Midterm #1.
Physics 206, 2004.

1.1). ${}^{x}S_{5/2}$. $\ell = 0,\ j = 5/2 \Rightarrow s = 5/2$.

$x = 2s+1 = \underline{6}$.

2). ${}^{2}F_{y}$. $\ell = 3.\ s = 1/2 \Rightarrow j = 3 \pm 1/2$.

$y = j = \underline{5/2}$ or $\underline{7/2}$.

2. 1). Hilbert space $= 1 \otimes 1 \otimes 1$.

For each spin there are 3 linearly independent wave functions.

Hence, dimension $= 3 \times 3 \times 3 = \underline{27}$

2). $1 \otimes 1 \otimes 1 = (1 \otimes 1) \otimes 1 = (0 \oplus 1 \oplus 2) \otimes 1 =$

$= 1 \oplus (1 \otimes 1) \oplus (2 \otimes 1) = 1 \oplus (0 \oplus 1 \oplus 2) \oplus (1 \oplus 2 \oplus 3) =$

$= 0 \oplus 1 \oplus 1 \oplus 1 \oplus 2 \oplus 2 \oplus 3$.

$\underline{\ell = 0, 1, 2, 3}$.

3). $\ell = 3$, there are $2 \cdot 3 + 1 = \underline{7}$ states

$\ell = 2$, there are $2 \cdot (2 \cdot 2 + 1) = \underline{10}$ states

$\ell = 1$, there are $3(2 \cdot 1 + 1) = \underline{9}$ states

$\ell = 0$, $0 \cdot 2 + 1 = \underline{1}$ state

$\boxed{7 + 10 + 9 + 1 = 27}$

4). $\hat{H} = J(\vec{S}_1 \vec{S}_2 + \vec{S}_2 \vec{S}_3 + \vec{S}_3 \vec{S}_1) = \frac{J}{2}(\vec{S}_1 + \vec{S}_2 + \vec{S}_3)^2 -$

$- J \frac{\vec{S}_1^2 + \vec{S}_2^2 + \vec{S}_3^2}{2} = -\frac{3}{2} J\, S_1(S_1 + 1) + \frac{J}{2} \vec{S}^2 =$

$= -3J + \frac{J}{2} S(S+1)$

$E_{S=0} = \underline{-3J}$ (only $\underline{1}$ such state)

$E_{S=1} = -3J + J = \underline{-2J}$ ($\underline{9}$ such states)

$E_{S=2} = -3J + 3J = \underline{0}$ ($\underline{10}$ such states)

$E_{S=3} = -3J + 6J = \underline{3J}$ ($\underline{7}$ such states)

3. This was discussed at the lectures.

$$\Theta_{\alpha\beta} = \sum e_i (3x_\alpha^{(i)} x_\beta^{(i)} - \delta_{\alpha\beta} \vec{x}^2).$$

One can form a spherical tensor $T_{\alpha\beta}$ of rank 2 from $\Theta_{\alpha\beta}$:

$$\langle \alpha_1 l_1 m_1 | T_{\alpha\beta} | \alpha_2 l_2 m_2 \rangle \neq 0$$

only if $|l_2 - 2| \leq l_1 \leq l_2 + 2$.

For $l_1 = l_2 = 0$ the above inequalities are not valid. Hence, $\langle \alpha_1 l_1 m_1 | T_{\alpha\beta} | \alpha_2 l_2 m_2 \rangle = 0$.
Hence, $\langle \alpha_1 l_1 m_1 | \Theta_{\alpha\beta} | \alpha_2 l_2 m_2 \rangle = 0$.

4.1) $[V_0 \delta(x)] = [\text{energy}]$. $\int \delta(x) dx = 1 \Rightarrow [\delta(x)] = \frac{1}{[\text{length}]} \Rightarrow$

$[V_0] = [\text{energy}][\text{length}] = \text{erg}\cdot\text{cm}$.

2) The ground state wave function is even. Hence, $b = a$.

3) $\psi(x) = x^2 - a^2, \quad -a < x < a.$

$$\int_{-a}^{a} \psi^2(x)\,dx = 2\int_0^a (x^2-a^2)^2 dx = 2\int_0^a (x^4 - 2a^2x^2 + a^4)dx = 2\left(\frac{a^5}{5} - \frac{2a^5}{3} + a^5\right) =$$

$$= 2\left(\frac{a^5}{3} + \frac{a^5}{5}\right) = \frac{16a^5}{15}.$$

$$\langle \psi | H | \psi \rangle = -V_0 a^4 - \frac{\hbar^2}{2m}\int_{-a}^{a} (x^2-a^2)(x^2-a^2)'' dx =$$

$$= -V_0 a^4 - \frac{\hbar^2}{m}\int_{-a}^{a} (x^2-a^2)dx = -V_0 a^4 + \frac{2\hbar^2}{m}\int_0^a (a^2-x^2)dx =$$

$$= -V_0 a^4 + \frac{2\hbar^2}{m}\left(a^3 - \frac{a^3}{3}\right) = -V_0 a^4 + \frac{4}{3}\frac{\hbar^2}{m}a^3.$$

$$E(a) = \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{15}{16a^5}\left(-V_0 a^4 + \frac{4}{3}\frac{\hbar^2}{m}a^3\right) = \frac{15}{16}\left(-\frac{V_0}{a} + \frac{4}{3}\frac{\hbar^2}{ma^2}\right)$$

$$\frac{dE}{da} = 0 \Rightarrow \frac{V_0}{a^2} - \frac{8}{3}\frac{\hbar^2}{ma^3} = 0 \Rightarrow a_0 = \frac{8}{3}\frac{\hbar^2}{mV_0}.$$

$$E(a_0) = \frac{15}{16}\left(-\frac{3}{8}\frac{mV_0^2}{\hbar^2} + \frac{4}{3}\frac{9}{64}\frac{mV_0^2}{\hbar^2}\right) = \frac{15}{16}\frac{mV_0^2}{\hbar^2}\left(-\frac{3}{8} + \frac{3}{16}\right) =$$

$$= \frac{15}{16}\frac{mV_0^2}{\hbar^2}\left(-\frac{3}{16}\right) = -\frac{45}{256}\frac{mV_0^2}{\hbar^2}$$

Dimensions: $\frac{[g][erg]^2[cm]^2}{[erg]^2[s]^2} = [erg]$.