Physics 206 May 13, 2002

Final Exam
35 points total

9:00 am – 12:00 noon

Calculators may be used, but no books or other written materials are permitted. Show the steps leading to your answers. Credit will be given based on evidence of your understanding of the material. Be sure to express all quantities used in the proper physical units.

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Possibly Useful Information

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\begin{aligned}
h &= 6.63 \times 10^{-34} J \cdot s = 4.14 \times 10^{-15} eV \cdot s \\
\hbar &= h/(2\pi) = 1.05 \times 10^{-34} J \cdot s = 1.05 \times 10^{-27} erg \cdot s \\
m &= 9.11 \times 10^{-31} kg = 9.11 \times 10^{-28} g \\
G &= 6.67 \times 10^{-11} N \cdot m^2/kg^2 \\
e &= 1.602 \times 10^{-19} C = 4.803 \times 10^{-10} esu \\
\frac{1}{4\pi\varepsilon_0} &= 8.98 \times 10^9 N \cdot m^2/C^2 \\
c &= 2.998 \times 10^8 m/s \\
\mu_0 &= 4\pi \times 10^{-7} Wb/(A \cdot m) \\
1eV &= 1.602 \times 10^{-19} J = 1.602 \times 10^{-12} erg \\
1\mathring{A} &= 10^{-10} m \\
1 \text{ tesla} &= 1\ N \cdot s/(C \cdot m) \\
\alpha &\equiv \frac{e^2}{\hbar c} \approx 1/137 \\
a_0 &= \frac{\hbar^2}{me^2} = 0.529\mathring{A}
\end{aligned}
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Read this: The grading of the problems will be based on (1) qualitative correctness, (2) quantitative correctness of your **numerical** answers (where required) and (3) your willingness to check the cancellation of units by explicit calculation (where required). Needless to say, show your work; all quantities must have units attached to them, unless they are dimensionless. Cancel out units to the full extent possible, ie. the combination $N/(kg \cdot m/s^2)$ is actually *dimensionless*. Hint: Ask yourself, "is my answer physically reasonable?"

1. (12 points total) Consider the low-energy scattering of electrons off hydrogen atoms in their ground state. Work only to lowest-order in the Born approximation. Assume that the incoming electrons are polarized spin-up, and the hydrogen electrons are polarized spin-down (this is to make them distinguishable, avoiding complications due to Fermi statistics.)

a) (5 points) First find the interaction potential $V(r)$ that the electrons see due to the hydrogen atom. As the scattering electrons are moving slowly, you may assume that they just see the average charge density of the electron cloud, as well as the Dirac $\delta(r)$-function charge density of the proton. **Use Poisson's equation** $\vec{\nabla}^2\phi(\vec{r}) = -4\pi\rho(\vec{r})$ to show that

$$\phi(\vec{r}) = \left(\frac{e}{r} + \frac{e}{a_0}\right) e^{-2r/a_0} \tag{1}$$

is the correct electrostatic potential seen by the spin-up electron when it encounters the hydrogen atom. Don't forget that $\rho(\vec{r})$ has contributions both from the spin-down electron cloud, and the nucleus. Show explicitly that you obtain the correct electron density. Hint:

$$\vec{\nabla}^2 = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \{\text{angular terms}\} \ . \tag{2}$$

b) (4 points) Now calculate the differential scattering cross section in the low-energy limit. Is there any angular dependence? Hint: the following integral may be useful:

$$\int_0^\infty e^{-2r/a_0}\, r^n\, dr = \left(\frac{a_0}{2}\right)^{n+1} n! \tag{3}$$

c) (3 points) What is the total scattering cross section, σ, in the low energy limit? How does it compare to scattering off a hard sphere of radius a_0? I'm looking for a real, actual, number here with correct units and explicit dimensional analysis, not an abstract algebraic expression.

2. (13 points total) The Casimir effect.

a) (2 points) The Hamiltonian for electromagnetism is:

$$\hat{H} = \frac{1}{8\pi}\int \left\{16\pi^2 c^2 |\vec{\Pi}(\vec{r})|^2 + |\vec{\nabla}\times\vec{A}(\vec{r})|^2\right\} d^3r \ . \tag{4}$$

What is the physical meaning of each of the two terms that appear in the integrand?

b) (4 points) Re-express the Hamiltonian in terms of the creation $\hat{a}^\dagger(\vec{k},\lambda)$ and annihilation $\hat{a}(\vec{k},\lambda)$ operators by making the substitution:

$$\begin{aligned}
\vec{A}(\vec{r}) &= \sum_\lambda \int \sqrt{\frac{\hbar c^2}{4\pi^2\omega(\vec{k})}} \left\{\hat{a}(\vec{k},\lambda)\vec{\epsilon}(\vec{k},\lambda)e^{i\vec{k}\cdot\vec{r}} + H.c.\right\} d^3k \\
\vec{\Pi}(\vec{r}) &= -i\sum_\lambda \int \sqrt{\frac{\hbar\omega(\vec{k})}{64\pi^4 c^2}} \left\{\hat{a}(\vec{k},\lambda)\vec{\epsilon}(\vec{k},\lambda)e^{i\vec{k}\cdot\vec{r}} - H.c.\right\} d^3k
\end{aligned} \tag{5}$$

where "H.c" is shorthand for Hermitian conjugate, or taking the adjoint. Also the dispersion is $\omega(\vec{k}) = c|\vec{k}|$. The polarization vector has unit length and satisfies the transversality constraint $\vec{k}\cdot\vec{\epsilon}(\vec{k},\lambda) = 0$.

As you did previously for the simple harmonic oscillator, identify the zero-point energy by ordering the creation and annihilation operators into the form $[\hat{a}^\dagger\hat{a} + 1/2]$.

c) (2 points) Show that the above formulas for $\vec{A}$ and $\vec{\Pi}$ respect the gauge condition $\vec{\nabla} \cdot \vec{A} = 0$ as well as Gauss's law $\vec{\nabla} \cdot \vec{\Pi} = 0$.

d) (5 points) Two flat, square, conducting metal plates, of length $L = 1$ cm on a side, are separated by a small distance $a = 1\mu$m. Use physical reasoning, combined with dimensional analysis to **estimate** the force between the plates, in Newtons. Do not attempt to calculate any sums or integrals, just obtain a rough answer from dimensional analysis. (The exact answer, which agrees with the one obtained by dimensional analysis up to a factor of order unity, will be given in the exam solution.) Hint: the force is proportional to the area of the plates. Is the force attractive or repulsive? I'm looking for a real, actual, number here (not an abstract algebraic expression.) Verify the dimensional correctness of your answer.

3. (10 points total) Free particle solutions of the Dirac equation. Find the energy eigenvalues and eigenstates of the free Dirac equation,

$$\hat{H}|\psi\rangle = E|\psi\rangle \tag{6}$$

where

$$\hat{H} = c\vec{\alpha} \cdot \hat{\vec{p}} + \beta mc^2 \ . \tag{7}$$

a) (3 points) First find the energy eigenvalues E for a particle with momentum $\vec{p}$ by considering the square of the Hamiltonian, $(\hat{H})^2$. What (representation-independent) properties of the $\vec{\alpha}$ and β matrices allow one to find a simple expression for $(\hat{H})^2$?

b) (4 points) Now that you've determined the energy eigenvalues, find the eigenstates. To simplify matters, let the particle have momentum only in the z-direction, and take it to be spin-up. Also, choose a particular representation for the matrices $\vec{\alpha}$ and β given by:

Thus, $|\psi\rangle$ may be written as:

Find the constant a for **both** positive and negative energy solutions. Normalize the wavefunctions by choosing $\mathcal{N}$ appropriately.

c) (3 points) Which of the following expressions is a suitable expression for the **charge** density $\rho(\mathbf{x}, t)$: $e * \psi^\dagger(\mathbf{x}, t) \cdot \psi(\mathbf{x}, t)$ or $e * \psi^\dagger(\mathbf{x}, t)\beta\psi(\mathbf{x}, t)$? Why?

Have a great summer!