On the QCD Effective Locality property

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QCD generating functional

$$\mathcal{Z}[j,\eta,\bar\eta] = e^{\frac{i}{2}\int j\mathbf{D}_c(\zeta)j}\,\mathcal{N}\int \mathrm{d}[\chi]\; e^{\frac{i}{4}\int\chi^2}\; e^{\mathfrak{D}_A}\; e^{\frac{i}{2}\int\chi\mathbf{F}+\frac{i}{2}\int A\mathbf{D}_c^{-1}(\zeta)A} \times e^{i\int\bar\eta\mathbf{G}_c[A]\eta}\; e^{\mathbf{L}[A]}\Big|_{A=\int\mathbf{D}_c(\zeta)j}$$

whith '*Linkage Operator*'

$$\mathfrak{D}_A = -\frac{i}{2}\int \frac{\delta}{\delta A^a_\mu}\,\mathbf{D}_c^{(\zeta)}\Big|^{ab}_{\mu\nu}\frac{\delta}{\delta A^b_\nu}$$

and

$$\mathbf{D}_c^{(\zeta)}\Big|^{ab}_{\mu\nu} = \delta^{ab}(-\partial^2)^{-1}\left[g_{\mu\nu}-(1-\zeta)\frac{\partial_\mu\partial_\nu}{\partial^2}\right]$$

In order to display Effective Locality ..

.. Schwinger-Fradkin's representations are used

$$\langle p|\mathbf{G}_c[A]|y\rangle = e^{-ip\cdot y}\, i\int_0^\infty ds\, e^{-ism^2}\, e^{-\frac{1}{2}\, Tr\ln(2h)}$$
$$\times\int d[u]\{m - i\gamma\cdot[p - gA(y-u(s))]\}\, e^{\frac{i}{4}\int_0^s ds'\,[u'(s')]^2}\, e^{ip\cdot u(s)}$$
$$\times\boxed{\left(e^{g\int_0^s ds'\sigma\cdot F(y-u(s'))}\, e^{-ig\int_0^s ds'\, u'(s')\cdot A(y-u(s'))}\right)_+}$$

with Fradkin's (classical) fields

$$u_\mu(s) = \int_0^s \mathrm{d}s'\, v_\mu(s')\,, \quad u_\mu(0) = 0$$

As a typical and most important part of a 4-point fermionic function, one gets in an exponential the argument ..

$$
\begin{aligned}
&+\frac{i}{2}g\int \mathrm{d}^4 w_1 \int_0^{s} \mathrm{d}s_1 \int_0^{\bar{s}} \mathrm{d}s_2\; u'_\mu(s_1)\,\bar{u}'_\nu(s_2) \\
&\times \Omega^a(s_1)\,\bar{\Omega}^b(s_2)\;\left(f\cdot\chi(w_1)\right)^{-1}\Big|^{\mu\nu}_{ab} \\
&\times \delta^{(4)}(w_1 - y_1 + u(s_1))\,\delta^{(4)}(w_1 - y_2 + \bar{u}(s_2))
\end{aligned}
\qquad (1)
$$

as the generic structure.

But how to think of ..

$$\delta^{(4)}(w_1 - y_1 + u(s_1))\,\delta^{(4)}(y_1 - y_2 + \bar{u}(s_2) - u(s_1)) \qquad (2)$$

That is, basically, how should we interpret

$$\delta^{(4)}(\bar{u}(s_2) - u(s_1)) \qquad ? \qquad (3)$$

A heuristic manipulation suggests that ..

$$\delta(\bar{u}_0(s_2) - u_0(s_1))\delta(\bar{u}_{\mathrm{L}}(s_2) - u_{\mathrm{L}}(s_1)) = \frac{\delta(s_1)\delta(s_2)}{|u'_{\mathrm{L}}(0)||\bar{u}'_0(0)|} \quad (4)$$

assuming $\mathcal{C}^1\left([0,s],[0,\bar{s}] \to \mathbb{R}^4\right)$ -$u, \bar{u}$ Fradkin's fields, and leading to the remaining (out of $\delta^{(4)}(\bar{u}(s_2) - u(s_1))$)

$$\delta^{(2)}(\vec{y}_{1\perp} - \vec{y}_{2\perp}) = \delta^{(2)}(\vec{b})$$

where $\vec{b}$ is the impact parameter, or transverse distance between the two scattering quarks.

A physical stake .. (H.M. Fried's talk)

the necessary change of $\delta^{(2)}(\vec{b})$ into a transverse 'Levy-flight' distribution for bound quarks

$$\delta^{(2)}(\vec{b}) \to \varphi(b) = \frac{\mu^2}{\pi} \frac{1+\xi/2}{\Gamma(\frac{1}{1+\xi/2})} e^{-(\mu b)^{2+\xi}}, \quad \xi \ll 1$$

.. so that .. starting from quark propagation as ordinarily (perturbatively) conceived, *à la* $G_c(x,y|A)$, in the non-perturbative bound context, one could be lead to think of quark propagation in terms of Levy-flights !

But what mathematicians say ? ..

- ‘*That it is crazy!*’

- What do physicists reply?

- ‘‘*That it works! The equation*

$$\delta(\bar{u}_0(s_2) - u_0(s_1))\delta(\bar{u}_{\mathrm{L}}(s_2) - u_{\mathrm{L}}(s_1)) = \frac{\delta(s_1)\delta(s_2)}{|u'_{\mathrm{L}}(0)||\bar{u}'_0(0)|}$$

complies with an Eikonal approximation’ ($u(s_1) \longrightarrow s_1 p_1$, $\bar{u}(s_2) \longrightarrow s_2 p_2$, ..)
And so mathematicians skip to the most achieved realization of a functional space they master:

The Wiener functional space

Theorem

For all couple $(s_1, s_2) \in]0, s] \times]0, \bar{s}]$,

$$m \otimes m \left(\{ (u, \bar{u}) \in \mathcal{C}_0^{0,s} \times \mathcal{C}_0^{0,\bar{s}} \mid u(s_1) = \bar{u}(s_2) \} \right) = 0$$

$$m \otimes m \left(\{ (u, \bar{u}) \in \mathcal{C}_0^{0,s} \times \mathcal{C}_0^{0,\bar{s}} \mid u(0) = \bar{u}(0) = 0 \} \right) = 1$$

$\mathcal{C}_0^{0,s}$= $\mathcal{C}(]0, s], \mathbb{R}^1)$ space of continuous functions $u, \bar{u}$, from $]0, s]$ into $\mathbb{R}^1$ / $u(0) = 0$. (the c-functions $u, \bar{u}$, represent either of the 2 possibilities u_0 and u_L).

m = Wiener measure on $\mathcal{C}_0^{0,s}$

$m \otimes m$ = Wiener measure on $\mathcal{C}_0^{0,s} \times \mathcal{C}_0^{0,\bar{s}}$ endowed with the topology product ($u, \bar{u}$ independent).

Proof ..

Let A be the set $\left\{(u,\bar{u}) \in \mathcal{C}_0^{0,s} \times \mathcal{C}_0^{0,\bar{s}} \mid u(s_1) = \bar{u}(s_2)\right\}$.

One has $A = \bigcap_{n=1}^{\infty} A_n$, where

$$A_n = \left\{(u,\bar{u}) \in \mathcal{C}_0^{0,s} \times \mathcal{C}_0^{0,\bar{s}} \mid -\frac{1}{n} \leq u(s_1) - \bar{u}(s_2) \leq +\frac{1}{n}\right\}$$

Because of the obvious inclusion,

$$\forall n, \quad A_{n+1} \subset A_n$$

one can write,

$$m \otimes m(A) = \lim_{n \to \infty} m \otimes m(A_n)$$

Proof ..

Now, $X_n \equiv m{\otimes}m(A_n)$ is given by

$$X_n = m{\otimes}m\{(u,\bar{u}) \in \mathcal{C}_0^{0,s} \times \mathcal{C}_0^{0,\bar{s}} \mid u(s_1) - \frac{1}{n} \leq \bar{u}(s_2) \leq u(s_1) + \frac{1}{n}\}$$

$$= \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{\sqrt{2\pi s_1}}\, e^{-\frac{x^2}{2s_1}} \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} \frac{\mathrm{d}y}{\sqrt{2\pi s_2}}\, e^{-\frac{y^2}{2s_2}}$$

$$\equiv \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{\sqrt{2\pi s_1}}\, e^{-\frac{x^2}{2s_1}}\, f_n(x)$$

Since

$$\left|\frac{1}{\sqrt{2\pi s_1}}\, e^{-\frac{x^2}{2s_1}}\, f_n(x)\right| \leq \frac{1}{\sqrt{2\pi s_1}}\, e^{-\frac{x^2}{2s_1}}$$

one gets $X_\infty = 0$, by the *Dominated convergence theorem*.

Equivalences

A long known abelian equivalence (∼Wick's theorem underlying structure):

$$e^{-\frac{i}{2}\int \frac{\delta}{\delta A} D_c \frac{\delta}{\delta A}}\ \mathcal{F}(A)\big|_{A=0} = N \int \mathrm{d}[\mathrm{A}]\, \mathrm{e}^{\frac{\mathrm{i}}{2}\int \mathrm{A} \mathrm{D}_c^{-1} \mathrm{A}}\, \mathcal{F}(\mathrm{A})$$

First: can be proven to extend to the non-abelian case with '*some consequences*' for non-perturbative QCD:

- EL.

- Itself, EL, confirming the intrinsic perturbative character of the QCD BRST generating functional .. as advocated by B ! http://www.scholarpedia.org/article/Becchi-Rouet-Stora-Tyutin-symmetry.

- Shedding new light on the Gribov-Singer copies problem (alternative? Work in progress ..)

Another non abelian simplification!

EL allows one to define:

The infinite dimensional functional integrations over Halpern's field $\chi^a_{\mu\nu}$ configuration space:

$$\int \mathrm{d}[\chi] = \prod_{i\in\mathcal{M}} \prod_{a=1}^{N^2-1} \prod_{0=\mu<\nu}^{3} \int \mathrm{d}[\chi^a_{\mu\nu}](w_i) \qquad (5)$$

reducing it to:

Ordinary Lebesgue integrations over finite-dimensional $\mathbb{R}^n$ spaces.

Such a reduction is not possible in general:

Here a consequence of non-perturbative EL.

Proof

$w_i \longrightarrow w_0 \equiv O$, a choosen origin of $\mathcal{M}$.

Intuitively clear, a proof can be given in Wiener space making use of <u>an adapted form</u> of '*Wiener's integration formula*' :

$$\int_{\mathcal{W}} f\left(x(s_1), \, .. \, , x(s_n)\right) dm(x) = \int_{\mathbb{R}^n} \frac{f\left(u_1, \, .., u_n\right) e^{-\sum_1^n \frac{(u_j - u_{j-1})^2}{2(s_j - s_{j-1})}}}{\sqrt{s_1 .. (s_n - s_{n-1})}} d\vec{U}_n$$

a consequence of the '*measure image theorem*'.

Further on, the $\chi^a_{\mu\nu}$ can be $SU(3)$- Lie-algebra valuated:

$$\chi^a_{\mu\nu} \longrightarrow \sum_{a=1}^{N^2-1} \chi^a_{\mu\nu} T^a$$

the full power of '*Random Matrix*' can be used, yielding ..

A short sequence of fruitful equalities ..

$$e^{-\frac{i}{2}\int \frac{\delta}{\delta A} D_c \frac{\delta}{\delta A}}\ \mathcal{F}(A)\Big|_{A=0} \overset{A\to NA}{=} N \int \mathrm{d}[\mathrm{A}]\, \mathrm{e}^{\frac{\mathrm{i}}{2}\int \mathrm{A}\mathrm{D}_{\mathrm{c}}^{-1}\mathrm{A}}\ \mathcal{F}(\mathrm{A})$$

$$\overset{EL}{=} N' \int_{\mathcal{W}^n} \prod \mathrm{d}\,[\chi^a_{\mu\nu}(w_0)]\, e^{\frac{i}{4}\Sigma(\chi^a_{\mu\nu}(w_0))^2}\ \mathcal{G}(\chi$$

$$= N'' \int_{\mathcal{M}_{n'}(\mathbb{C})} \mathrm{d}[H]\,\delta(Tr\,H)\ e^{\frac{i}{4C_A} Tr\,(H)^2}\ \mathcal{G}(H$$

$H \in \mathcal{M}_{n'}(\mathbb{C})$, algebra of hermitian $n' \times n'$ traceless random matrices

$$d[H] = \mathrm{d}\Theta_1 \,..\, \mathrm{d}\Theta_{n'} \prod_{1\le i<j\le n'} (\Theta_i - \Theta_j)^2$$

$$\times \quad f(p)\, \mathrm{d}p_1 \,..\, \mathrm{d}p_l$$