+\hat{k}}\right) \quad (27)$$

$$Q^c = A_i^c(y)(\delta_{x,y+\hat{\jmath}} - \delta_{x,y-\hat{\jmath}}) - A_j^c(x)(\delta_{x+\hat{\imath},y} - \delta_{x-\hat{\imath},y}) + \delta_{ij}\sum_k \{A_k^c(x) + A_k^c(y)\}(\delta_{x+\hat{k},y} - \delta_{x-\hat{k},y}) + 2F_{ij}^c(x)\delta_{x,y} \quad (28)$$

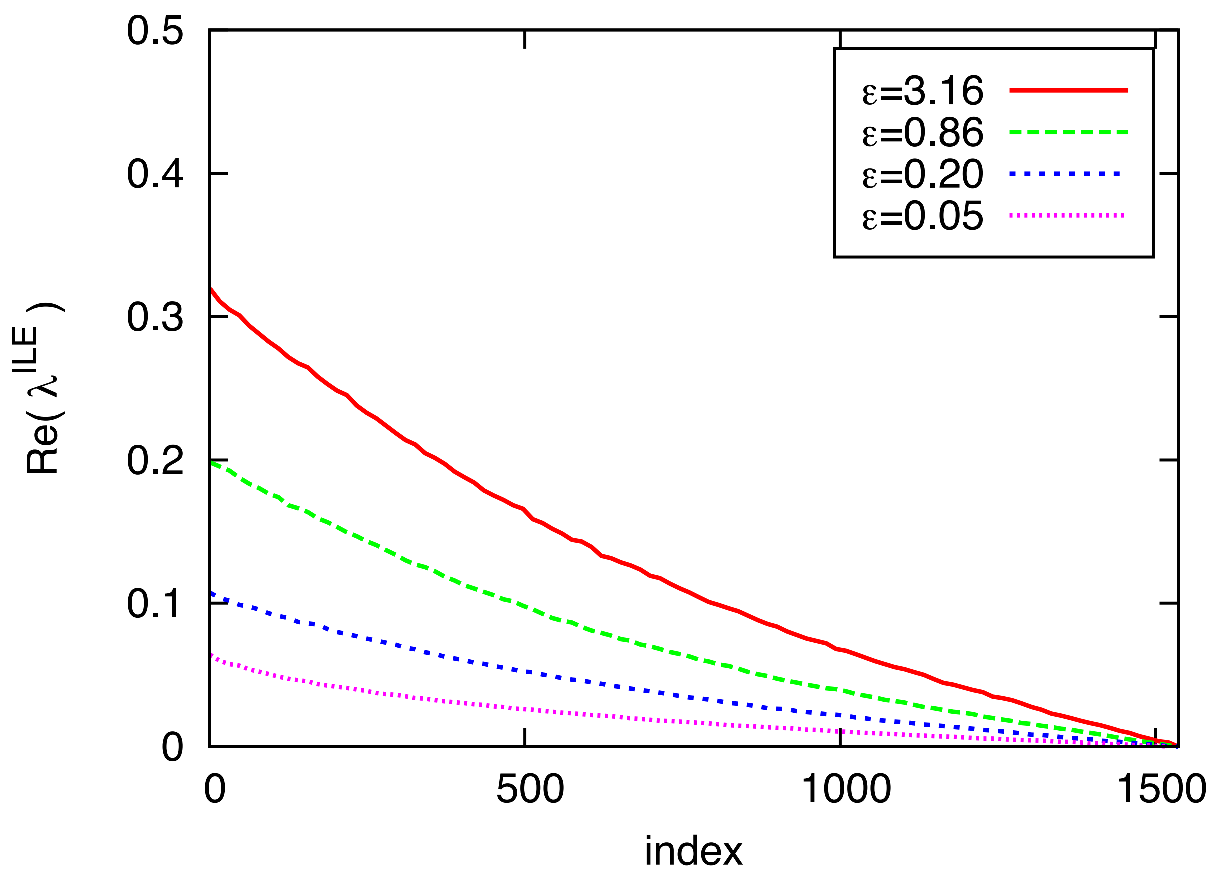
$$R^{de} = \{-A_i^e(x)A_j^d(x) + \delta_{ij}\sum_k A_k^d(x)A_k^e(x)\}\delta_{x,y} \ . \quad (29)$$

SU(3) YM

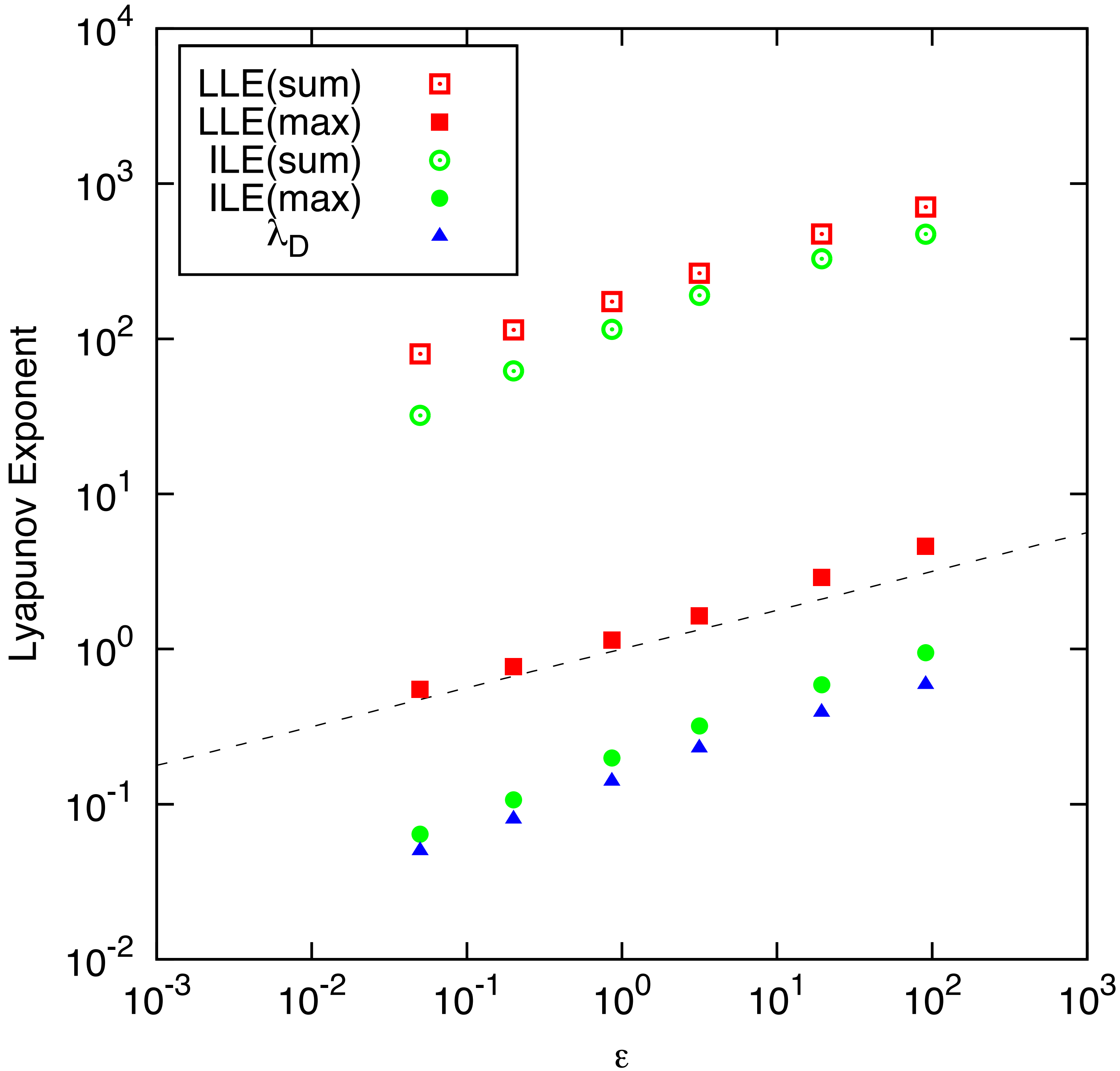
Infinitesimal fluctuations grow at exponential rate increasing with energy density



Intermediate time Lyapunov spectrum on 4^3 lattice



Lyapunov exponents



energy dependence of maximal Lyapunov exponent and KS entropy growth rate

$$\sum_i \lambda_i \sim \varepsilon^{1/4} L^3$$

$$\Rightarrow \tau_{eq} \sim 5/T \sim 2\text{–}3 \text{ fm/c}$$

Kunihiro, BM, Ohnishi, Schäfer, Takahashi & Yamamoto
Phys. Rev. D82 (2010) 114015

Holographic Thermalization

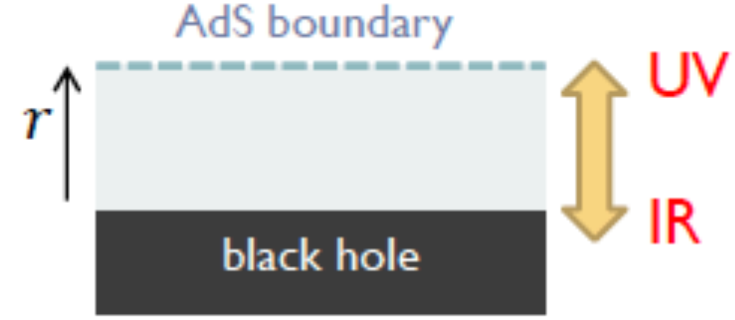
AdS/CFT dictionary

- Want to study strongly coupled phenomena in QCD
- Toy model: $\mathcal{N} = 4\ SU(N_c)$ SYM

$\mathcal{N} = 4$ SYM	String theory in AdS_5
Strong coupling	Weak coupling
Vacuum	Empty AdS_5
Thermal state (equilibrated plasma)	AdS BH
Thermalization	BH formation
HI collision	Energy injection

holo. dual ⟺ AdS/CFT

$$\left(\frac{R_{AdS}}{l_s}\right)^4 = g_{\mathrm{YM}}^2 N_c \equiv \lambda$$



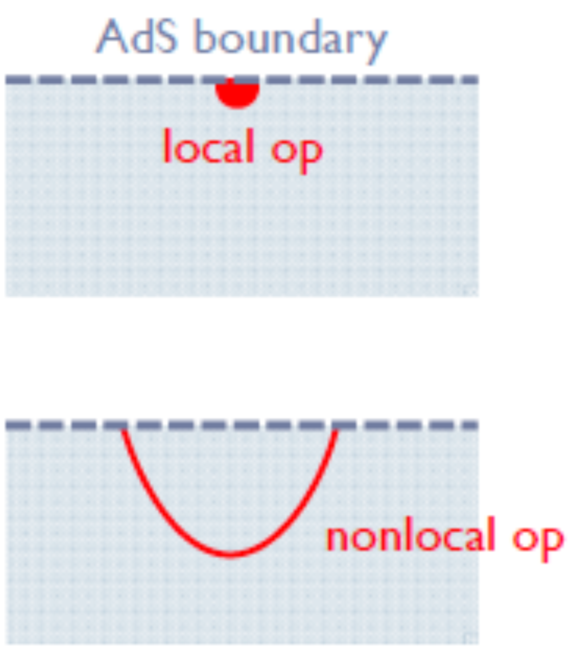
Holographic thermalization

- What is the measure of thermalization on the boundary?
  - Local operators are not sufficient

    $\langle T_{\mu\nu} \rangle$ etc.

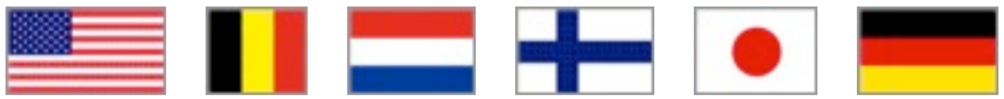
  - Nonlocal operators are more sensitive

    $\langle O(x) O(x') \rangle$ etc.



- What is the thermalization time?
  - When observables reach their thermal values

Vijay Balasubramanian (U Penn)
Alice Bernamonti, Ben Craps, Neil Copland, Wieland Staessens (VU Brussels)
Jan de Boer (Amsterdam)
Esko Keski-Vakkuri (Helsinki/Uppsala)
Masaki Shigemori (KMI Nagoya)
Andreas Schäfer (Regensburg)
Phys. Rev. Lett. 106 (2011) 191601; arXiv: *1010.4753*

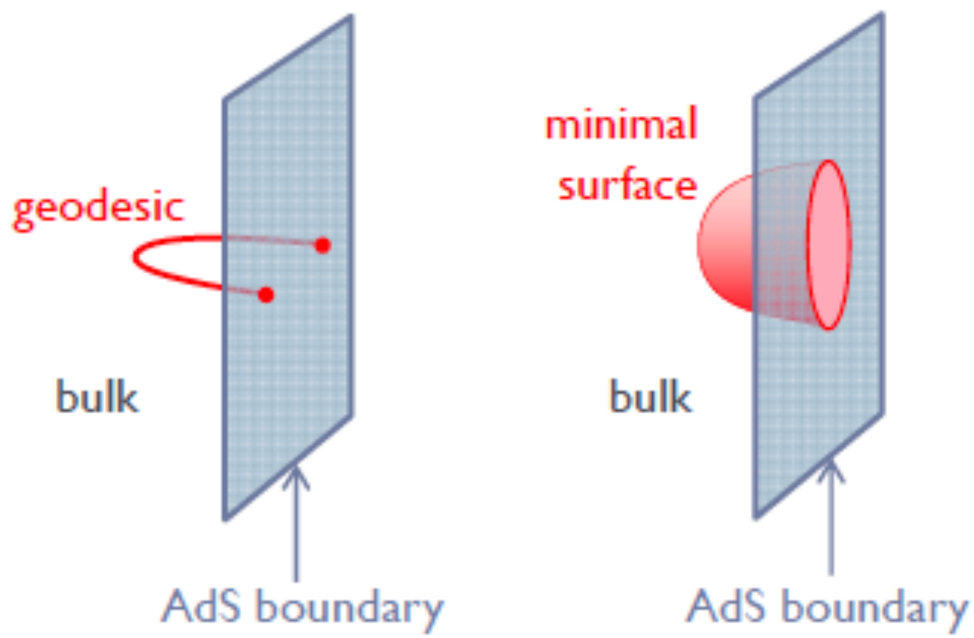


Thermality probes

- **Local operators** like $\langle T_{\mu\nu} \rangle$ measure moments of the momentum distribution of field excitations
  - e.g. $\langle k_x^2 \rangle$ vs. $\langle k_z^2 \rangle$
- **Nonlocal operators**, like the equal-time Green function, are sensitive to the momentum distribution and to the spectral density of excitations:
  - $$G(\vec{x}) = \int d\vec{k}\, dk^0 \sigma\left(k^0, \vec{k}\right)\left[n\left(\vec{k}\right)+1\right]\exp\left(i\vec{k}\cdot\vec{x}\right)$$
  -
- **Entropy** is the "gold standard" of thermalization:
  - $S = -\,\mathrm{Tr}[\rho \ln(\rho)]$ probes all degrees of freedom.
  - Coarse graining mechanism: ***Entanglement entropy***.

Probes we consider

- 2-point function
  - $\langle \mathcal{O}(x)\mathcal{O}(x)\rangle$
  - Bulk: geodesic (1D)
- Wilson line
  - $W = P\{\exp[\int_C A_\mu(x)dx^\mu]\}$
  - Bulk: minimal surface (2D)
- Entanglement entropy
  - $S_A = -\mathrm{Tr}_A[\rho_A \log \rho_A], \quad \rho_A = \mathrm{Tr}_B[\rho_{tot}]$
  - Bulk: codim-2 hypersurface (same dimension as boundary space)



Use semiclassical approximation

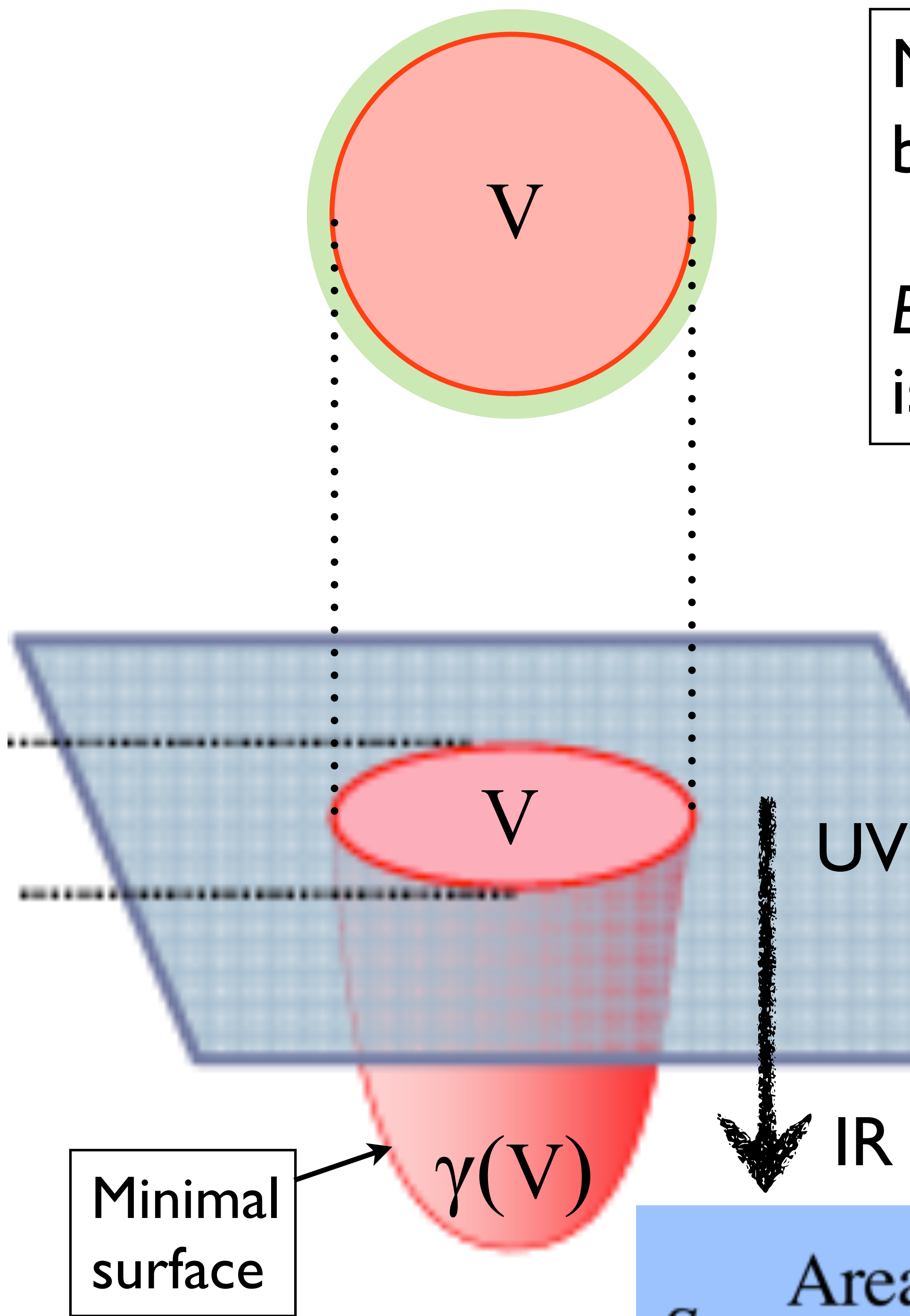
For details: *V. Balasubramanian, et al.*, PRL **106**, 191601 (2011); arXiv:1103.2683

See also: S. Caron-Huot, P.M. Chesler & D. Teaney, arXiv:1102.1073

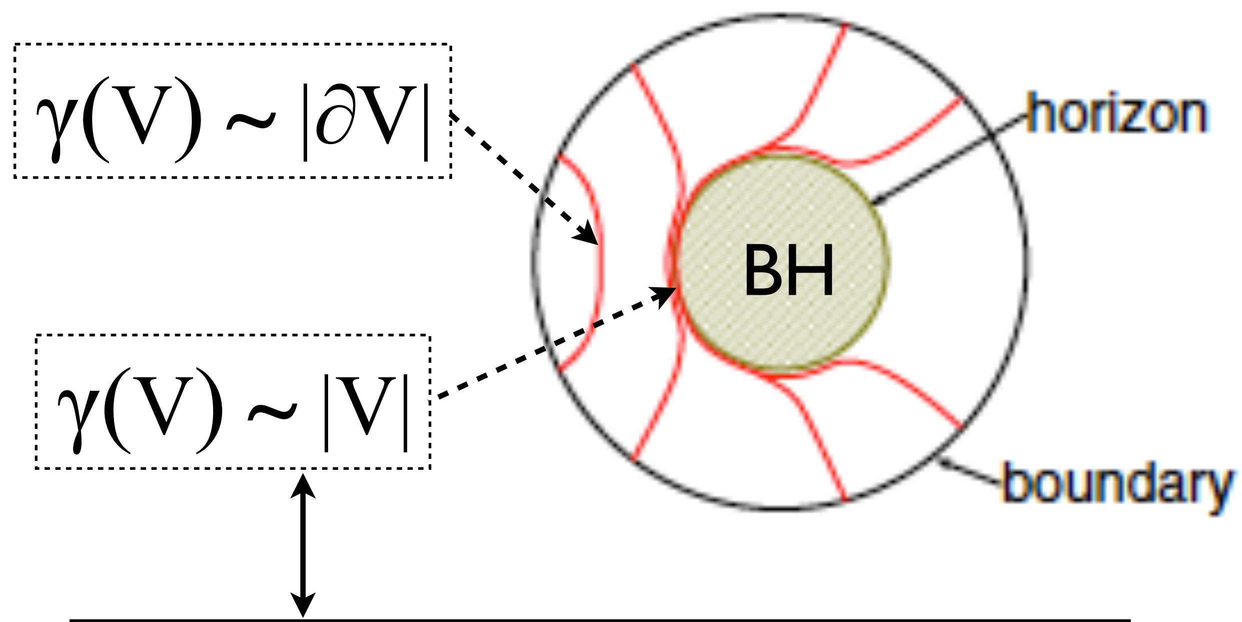
Entanglement entropy

Modes with momentum k “leak” into surrounding by $\Delta x \sim 1/k$ ➠ *entanglement* with environment

Entanglement entropy of localized vacuum domain is proportional to surface area (*Srednicki* 1994).



$$S = \frac{\mathrm{Area}[\gamma(V)]}{4G_N}$$



T ≠ 0: *S* proportional to volume
⇔ area of horizon of dual BH

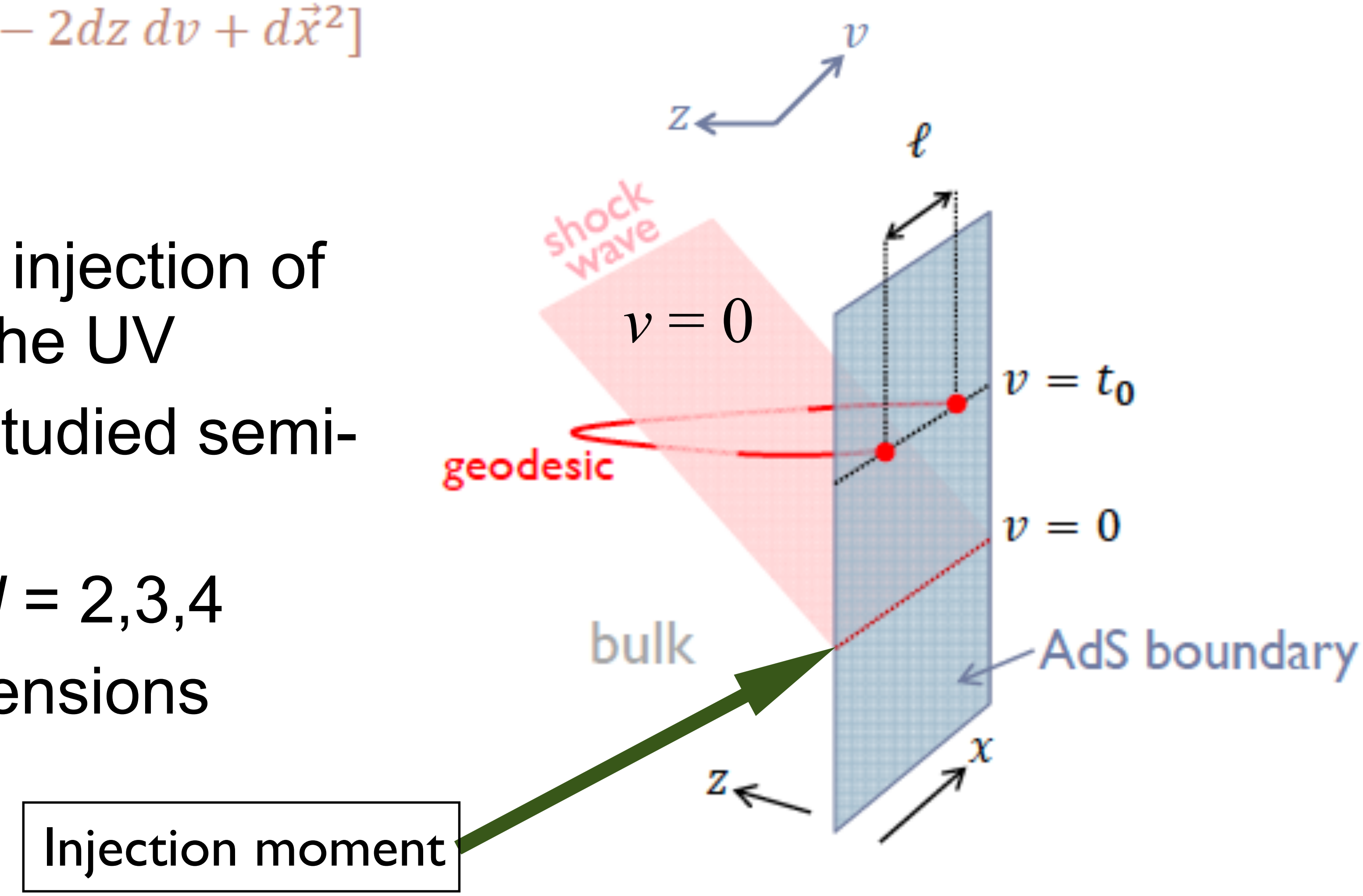
(*Ryu & Takayanagi* 2006)

Vaidya-AdS geometry

- Light-like (null) infalling energy shell in AdS (shock wave in bulk)
  - *Vaidya-AdS space-time* (analytical)

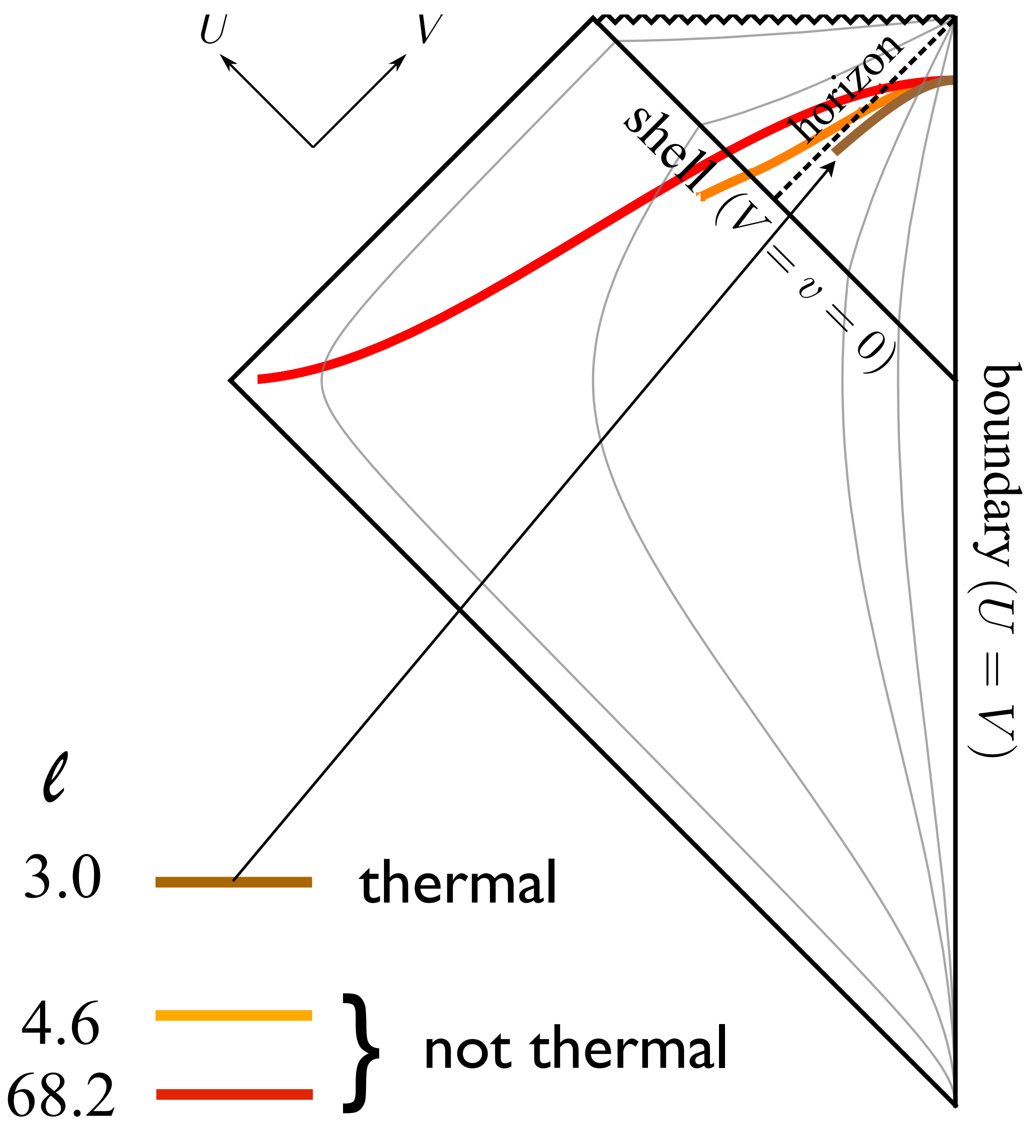
$$ds^2 = \frac{1}{z^2}\left[-\left(1 - m(v)z^d\right)dv^2 - 2dz\,dv + d\vec{x}^2\right]$$

  - $z = 0$: UV $z = \infty$: IR
  - Homogeneous, sudden injection of entropy-free energy in the UV
  - Thin-shell limit can be studied semi-analytically
  - We studied AdS$_{d+1}$ for d = 2,3,4
  - ⇔ Field theory in d dimensions



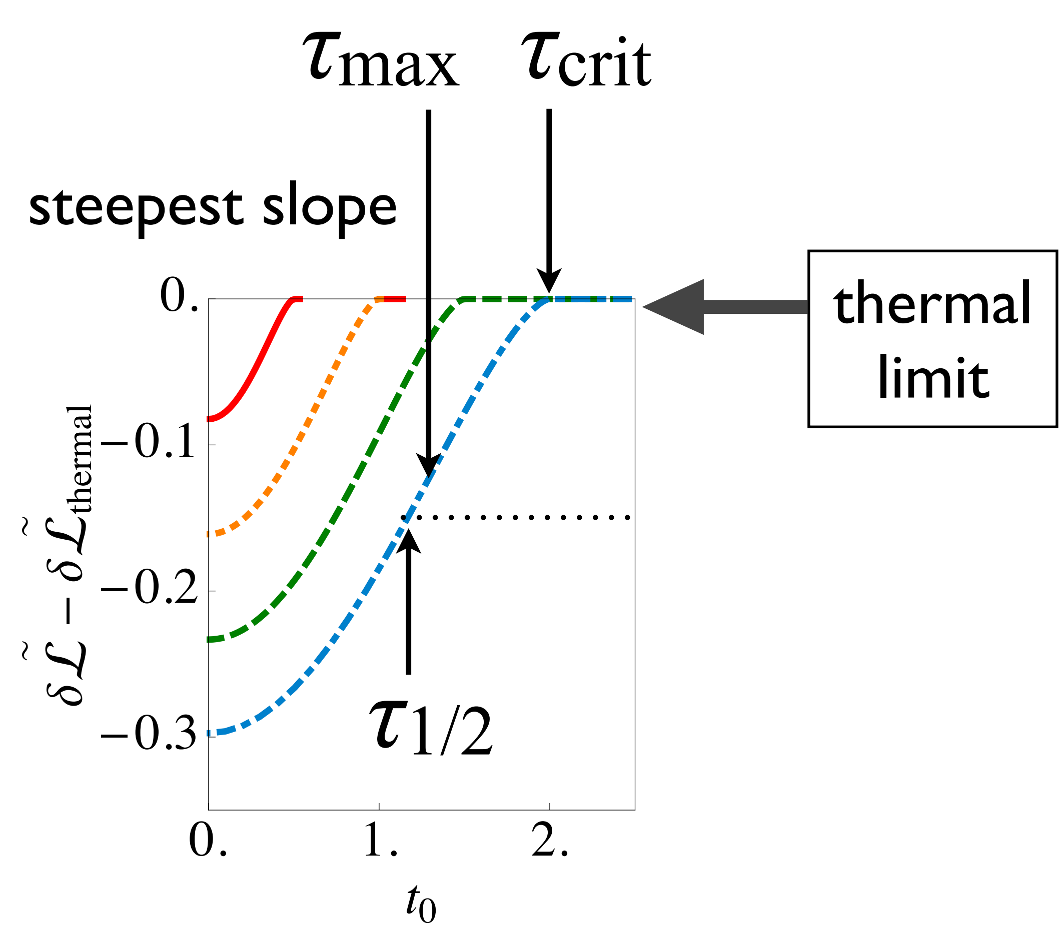
Probing thermalization

Equal-time geodesics for fixed $t_0 = 2$ and $\ell = 3.0, 4.6, 68.2$

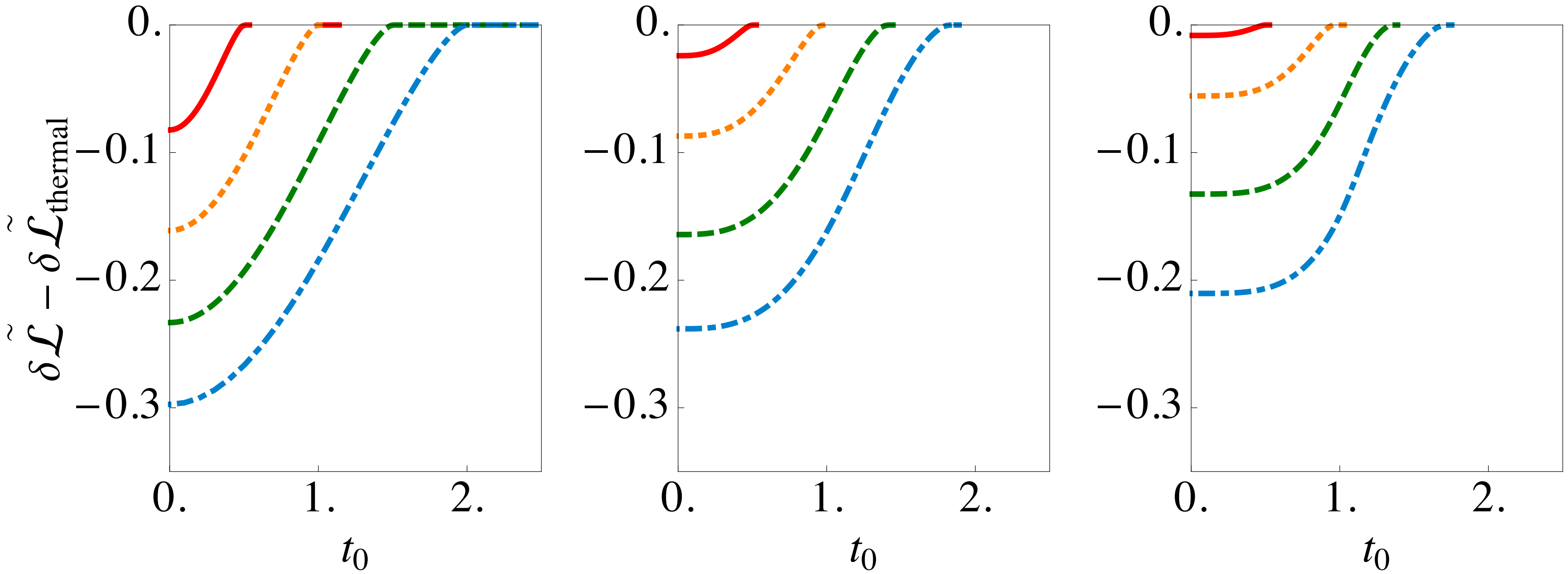


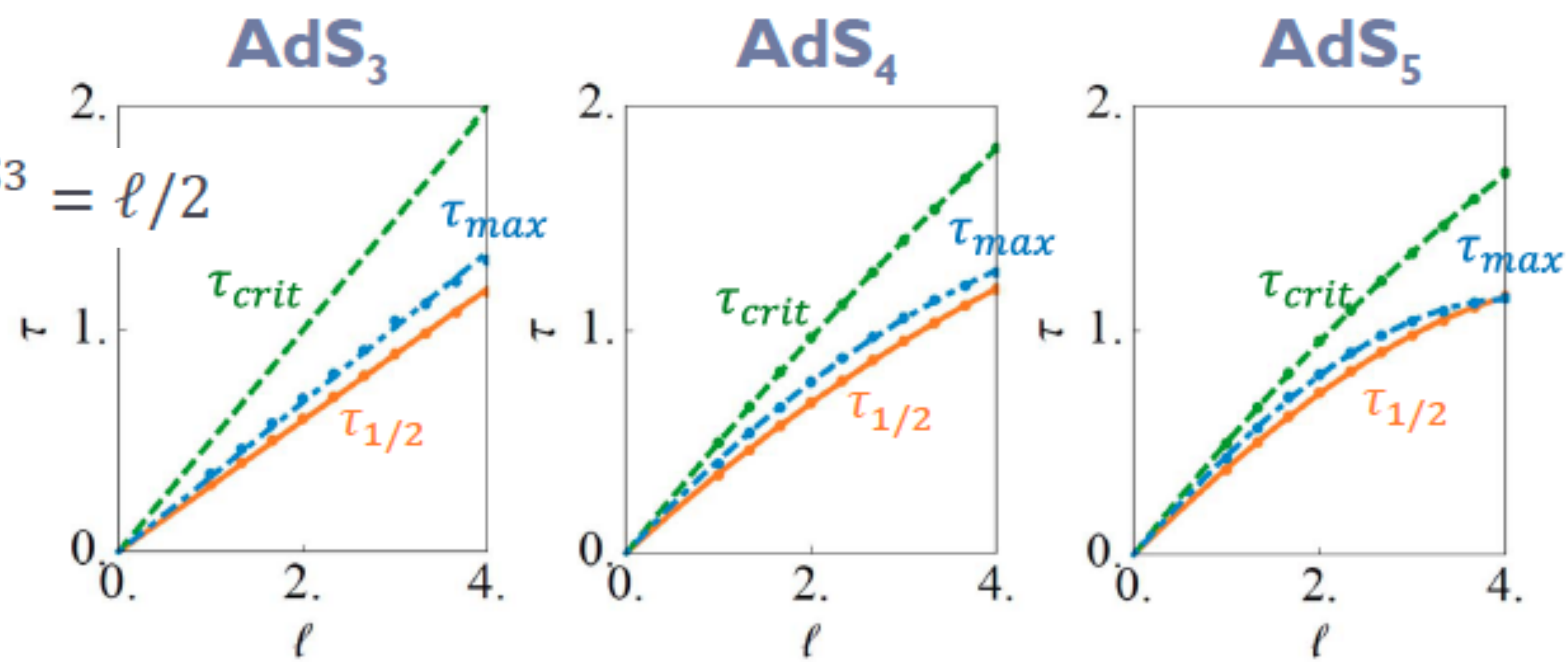
Geodesics staying outside the falling shell only probe "thermal" part of bulk space
➠ 2-point function is thermalized

$$\langle O(x)O(x')\rangle \sim \exp[-\delta\mathcal{L}] \quad \text{with} \quad \delta\mathcal{L} = \mathcal{L} - \mathcal{L}_{AdS}$$



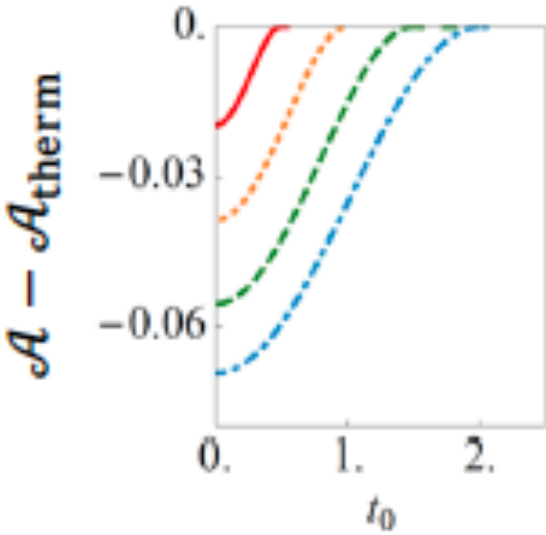
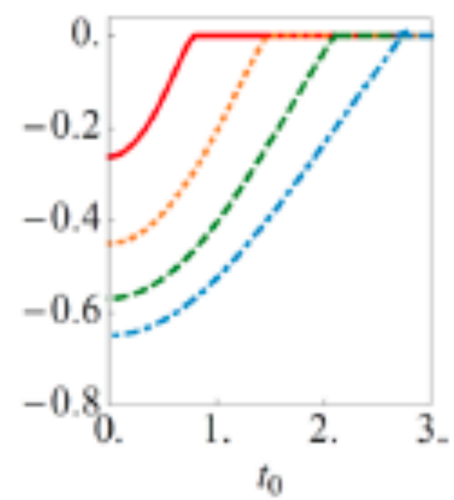
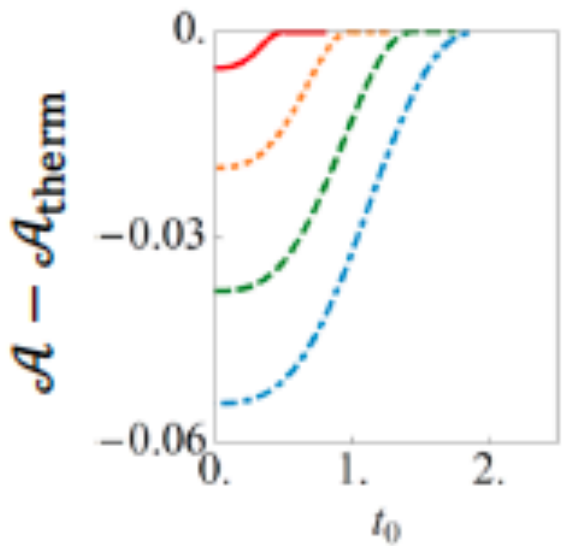
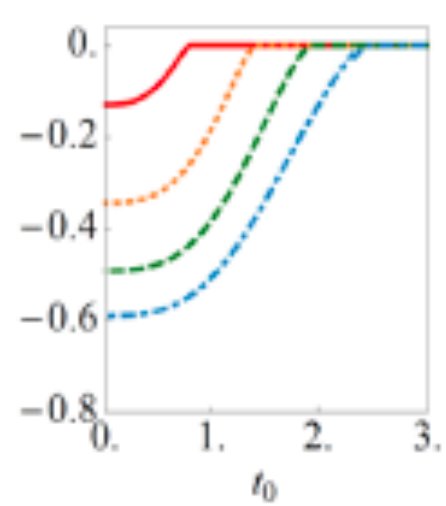
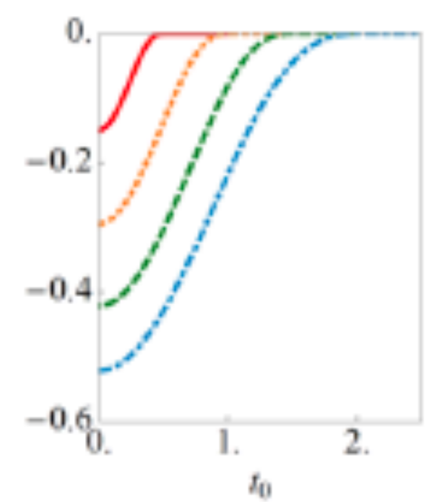
2-point functions



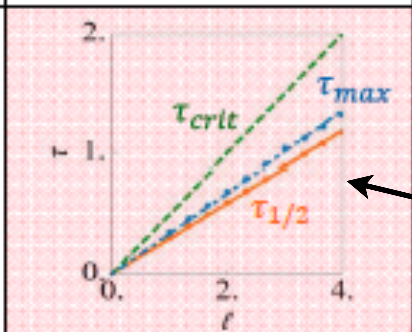
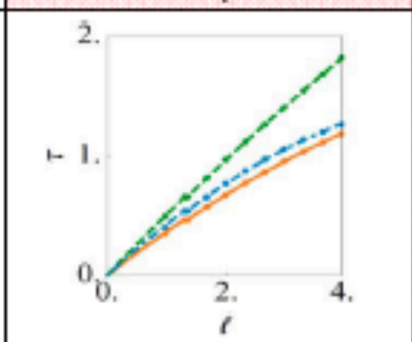
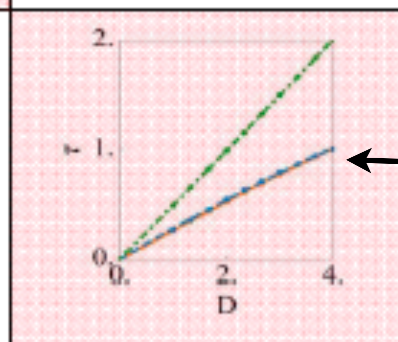
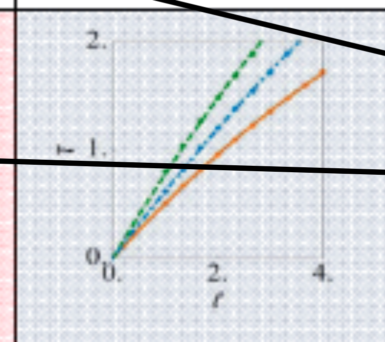
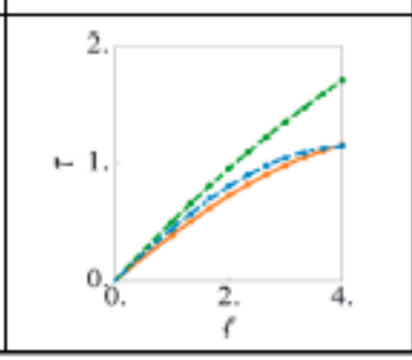
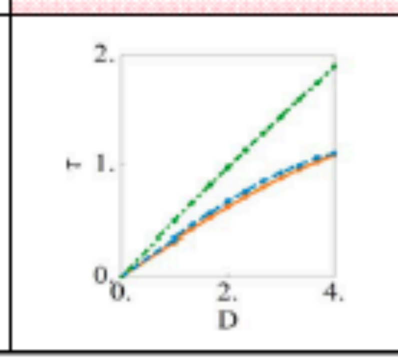
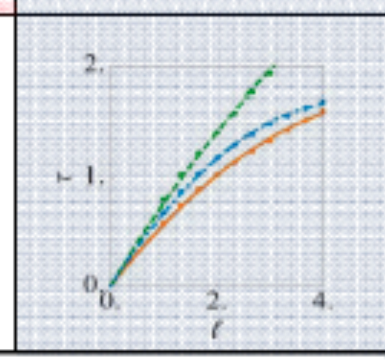
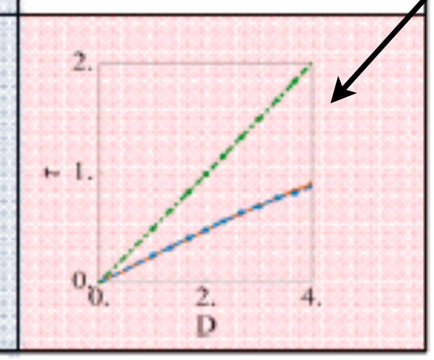


$\tau_{\text{crit}}^{\text{AdS3}} = \ell/2$

Higher dim. observables

	Circular Wilson loop	Rectangular Wilson loop	Wilson "sphere"
AdS_4			N/A
AdS_5			

Entropy thermalizes slowest

	geodesic	circular Wilson loop	Rectangular Wilson loop	EE for a sphere
AdS_3		N/A	N/A	N/A
AdS_4				N/A
AdS_5				

Entanglement entropy of spherical volume in d = 2, 3, 4

$$\tau_{\mathrm{crit}} = \ell/2$$

Thermalization time for entanglement entropy
= time for light to escape from the center of the volume to the surface

All other observables thermalize even faster.

Conclusions

- Long-distance observables sensitive to IR modes take longer to thermalize
  - Top-down rather than bottom-up thermalization
- Entropy is the last observable to reach thermal value
- Thermalization proceeds as fast as constrained by causality i.e. at the speed of light
  - True for homogeneous energy injection; speed of sound will govern equilibration of spatial inhomogeneities

(Very crude) phenomenology:

$$\tau_{crit} \sim 0.5\ \hbar/T \approx 0.3\ \text{fm/c} \quad \text{for} \quad T = 300 - 400\ \text{MeV}$$

The End !