Low-energy limit of QCD at finite temperature

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Plan of the talk

- Classical field theory
    - Scalar field theory
    - Yang-Mills theory
    - Mapping theorem: Formulation
    - Mapping theorem: Yang-Mills-Green function
    - Mapping theorem: A comparison
    - Mapping theorem: Next-to-leading order
- Quantum field theory
    - Scalar field theory
    - Yang-Mills theory
    - QCD in the infrared limit
    - Bosonization
    - Thermalization
- Conclusions

Classical field theory: Scalar field

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$$\phi = \mu\left(\frac{2}{\lambda}\right)^{\frac{1}{4}} \mathrm{sn}(p\cdot x + \theta, i)$$

being sn an elliptic Jacobi function and μ and θ two constants. This solution holds provided the following dispersion relation holds

$$p^2 = \mu^2\sqrt{\frac{\lambda}{2}}$$

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- Mass arises from the nonlinearities when λ is taken to be finite rather than going to zero.

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$$\delta\phi = \kappa^2 \lambda \int d^4x' d^4x'' \Delta(x - x')[\Delta(x' - x'')]^3 j(x') + O(j(x)^3)$$

with the identification $\kappa = \mu$, the same of the exact solution, and $\Box\Delta(x - x') + \lambda[\Delta(x - x')]^3 = \mu^{-1}\delta^4(x - x')$.

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- This implies that the corresponding quantum field theory, in a very strong coupling limit, takes a Gaussian form and is trivial (triviality of the scalar field theory in the infrared limit).
- All we need now is to find the exact form of the propagator $\Delta(x - x')$ and we have completely solved the classical theory for the scalar field in a strong coupling limit.

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- In order to solve the equation

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- It is straightforwardly obtained the Fourier transformed solution

$$\Delta_0(\omega)=\sum_{n=0}^{\infty}(2n+1)\frac{\pi^2}{K^2(i)}\frac{(-1)^n e^{-(n+\frac{1}{2})\pi}}{1+e^{-(2n+1)\pi}}\frac{1}{\omega^2-m_n^2+i\epsilon}$$

being $m_n=(2n+1)\frac{\pi}{2K(i)}\left(\frac{\lambda}{2}\right)^{\frac{1}{4}}\mu$ and $K(i)\approx 1.3111028777$ an elliptic integral.

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- We are able to recover the full covariant propagator by boosting from the rest reference frame obtaining finally

$$\Delta(p)=\sum_{n=0}^{\infty}(2n+1)\frac{\pi^2}{K^2(i)}\frac{(-1)^n e^{-(n+\frac{1}{2})\pi}}{1+e^{-(2n+1)\pi}}\frac{1}{p^2-m_n^2+i\epsilon}.$$

This shows that our solution given above indeed represents a strong coupling expansion being meaningful for $\lambda\rightarrow\infty$.

Classical field theory: Yang-Mills field

- A classical field theory for the Yang-Mills field is given by

$$\partial^\mu \partial_\mu A^a_\nu - \left(1 - \frac{1}{\alpha}\right)\partial_\nu (\partial^\mu A^a_\mu) + g f^{abc} A^{b\mu}(\partial_\mu A^c_\nu - \partial_\nu A^c_\mu) + g f^{abc} \partial^\mu (A^b_\mu A^c_\nu) + g^2 f^{abc} f^{cde} A^{b\mu} A^d_\mu A^e_\nu = -j^a_\nu .$$

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- For the homogeneous equation, we want to study it in the formal limit $g\rightarrow\infty$. We note that a class of exact solutions exists if we take the potential A^a_μ just depending on time, after a proper selection of the components [see Smilga (2001)]. These solutions are the same of the scalar field when spatial coordinates are set to zero (rest frame).

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- Differently from the scalar field, we cannot just boost away these solutions to get a general solution to Yang-Mills equations due to gauge symmetry. Anyhow, one can prove that the mapping persists but is just approximate in the limit of a very large coupling.
- This mapping would imply that we will have at our disposal a starting solution to build a quantum field theory for a strongly coupled Yang-Mills field. This solution has a mass gap already at a classical level!

Classical field theory: Yang-Mills field

- Exactly as in the case of the scalar field we assume the following solution to our field equations

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- This implies that the corresponding quantum theory, in a very strong coupling limit, takes a Gaussian form and is trivial.
- The crucial point, as already pointed out in the eighties [T. Goldman and R. W. Haymaker (1981), T. Cahill and C. D. Roberts (1985)], is the exact determination of the gluon propagator in the low-energy limit. Then, a lot of physics will be at our hands!

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- The mapping theorem helps to solve this problem definitely.

Mapping theorem: Formulation

- Exact determination of the gluon propagator can be largely simplified if we are able to map Yang-Mills theory on a theory with known results. With this aim in mind the following theorem has been proved:

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- **MAPPING THEOREM:** *An extremum of the action*

$$S = \int d^4x \left[\frac{1}{2}(\partial\phi)^2 - \frac{\lambda}{4}\phi^4\right]$$

is also an extremum of the SU(N) Yang-Mills Lagrangian when one properly chooses A_μ^a *with some components being zero and all others being equal, and* $\lambda = Ng^2$*, being* g *the coupling constant of the Yang-Mills field, when only time dependence is retained. In the most general case the following mapping holds*

$$A_\mu^a(x) = \eta_\mu^a \phi(x) + O(1/\sqrt{N}g),$$

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- This theorem was proved in the following papers: M. Frasca, Phys. Lett. **B670**, 73-77 (2008) [0709.2042]; Mod. Phys. Lett. A **24**, 2425-2432 (2009) [0903.2357] after considering a criticism by Terry Tao. Tao agreed with the latest proof.

Mapping theorem: Yang-Mills-Green function

- The mapping theorem permits us to write down immediately the propagator for the Yang-Mills equations in the Landau gauge for SU(N):

$$D_{\mu\nu}^{ab}(p)=\delta_{ab}\left(\eta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\sum_{n=0}^{\infty}\frac{B_n}{p^2-m_n^2+i\epsilon}+O\left(\frac{1}{\sqrt{N}g}\right)$$

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- All this classical analysis could be easier to work out on the lattice than the corresponding quantum field theory and would already be an important step beyond.

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- This equation can be solved exactly notwithstanding its appearance. Indeed, one can generally solve the equation for the Green function

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- This Green function maintains identical poles as the one at the leading order.

Quantum field theory: Scalar field (1)

- We can formulate a quantum field theory for the scalar field starting from the generating functional

$$Z[j] = \int [d\phi] \exp\left[i \int d^4x \left(\frac{1}{2}(\partial\phi)^2 - \frac{\lambda}{4}\phi^4 + j\phi\right)\right].$$

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  Then we can seek for a solution series as $\phi = \sum_{n=0}^{\infty}\lambda^{-n}\phi_n$ and rescale the current $j \to j/\lambda$ being this arbitrary.

- It is not difficult to see that the leading order correction can be computed solving the classical equation

$$\Box\phi_0 + \phi_0^3 = j$$

  that we already know how to manage. This is completely consistent with our preceding formulation [M. Frasca (2006)] but now all is fully covariant. We are just using our ability to solve the classical theory.

Quantum field theory: Scalar field (2)

- Using the approximation holding at strong coupling

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it is not difficult to write the generating functional at the leading order in a Gaussian form

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- This conclusion is really important: It says that the scalar field theory in d=3+1 is trivial in the infrared limit!
- This functional describes a set of free particles with a mass spectrum

$$m_n = (2n+1)\frac{\pi}{2K(i)}\left(\frac{\lambda}{2}\right)^{\frac{1}{4}}\mu$$

that are the poles of the propagator, the one of the classical theory.

Quantum field theory: Yang-Mills field (1)

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- and the corresponding current terms

$$S_c = \int d^4x j^a_\mu(x) A^{a\mu}(x) + \int d^4x \left[\bar{c}^a(x)\varepsilon^a(x) + \bar{\varepsilon}^a(x) c^a(x)\right].$$

Quantum field theory: Yang-Mills field (2)

- We now use the mapping theorem fixing the form of the propagator in the infrared, e.g. in the Landau gauge, as

$$D^{ab}_{\mu\nu}(p)=\delta_{ab}\left(\eta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\sum_{n=0}^\infty\frac{B_n}{p^2-m_n^2+i\epsilon}+O\left(\frac{1}{\sqrt{N}g}\right)$$

but this can be recomputed in any gauge by the classical equations with the mapping theorem.

Quantum field theory: Yang-Mills field (2)

- We now use the mapping theorem fixing the form of the propagator in the infrared, e.g. in the Landau gauge, as

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- and we note that, in this approximation, the ghost field just decouples and becomes free and one finally has at the leading order

$$Z_0[j]=N\exp\left[\frac{i}{2}\int d^4x'd^4x''j^{a\mu}(x')D^{ab}_{\mu\nu}(x'-x'')j^{b\nu}(x'')\right].$$

  This functional describes free massive glueballs that are the proper states in the infrared limit. Yang-Mills theory is <u>trivial</u> in the limit of the coupling going to infinity and we expect the running coupling to go to zero lowering energies.

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- Our conclusion is that, in a strong coupling expansion $1/\sqrt{N}g$, we get the so called decoupling solution.

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- We recognize here an explicit Yukawa interaction and a Nambu-Jona-Lasinio non-local term. Already at this stage we are able to recognize that NJL is the proper low-energy limit for QCD at zero temperature.

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- This action can be the starting point for our analysis at finite temperature. But before doing this, we want to give explicitly the contributions from gluon resonances. In order to do this, we introduce the bosonic currents $j_\mu^a(x) = \eta_\mu^a j(x)$ with the current $j(x)$ that of the gluonic excitations after mapping.

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- Using the relation $\eta^a_\mu \eta^{\mu a} = 3(N_c^2 - 1)$ we get in the end

$$S_{gf} = \frac{3}{2}(N_c^2 - 1)\int d^4x' d^4x'' \left[j(x')\Delta(x' - x'')j(x'') + O\left(\frac{1}{\sqrt{N}g}\right) + O\left(j^3\right)\right]$$

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- This means the we can write the bosonic currents contribution as coming from a boson field and written down as $\sigma(x) = \sqrt{3(N_c^2 - 1)/B_0} \int d^4x' \Delta(x - x')j(x')$.

QCD at the infrared limit (4)

- So, the model we consider for our finite temperature analysis, directly derived from QCD, is [Weise et al., Phys. Rev. D79, 014022 (2009), arXiv:0810.1099v2 [hep-ph]]

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- Now, we recover the non-local model of Weise et al. directly from QCD ($2\mathcal{G}(0) = G$ is the standard NJL coupling)

$$\mathcal{G}(p) = -\frac{1}{2}g^2\sum_{n=0}^{\infty}\frac{B_n}{p^2-(2n+1)^2(\pi/2K(i))^2\sigma + i\epsilon} = \frac{G}{2}\mathcal{C}(p)$$

with $\mathcal{C}(0) = 1$ fixing in this way the value of G using the gluon propagator.

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- This holds together with the gap equations

$$M(p) = m_q + \mathcal{C}(p)v$$

$$v = \frac{4NN_f}{m_0^2 + 1/G} \int \frac{d^4p}{(2\pi)^4} \mathcal{C}(p) \frac{M(p)}{p^2 + M^2(p)}$$

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- The next-to-leading order term is given by (a correction to mass m_0 for the σ field)

$$S^{(2)} = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\left[F_+(p^2)\delta\sigma(p)\delta\sigma(-p) + F_-(p^2)\delta\boldsymbol{\pi}(p)\delta\boldsymbol{\pi}(-p)\right]$$

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- Till now there are two novelties really implied with respect to the work of Weise et al.: The model is exactly obtained from QCD and the expression of the form factor $\mathcal{C}(p)$ is properly fixed through the exact gluon propagator at infrared.

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- The next step is to consider the case of finite temperature. This can be easily accomplished with the exchange

$$\int \frac{d^4p}{(2\pi)^4} \to \beta^{-1} \sum_{k=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3}$$

being the sum over k that on Matsubara frequencies $\omega_k = 2k\pi/\beta$ for bosons and $\omega_k = (2k+1)\pi/\beta$ for fermions.

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- At small temperatures we are able to get the critical temperature

$$T_c^2 \approx \frac{3}{\pi^2}\left[\Lambda^2 - \frac{\pi^2}{NN_f}\left(m_0^2 + \frac{1}{G}\right)\right]$$

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- Assuming the integral regularized by a cut-off Λ and noting that $\mathcal{C}(p)$ is practically 1 in the low-energy range, we can prove the existence of a critical temperature where the chiral symmetry is restored.
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- This expression is very similar to the one obtained in [D. Gomez Dumm and N. N. Scoccola, Phys. Rev. C72 (2005) 014909]

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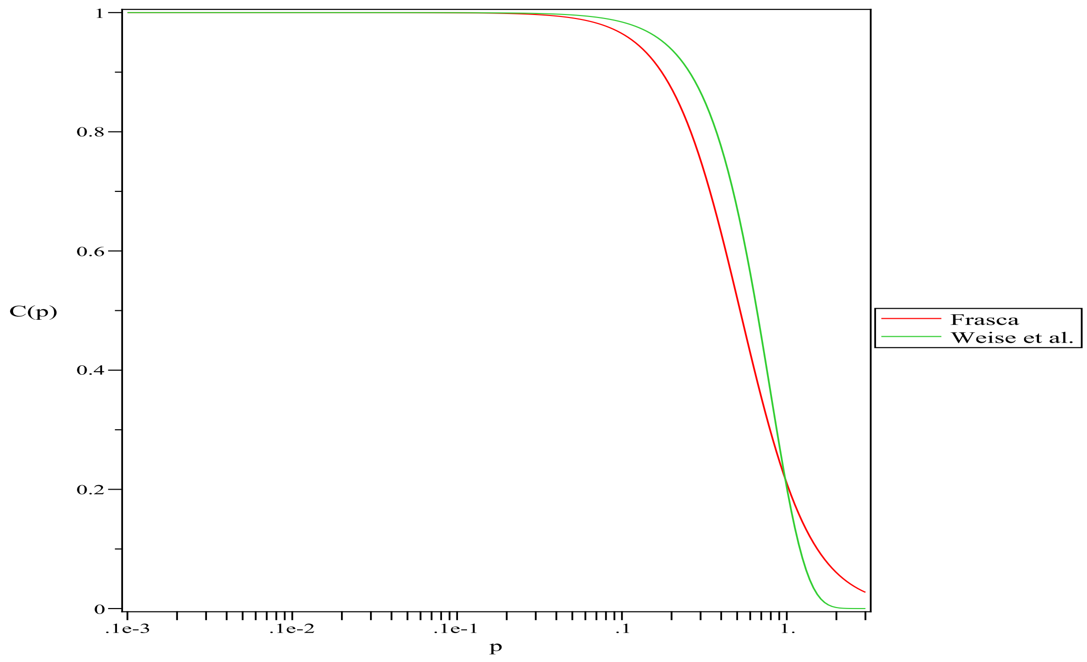

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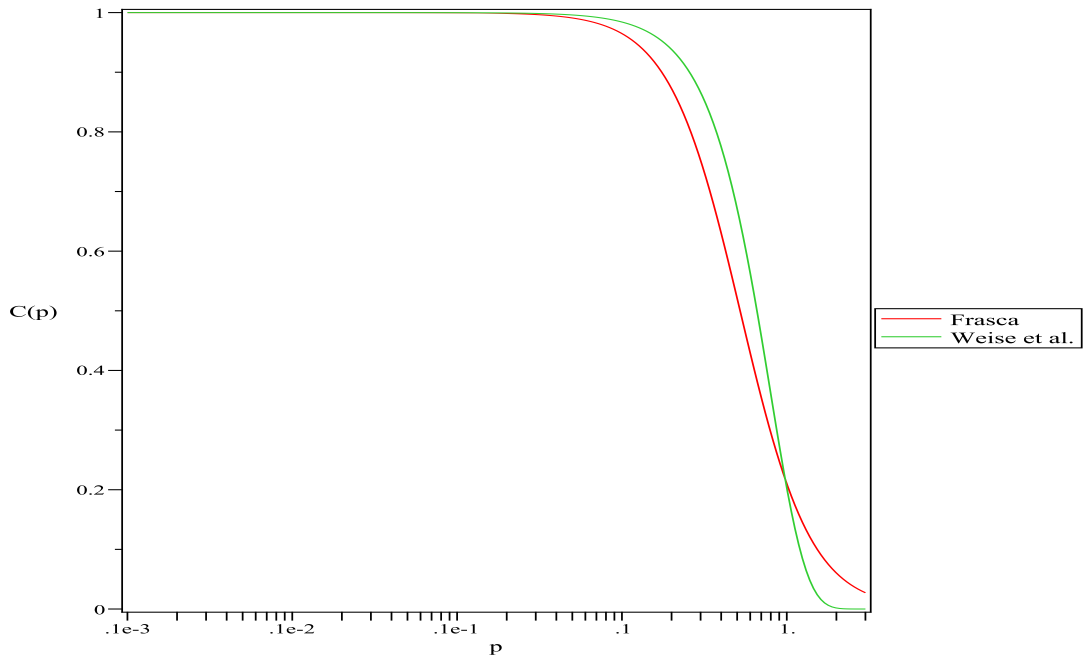

- Istanton liquid approximation is a good one indeed in describing the ground state of Yang-Mills theory!

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