11th Workshop on Non-Perturbative QCD

Paris, IAP, 6-10 June 2011

The gauge invariant quark Green's function in QCD2

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Gauge invariant Green's functions

Gauge invariant Green's functions are expected to provide more reliable informations about the physical properties of observables than the gauge variant ones.

For quarks, the gauge invariant two-point Green's function is defined as

$$S_{\alpha\beta}(x, x'; C_{x'x}) = -\frac{1}{N_c} \langle \overline{\psi}_\beta(x')\, U(C_{x'x}; x', x)\, \psi_\alpha(x) \rangle,$$

where U is a path-ordered gluon field phase factor along a line $C_{x'x}$ joining a point x to a point x':

$$U(C_{x'x}; x', x) \equiv U(x', x) = Pe^{-ig \int_x^{x'} dz^\mu A_\mu(z)}.$$

Green's functions with paths along skew-polygonal lines are of particular interest. Straight line segments have Lorentz invariant forms.

For skew-polygonal lines with n sides and $n-1$ junction points y_1, y_2, ...,y_{n-1} between the segments, we define:

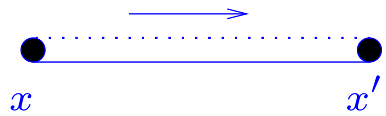
$$S_{(n)}(x, x'; y_{n-1}, \ldots, y_1) = -\frac{1}{N_c} \langle \overline{\psi}(x') U(x', y_{n-1}) \ldots U(y_1, x) \psi(x) \rangle,$$

where each U is along a straight line segment.

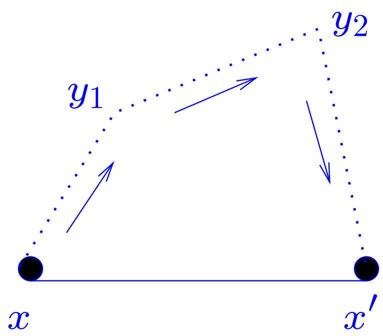
For one straight line, one has:

$$S_{(1)}(x, x') \equiv S(x, x') = -\frac{1}{N_c} \langle \overline{\psi}(x')\, U(x', x)\, \psi(x) \rangle.$$

Pictorially:



$$S(x,x') \equiv S_{(1)}(x,x') = -\frac{1}{N_c} < \overline{\psi}(x')\, U(x',x)\, \psi(x) >$$



$$S_{(3)}(x,x';y_2,y_1) = -\frac{1}{N_c} < \overline{\psi}(x')\, U(x',y_2) U(y_2,y_1) U(y_1,x)\, \psi(x) >$$

Quark propagator in the external gluon field

A two-step quantization. One first integrates with respect to the quark fields. This produces in various terms the quark propagator in the presence of the gluon field. Then one integrates with respect to the gluon field through Wilson loops.

We use for the quark propagator in extenal field a representation which involves phase factors along straight lines together with the full quark Green's function. Generalization of a representation introduced by Eichten and Feinberg, 1981, for heavy quarks.

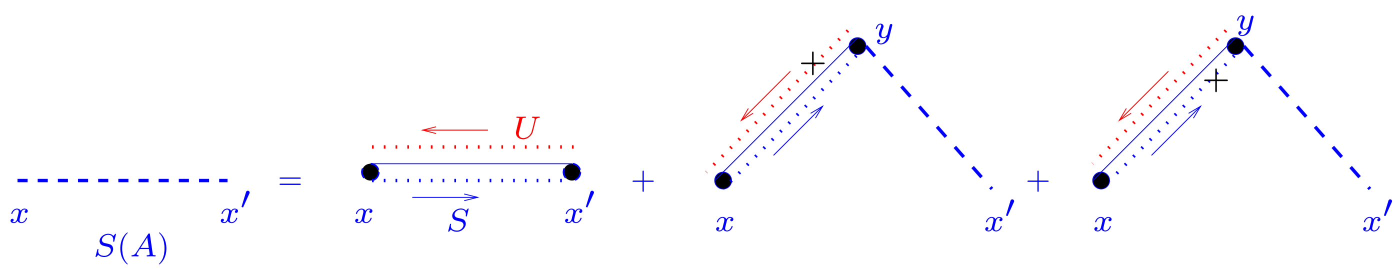
The quark propagator in the external gluon field is expanded around the following gauge covariant quantity:

$$S(x,x')\Big[U(x,x')\Big]^{a}_{\ b}.$$

[$S(x,x')$ is the gauge invariant Green's function along one straight line segment.]

$$S(x,x';A) = S(x,x')U(x,x') + \left(S(x,y)\frac{\bar{\delta}U(x,y)}{\bar{\delta}y^{\alpha -}} + \frac{\bar{\delta}S(x,y)}{\bar{\delta}y^{\alpha +}}U(x,y) \right)\gamma^{\alpha}S(y,x';A).$$

Pictorially:



This yields an expansion of $S(A)$ in terms of the gauge invariant Green's function S and explicit phase factors along straight lines.

Its systematic use leads to the derivation of functional relations between the Green's functions $\boldsymbol{S_{(n)}}$ (polygonal line with $\boldsymbol{n}$ segments) and $\boldsymbol{S}$ (one segment).

Using then the quark equations of motion and the functional relations between Green's functions, one establishes the following integrodifferential equation for the Green's function $S(x,x')$:

$$(i\gamma.\partial_{(x)}-m)S(x,x') = i\delta^4(x-x')+i\gamma^\mu\Big\{K_{2\mu}(x',x,y_1)\,S_{(2)}(y_1,x';x)$$

$$+\sum_{n=3}^{\infty}K_{n\mu}(x',x,y_1,\ldots,y_{n-1})\,S_{(n)}(y_{n-1},x';x,y_1,\ldots,y_{n-2})\Big\},$$

where the kernel K_n contains globally n derivatives of Wilson loops with a $(n+1)$-sided skew-polygonal contour and also the Green's function S and its derivative.

The Green's functions $S_{(n)}$ themselves are related to the simplest Green's function S with functional relations.

Interest of the quark gauge invariant Green's function

Interest related to its particular status.

If the theory is confining, it is not possible to cut the Green's function and to saturate it with a complete set of physical states (hadrons). Intermediate states are necessarily colored states.

$$\langle\overline{\psi}(x')\,U(x',x)\,\psi(x)\rangle.$$

This would suggest that the Green's function does not have singularities.

However, the equation that it satisfies, derived from the QCD Lagrangian, contains singularities generated by the free quark propagator.

This paradoxical situation is overcome with the acceptance that the quarks and gluons continue forming a complete set of states with positive energies and could be used for any saturation scheme of intermediate states. It is up to the theory to indicate to us at the end how the related singularities combine to form the complete solutions.

Therefore, the knowledge of the gauge invariant quark Green's function provides us a direct information about the effect of confinement in the colored sector of quarks.

Spectral functions

Green's functions with paths along straight lines are dependent only on the end points of the paths. The transition is then simple to momentum space by Fourier transformation.

Using then the spectral analysis with intermediate states and causality, one arrives at a generalized form of the Källén–Lehmann representation for the Green's function S in momentum space, in which the cut starts on the real axis from the quark mass squared m^2 and extends to infinity.

$$S(x, x') = S(x - x') = \int \frac{d^4p}{(2\pi)^4} \, e^{-ip.(x - x')} \, S(p).$$

$S(p)$ has the following representation in terms of real spectral functions $\rho_1^{(n)}$ and $\rho_0^{(n)}$ $(n = 1, \ldots, \infty)$:

$$S(p) = i \int_0^\infty ds' \sum_{n=1}^{\infty} \frac{\left[\, \gamma.p \, \rho_1^{(n)}(s') + \rho_0^{(n)}(s') \,\right]}{(p^2 - s' + i\varepsilon)^n}.$$

Two-dimensional QCD

Many simplifications in two-dimensional QCD at large N_c. In two dimensions, Wilson loop averages are exponential functionals of the areas of the surfaces enclosed by the contours. At large N_c, crossed diagrams and quark loop contributions disappear. ('t Hooft, 1974.)

Equation of S with the lowest-order kernel becomes an exact equation. In two dimensions, the second-order derivative of the logarithm of the Wilson loop average is a delta-function.

$$(i\gamma.\partial - m)S(x) = i\delta^2(x) - \sigma\gamma^\mu(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha})x^\nu x^\beta$$

$$\times\Big[\int_0^1 d\lambda\lambda^2 S((1-\lambda)x)\gamma^\alpha S(\lambda x) + \int_1^\infty d\xi S((1-\xi)x)\gamma^\alpha S(\xi x)\Big].$$

$$S(p) = \gamma.pF_1(p^2) + F_0(p^2).$$

$$S(x) = \frac{1}{2\pi}\Big(\frac{i\gamma.x}{r}\widetilde{F}_1(r) + \widetilde{F}_0(r)\Big), \quad r = \sqrt{-x^2}.$$

One obtains two coupled equations. Their resolution proceeds through several steps, based mainly on the spectral representation and the related analyticity properties.

The equations can be solved explicitly.

The covariant functions $F_1(p^2)$ and $F_0(p^2)$ are:

$$F_1(p^2) = -i\frac{\pi}{2\sigma}\sum_{n=1}^{\infty} b_n \frac{1}{(M_n^2 - p^2)^{3/2}},$$

$$F_0(p^2) = i\frac{\pi}{2\sigma}\sum_{n=1}^{\infty}(-1)^n b_n \frac{M_n}{(M_n^2 - p^2)^{3/2}}.$$

The threshold singularities or branch points $M_1^2,\ M_2^2, \ldots,\ M_n^2, \ldots$ are labelled with increasing values with respect to the index n; in particular $M_1 > m$.

For large n:

$$M_n^2 \simeq \sigma\pi n, \quad b_n \simeq \frac{\sigma^2}{M_n}, \quad \text{for } \sigma\pi n \gg m^2.$$

In $x-$space ($r = \sqrt{-x^2}$):

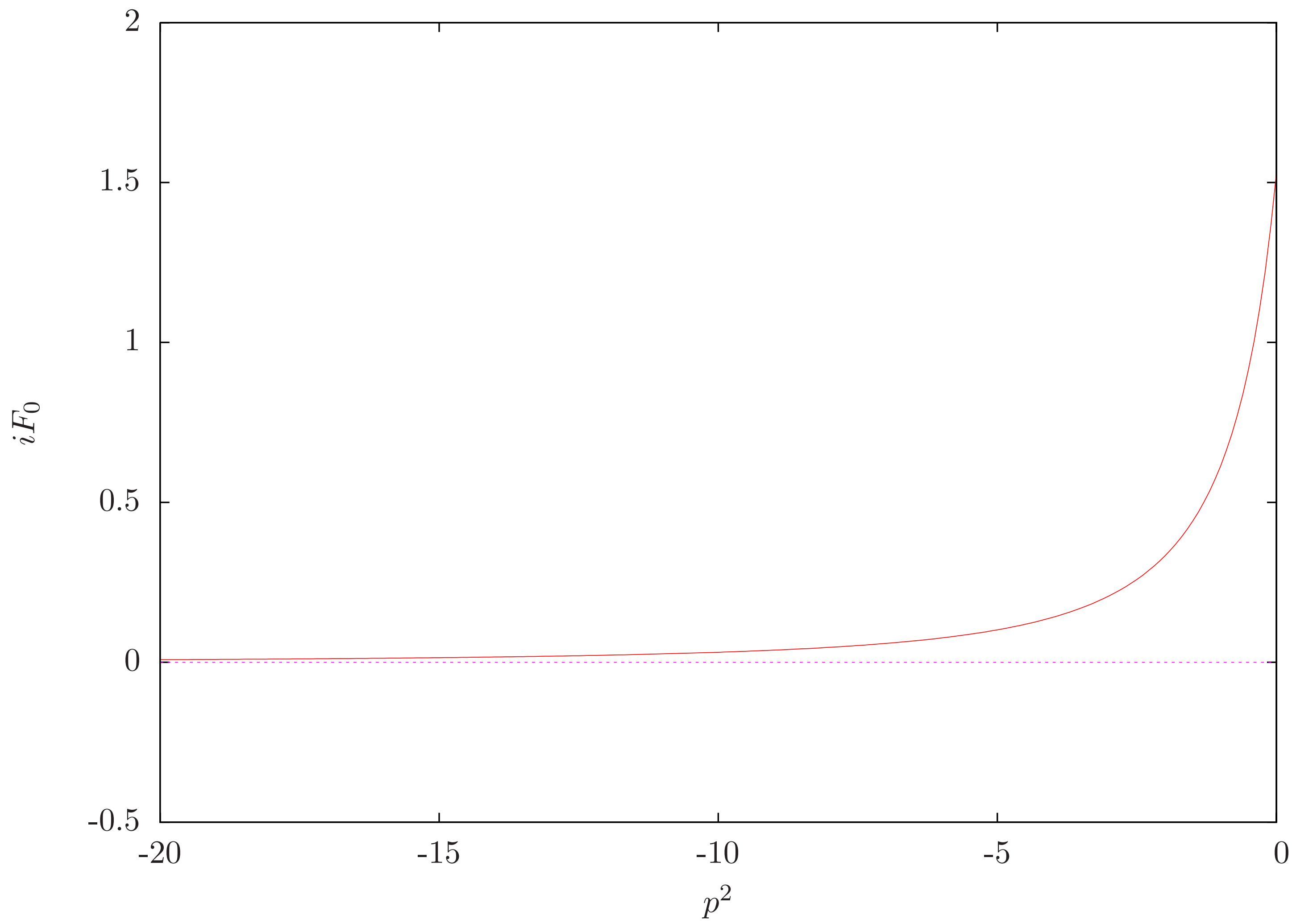
$$\widetilde{F}_1(r) = \frac{\pi}{2\sigma}\sum_{n=1}^{\infty} b_n e^{-M_n r}, \quad \widetilde{F}_0(r) = \frac{\pi}{2\sigma}\sum_{n=1}^{\infty}(-1)^{n+1} b_n e^{-M_n r}.$$

Asymptotic behaviors:

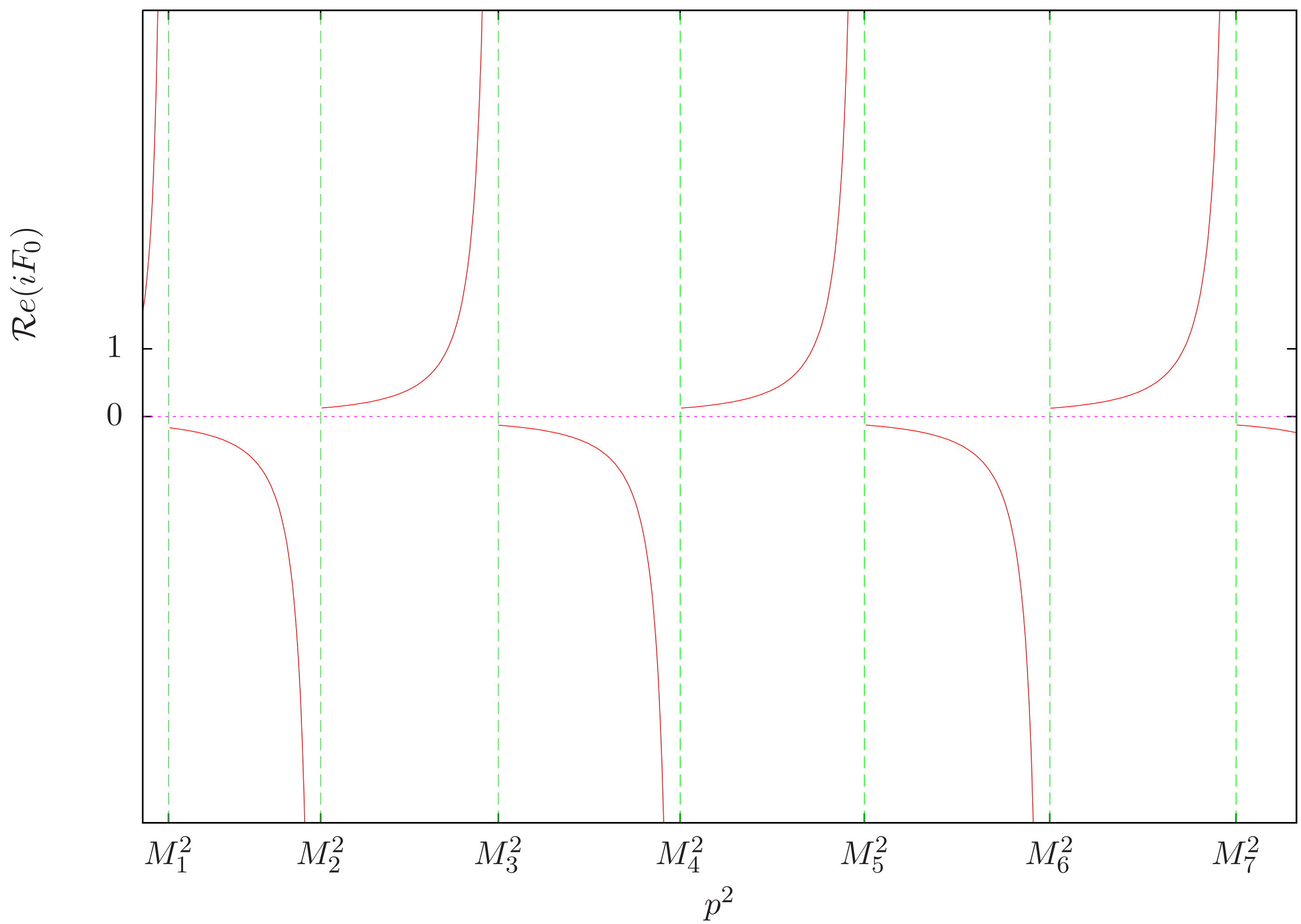
$$F_1(p^2) \underset{|p^2|\to\infty}{=} \frac{i}{p^2},$$

$$F_0(p^2) \underset{|p^2|\to\infty}{=} \frac{im}{p^2}, \qquad m \neq 0,$$

$$F_0(p^2) \underset{|p^2|\to\infty}{=} \frac{2i\sigma}{N_c} \frac{\langle \overline{\psi}\psi \rangle}{(p^2)^2}, \qquad m = 0.$$



2
1.5
1
0.5
0
-0.5
iF_0
-20
-15
-10
-5
0
p^2



$\mathcal{R}e(iF_0)$
1
0
M_1^2
M_2^2
M_3^2
M_4^2
M_5^2
M_6^2
M_7^2
p^2

Conclusion

1) The spectral functions are infrared finite and lie on the positive real axis of p^2. No singularities in the complex plane or on the negative real axis have been found. $\Longrightarrow$ Quarks contribute with positive energies.

2) The singularities are represented by an infinite number of threshold type singularities, characterized by positive masses M_n ($n = 1, 2, \ldots$). The corresponding singularities are stronger than simple poles and this feature might prevent observability of quarks as free particles.

3) The threshold masses M_n represent dynamically generated masses and maintain the scalar part of the Green's function at a nonzero value.