Tree graphs with causal structure

Piotr Białas

Faculty of Physics, Astronomy and Computer Science, Jagellonian University

Eleventh Workshop on Non-Perturbative QCD
l'Institute d'Astrophysique de Paris
10 June 2011

Z. Burda, J. Jurkiewicz, A. Krzywicki

Plan

- Random graphs
- Growing networks
- Causal trees

Ensemble of random graphs

- The graph G is chosen from some ensemble $\mathcal{G}$ with probability $P(G)$.
- We can calculate various (average) properties of the distribution $P(G)$.

$$\langle O \rangle = \sum_{G \in \mathcal{G}} O(G) P(G)$$

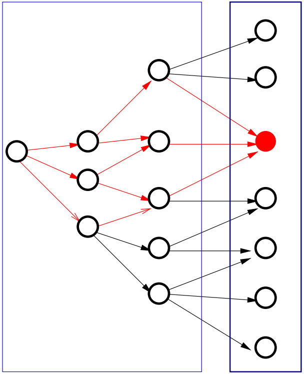
  - Degree distribution
  - Diameter

Graph probability $P(G)$ can be given

- explicitly
- or implicitly by *eg* specifying the construction process (growing networks)

Growth process

- What kind of ensemble is generated by a growth proces?
- What is the probability for a given given graph to occur?



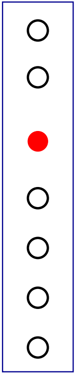
$$P(\text{final}) = \sum_{\text{paths}} P(\text{path})$$

$$= Z^{-1} \frac{1}{S(T)} \rho(T)$$

What are symmetry factors $S(T)$ and weights ρ ?

Growth process

- What kind of ensemble is generated by a growth proces?
- What is the probability for a given given graph to occur?



$$P(\text{final}) = Z^{-1}\frac{1}{S(T)}\rho(T)$$

What are symmetry factors $S(T)$ and weights ρ ?

Labeled graphs

- Usually we deal with labeled graphs *ie* with distinguishable vertices.
- A graph with N vertices can be labeld in $N!$ ways (but some of them are indistinguishable). As in Boltzman statistics we factor this out:
- $$P(G) = \frac{1}{N(G)!}\frac{\rho(G)}{Z}, \qquad Z = \sum_{G \in \mathcal{G}} \frac{1}{N(G)!}\rho(G)$$

Unlabeled graphs

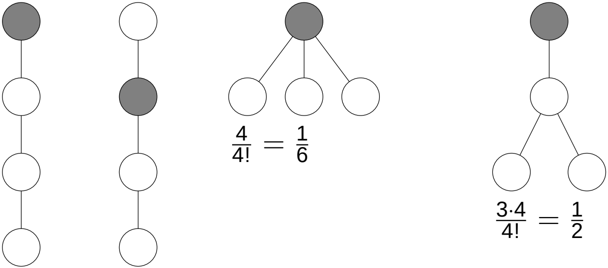
- We assume that $\rho(G)$ does not depend on the labeling

$$Z = \sum_{G \in \mathcal{G}_{labelled}} \frac{1}{N(G)!} \rho(G) = \sum_{G \in \mathcal{G}_{unlabelled}} \frac{\overbrace{L(G)}^{\frac{1}{S(T)}}}{N(G)!} \rho(G)$$

- $L(G)$ is the number of distinct labelings of graph G.
- symetry factors $1/S(T)$ – kinematics, “phase space”

  weights $\rho(T)$ – dynamics

Labelings (rooted graphs)



$\frac{4!}{4!} = 1$ $\quad$ $\frac{4!}{4!} = 1$ $\quad$ $\frac{4}{4!} = \frac{1}{6}$ $\quad$ $\frac{3 \cdot 4}{4!} = \frac{1}{2}$

Random Graphs Ensemble

To specify the random graphs ensemble explicitly we need:

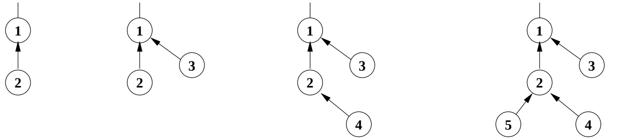
- Ensemble
- Symmetry factors (equality)
- Weights

Simplest weight is of the form (factorizes):

$$\rho(G) = \prod_{i \in G} \rho_{q_i}$$

Growing random networks

- We start with a single vertex
- At each step we attach a new vertex.
- The father is choosed with probability proportional to A_q where q is its degree *ie* the number of links attached to it.
- A_q is called an attachment kernel.

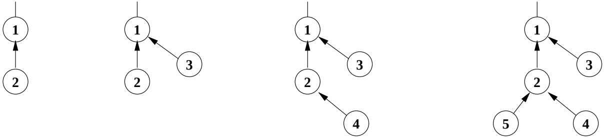


$$\frac{A_1}{A_1} \qquad \frac{A_2}{A_1+A_2} \qquad \frac{A_1}{2A_1+A_2} \qquad \frac{A_2}{A_3+A_2+2A_1}$$

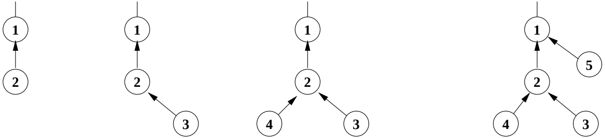
L. Barabasi, R. Albert, Science **286**, 509 (1999)

P.L. Krapivsky, S. Redner, Phys. Rev. **E62**, 66123, (2001)

Growing random networks



$$\frac{A_1}{A_1} \qquad \frac{A_2}{A_1+A_2} \qquad \frac{A_1}{2A_1+A_2} \qquad \frac{A_2}{A_3+A_2+2A_1}$$



$$\frac{A_1}{A_1} \qquad \frac{A_1}{A_1+A_2} \qquad \frac{A_2}{A_1+2A_2} \qquad \frac{A_2}{A_3+A_2+2A_1}$$

Attachment probability depends on labelings via normalisation

Mathew effect

The simple random model of graphs (ER) predicted Poisonian distribution.
The appearance of the power law in degree distribution can be explained by the preferential attachment rule.

> *For unto every one that hath shall be given, and he shall have abundance: but from him that hath not shall be taken away even that which he hath.*

Mt25:29, The Bible Authorised King James Version

H.A. Simon, Biometrica **42**, 425 (1995)

Preferential attachment

$$A_q = q + \omega, \quad \omega > -1$$

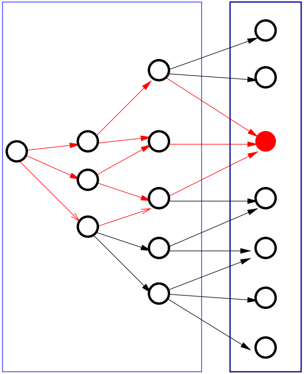
$$\pi_q = (2+\omega)\frac{\Gamma(3+2\omega)}{\Gamma(1+\omega)}\frac{\Gamma(q+\omega)}{\Gamma(q+3+2\omega)} \sim q^{-(3+\omega)}$$

In this case the norm does not depend on the tree:

$$\sum_i A_{q_i} = 2n + \omega n$$

Growth process

- What kind of ensemble is generated by a growth proces?
- What is the probability for a given given graph to occur?



$$P(\text{final}) = \sum_{\text{paths}} P(\text{path})$$
$$= Z^{-1} \frac{1}{S(T)} \rho(T)$$

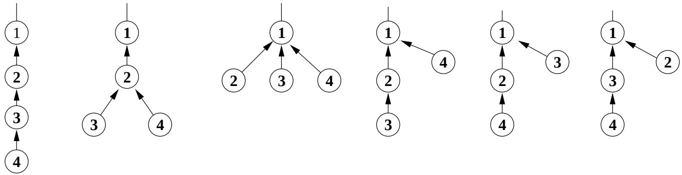
What are symmetry factors $S(T)$ and weights ρ ?

Causal trees

Can we describe the final ensemble?

- Labels represent “time of birth”.
  The label of the child must be greater then label of its father
- Causal labelings (increasing trees)

1 (24) 1 (12) 1 (4) 3 (24)



Causal trees – weights

- For general attachment kernel weights depend on the labeling
- For affine kernels probability of constructing a vertex with degree q is the probability of constructing the vertex of degree $q-1$ and attaching a new branch to it:

$$\rho_q = \rho_1 \prod_{k=1}^{q-1} A_k , \qquad \rho(T) = \prod_{i \in T} \rho_{q_i}$$

This gives model identical to GRN

Causal trees

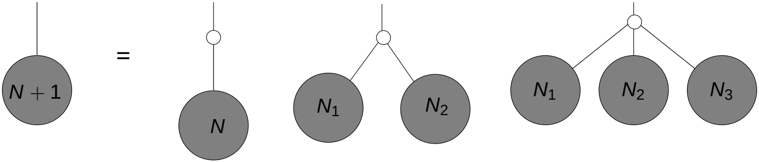
- Weights do not depend on the labels and factorise

$$\rho(T) = \prod_{i \in \text{vertices}} \rho_{q_i}$$

  ρ_q are arbitrary non-negative numbers. q_i is a degree of vertex i.

- How do the properties of trees with causal labelings differ form arbitraly labeled trees?

Partition function – trees

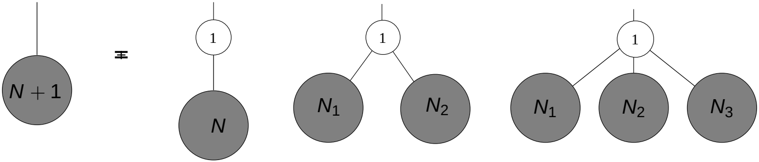


$$z_{N+1} = \frac{1}{(N+1)!} \sum_{T \in \mathcal{T}_{N+1}} \rho(T) = \frac{1}{(N+1)!} \sum_{k=1}^{\infty} \sum_{T_1,\ldots,T_k} \delta_{N_1+\cdots+N_k,N}$$

$$\frac{1}{k!} \boxed{\frac{(N+1)!}{N_1! \cdots N_k!}} \rho_{k+1} \prod_{i=1}^{k} \rho(T_i)$$

number of ways of distributing labels between trees

Partition function – trees – causal



$$z_{N+1} = \frac{1}{(N+1)!} \sum_{T \in \mathcal{T}_{N+1}} \rho(T) = \frac{1}{(N+1)!} \sum_{k=1}^{\infty} \sum_{T_1,\ldots,T_k} \delta_{N_1+\cdots+N_k,N}$$

$$\frac{1}{k!} \boxed{\frac{N!}{N_1! \cdots N_k!}} \rho_{k+1} \prod_{i=1}^{k} \rho(T_i)$$

number of ways of distributing labels between causal trees

Generating function (grand–canonical ensemble)

$$Z(\mu) = \sum_{N=1}^{\infty} e^{-\mu N} z_N \qquad \text{(Laplace transform)}$$

$$\boxed{Z(\mu) = e^{-\mu} \frac{1}{Z(\mu)} F(Z(\mu))}$$

$$F(Z) = \sum_{q=1}^{\infty} \rho_q Z^q$$

$$e^{\mu} = G(Z) \equiv \frac{F(Z)}{Z^2}$$

Generating function (grand–canonical ensemble)

$$Z(\mu) = \sum_{N=1}^{\infty} e^{-\mu N} z_N \qquad \text{(Laplace transform)}$$

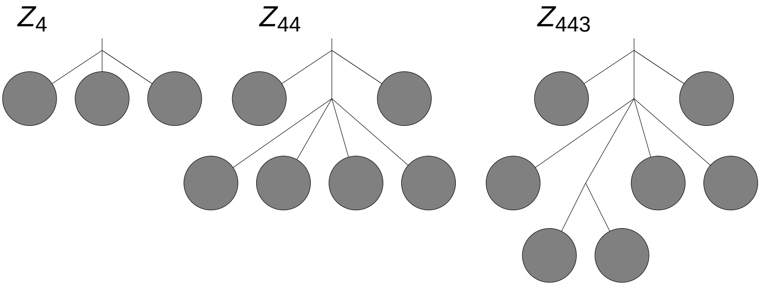
$$\boxed{Z'(\mu) = -e^{-\mu} \frac{1}{Z(\mu)} F(Z(\mu))}$$

$$F(Z) = \sum_{q=1}^{\infty} \rho_q Z^q$$

$$e^{-\mu} = G(Z) \equiv \int_0^Z \frac{x\mathrm{d}x}{F(x)}, \quad \bar{\mu} = -\log G(\overline{Z})$$

where $\overline{Z}$ is the radius of convergence of series $F(Z)$.

Correlations



$$G(r,\mu) \leq \bar{x}\,\frac{(\log F(Z(\mu)))^{r-1}}{(r-1)!}$$

$$\langle r\rangle_\mu \leq \text{const} \log \frac{1}{\Delta\mu} \qquad \langle r\rangle_N \leq \text{const} \log N$$

Weight of causality

$$z_N = \frac{1}{N!} \sum_{T \in labeled} \rho(T)$$

$$Z_N = \frac{1}{N!} \sum_{T \in causal} \rho(T) = \frac{1}{N!} \sum_{T \in labeled} w(T)\rho(T)\ ?$$

$$w(T) = \frac{\#\text{causal labelings}}{\#\text{all labelings}}$$

Weight of causality

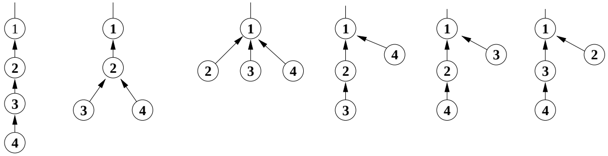
$$Z_{N+1} = \frac{1}{N+1} \sum_{k=1}^{\infty} \rho_{k+1} \sum_{N_1,\ldots,N_k} \delta_{N_1+\cdots+N_k,N} \prod_{i=1}^{k} Z_{N_i}$$

$$z_{N+1} = \sum_{k=1}^{\infty} \rho_{k+1} \sum_{N_1,\ldots,N_k} \delta_{N_1+\cdots+N_k,N} \prod_{i=1}^{k} z_{N_i}$$

$$w(T) = \prod_{\text{subtrees}} \frac{1}{\text{size of the subtree}}$$

$$= \prod_{i \in T} \frac{1}{\#\text{ of descendants of } i+1}$$

1 (24) 1 (12) 1 (4) 3 (24)



Weight of causality

Causal labeling supresses elongated trees

- Linear chain

$$w(T) = \frac{1}{N!}$$

- Star

$$w(T) = \frac{1}{N}$$

Summary

- One can embed growth information into random graphs ensemble using causal labelings.
- If we assume that the weights factorise the model can be solved for tree graphs.
- Generic causal random tree has an infinite Hausdorff dimension.

Summary

- Causal labelings on rooted trees can be reproduced by non–local weights

$$w(T) = \prod_{\text{subtrees}} \frac{1}{\text{size of the subtree}}$$

$$= \prod_{i \in T} \frac{1}{\#\text{ of descendants of } i + 1}$$