PH206 Final Exam 2002 Solution

1. $$\vec{\nabla}^2\phi(r) = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\left[\left(\frac{e}{r}+\frac{e}{a_0}\right)e^{-2r/a_0}\right]$$

Now $$r^2\frac{\partial}{\partial r}\left[\left(\frac{1}{r}+\frac{1}{a_0}\right)e^{-2r/a_0}\right] = -\left(1+\frac{2r}{a_0}+\frac{2r^2}{a_0^2}\right)e^{-2r/a}$$

So $$\frac{1}{r^2}\frac{\partial}{\partial r}\left[\left(1+\frac{2r}{a_0}+\frac{2r^2}{a_0^2}\right)e^{-2r/a_0}\right] = \frac{1}{r^2}\left[\cancel{\frac{-2}{a_0}}+\cancel{\frac{2}{a_0}}-\cancel{\frac{4r}{a_0^2}}+\cancel{\frac{4r}{a_0^2}}-\frac{4}{a}\right.$$

$$\Rightarrow \vec{\nabla}^2\phi(r) = \frac{4e}{a_0^3}e^{-2r/a_0} = +4\pi e\left|\frac{e^{-r/a_0}}{\sqrt{\pi}\,a_0^{3/2}}\right|^2$$

$$= +4\pi e\,|\psi_{100}(r,\theta,\phi)|^2$$

1s ground state electron.

What happened to the proton? Observe:

1) $\vec{\nabla}^2(1/r) \neq 0$ due to singularity at $r=0$.

2) $$\int_V d^3r\ \nabla^2(1/r) = \int_{\partial V} d^2\vec{S}\cdot\underbrace{\vec{\nabla}(1/r)}_{-\hat{r}/r^2} = -4\pi$$

3) $\Rightarrow \vec{\nabla}^2(1/r) = -4\pi\,\delta^3(\vec{r})$.

4) $\phi(r)\to 0$ as $r\to\infty$ exponentially fast, because the hydrogen atom is neutral $\Rightarrow$ proton already included!

So, $\rho(\vec{r}) = e\left[\delta^3(\vec{r}) - \dfrac{e^{-2r/a_0}}{\pi a_0^3}\right]$. Correct

b) Lowest-order Born Scattering:

$$f(\theta,\phi) = \frac{-m}{2\pi\hbar^2}\int e^{i(\vec{k}'-\vec{k})\cdot\vec{r}}\, V(\vec{r})\, d^3r$$

$$\xrightarrow[\text{low energy}]{} \frac{-m}{2\pi\hbar^2}\int V(\vec{r})\, d^3r$$

No angular dependence!

In the low-energy limit, can approximate $(\vec{k}'-\vec{k})\cdot\vec{r} \equiv \vec{q}\cdot\vec{r} = 0$ since $\vec{k}$ and $\vec{k}'$ are both tiny.

$$\Longrightarrow \quad f(\theta,\phi) = \frac{-m}{2\pi\hbar^2}\int \underbrace{-e\,\phi(r)}_{V(r)\text{ seen by } \uparrow \text{ electron}}\, d^3r$$

$$= \frac{me^2}{2\pi\hbar^2}\cdot 4\pi\int_0^\infty \left(\frac{1}{r}+\frac{1}{a_0}\right)e^{-2r/a_0}\, r^2\, dr$$

$$\frac{me^2}{\hbar^2} = \frac{1}{a_0}$$

$$= \frac{2me^2}{\hbar^2}\int_0^\infty \left(r + r^2/a_0\right)e^{-2r/a_0}\, dr$$

$$= \frac{2}{a_0}\left[\left(\frac{a_0}{2}\right)^2 + \frac{1}{a_0}\left(\frac{a_0}{2}\right)^3\cdot 2!\right] = \boxed{a_0}$$

$$\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2 = a_0^2$$ independent of angle.

c) $$\sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega = 4\pi a_0^2$$

Same as low-energy scattering off a <u>hard</u> sphere of radius a_0 ! $\Longrightarrow$ "Size" of hydrogen atom $= a_0$.

Numerically, $\sigma = 4\pi\left(0.529\times10^{-10}\,m\right)^2 \overset{N}{=} 3.5\times10^{-20}\,m^2$

$[\sigma] =$ Area, as it must!

2. a) $$\vec{E} = 4\pi c\, \vec{\Pi}\,. \qquad \vec{B} = \vec{\nabla}\times\vec{A}\,.$$ } Energy in E+M field due to both $\vec{E}^2$ and $\vec{B}^2$.

b) The key to working through this algebra is to break it down into "bite-sized" chunks! (Those who read Shankar with an active, alert, mind did this already!)

i) First, let's look at the prefactors

$$|\vec{\Pi}|^2 \sim \frac{1}{8\pi}\cdot 16\pi^2 c^2 \cdot \left(\sqrt{\frac{\hbar\omega}{64\pi^4 c^2}}\right)^2 = \frac{\hbar\,\omega(\vec{k})}{4\cdot(2\pi)^3}$$

$$|\vec{\nabla}\times\vec{A}|^2 \sim \frac{1}{8\pi}\cdot\left(\sqrt{\frac{\hbar c^2}{4\pi^2\omega}}\right)^2\cdot \underbrace{\vec{k}^2}_{\text{due to } \vec{\nabla}\times} = \frac{\hbar\,\omega(k)}{4\cdot(2\pi)^3}$$

(since $c^2\vec{k}^2 = \omega^2$)

ii) Next, let's look at the operators that can appear together. Since we integrate over $\vec{r}$ in $\hat{H}$, most terms drop out, due to oscillating $e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}$.

In fact $$\int e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}\, d^3r = (2\pi)^3\, \delta^3(\vec{k}-\vec{k}')\,.$$

Ah! See (i) above!

Both $|\vec{\Pi}|^2$ and $|\vec{\nabla}\times\vec{A}|^2$ contain terms like:

$$\hat{a}(\vec{k},\lambda)\,\hat{a}(-\vec{k},\lambda) \quad \text{and} \quad \hat{a}^\dagger(\vec{k},\lambda)\,\hat{a}^\dagger(-\vec{k},\lambda)$$

as well as $$\hat{a}^\dagger(\vec{k},\lambda)\,\hat{a}(\vec{k},\lambda) \quad \text{and} \quad \hat{a}(\vec{k},\lambda)\,\hat{a}^\dagger(\vec{k},\lambda)\,.$$

iii) Note that since $\hat{\epsilon}(\vec{k},\lambda)\cdot\epsilon(\vec{k},\lambda') = \delta_{\lambda\lambda'}$, there are <u>no</u> terms like $a^\dagger(\vec{k},\lambda)\, a(\vec{k},\lambda')$ with $\lambda \neq \lambda'$.

iv) Finally, what happens to the $\hat{a}\hat{a}$ and $\hat{a}^\dagger\hat{a}^\dagger$ terms? When $|\vec{\Pi}|^2$ is added to $|\vec{\nabla}\times\hat{A}|^2$ these cancel out, because they have opposite signs in each piece, due to the $-\vec{k}^2$ appearing in $|\vec{\nabla}\times\vec{A}|^2$ (<u>not</u> $\vec{k}^2$).

v) Putting everything together:

$$\hat{H} = \sum_\lambda \int d^3k \; \frac{\hbar\omega(\vec{k})}{4} \left[2\hat{a}^\dagger(\vec{k},\lambda)\,\hat{a}(\vec{k},\lambda) + 2\,\hat{a}(\vec{k},\lambda)\,\hat{a}^\dagger(\vec{k},\lambda) \right.$$

"Normal-order"

$$= \sum_\lambda \underbrace{\int d^3k}_{\text{Replace by } \sum_{\vec{k}} \text{ for modes between plates.}} \; \hbar\omega(\vec{k}) \left[\hat{a}^\dagger(\vec{k},\lambda)\,\hat{a}(\vec{k},\lambda) + \frac{1}{2} \right]$$

(Shankar Eq. 18.5.60)

zero-pt. energy.

Using the canonical commutation relation:

$$\left[\hat{a}(\vec{k}',\lambda') , \hat{a}^\dagger(\vec{k},\lambda) \right] = \delta_{\lambda\lambda'}\,\delta^3(\vec{k}-\vec{k}').$$

c) Clearly $\vec{\nabla}\cdot\vec{A} = \vec{\nabla}\cdot\vec{\Pi} = 0$ because $\vec{k}\cdot\vec{\epsilon}(\vec{k},\lambda) = 0$.

Acting with $\vec{\nabla}\cdot$ on $\vec{A}(\vec{r})$ just brings down $\pm\vec{k}$ from the $e^{i\vec{k}\cdot\vec{r}}$ and $e^{-i\vec{k}\cdot\vec{r}}$ terms. Same for $\vec{\Pi}(\vec{r})$.

d) Casimir Effect :

Since zero-pt. energy $\sim \hbar\omega \sim \hbar c k$

and $[k] = \frac{1}{\text{length}}$, we reason that

$$E_0 \sim \frac{\hbar c \cdot L^2}{a^3}$$ which is <u>proportional to Area</u> of plates.

$$\Rightarrow F \sim \frac{\partial E_0}{\partial a} \sim \boxed{\frac{\hbar c L^2}{a^4}}.$$

$[F] = \frac{J \cdot s \cdot \frac{m}{s} \cdot m^2}{m^4} = J/m = N.$

The force is <u>attractive</u> (pulling plates together) because as the plates get closer there are fewer allowed modes of oscillation, so zero-pt. energy decreases.

Exact Answer : $F = -\frac{\pi^2}{240} \frac{\hbar c}{a^4} L^2 \cong \frac{-1.3 \times 10^{-7}}{a^4 (\mu m)} \frac{N}{cm^2}$

"factor of order unity."

$$\boxed{= -1.3 \times 10^{-7} \text{ N}.}$$

See: Itzykson + Zuber, <u>Quantum Field Theory</u>, Ch. 3-2-4 "Vacuum Fluctuations" pp. 138-141, especially Eq. 3-150.

3. Free Dirac Eq. $\hat{H} = c\,\vec{\alpha}\cdot\hat{\vec{p}} + \beta m c^2$

a) $\hat{H}^2 = c^2 (\vec{\alpha}\cdot\hat{\vec{p}})^2 + \beta^2 m^2 c^4 + m c^3 (\vec{\alpha}\cdot\hat{\vec{p}}\,\beta + \beta\,\vec{\alpha}\cdot\hat{\vec{p}})$

Now $\alpha_i^2 = \beta^2 = I$

and $\{\alpha_i, \beta\} = 0$ and $\{\alpha_i, \alpha_j\} = 2\delta_{ij}$

So last term vanishes, and $(\vec{\alpha}\cdot\hat{\vec{p}})^2 = \hat{\vec{p}}^2$.

$\Rightarrow$ $\hat{H}^2 = c^2 \hat{\vec{p}}^2 + m^2 c^4$ already diagonal in 4-spinor space!

$\Rightarrow$ $\hat{H}|\psi\rangle = E|\psi\rangle \Rightarrow E = \pm\sqrt{c^2 p^2 + m^2 c^4}$.

b) Directly solve for $\boxed{a = \dfrac{cp}{E + mc^2} = \dfrac{E - mc^2}{cp}}$

Note: $|a| < 1$ for $E > 0$ $\quad \eta = 1 + a^2$
$|a| > 1$ for $E < 0$.

$$|\psi\rangle \longleftrightarrow \frac{e^{ipz/\hbar}}{\eta^{1/2}} \begin{pmatrix} 1 \\ 0 \\ a \\ 0 \end{pmatrix}$$

c) $\rho(\vec{x},t) = e\,\psi^\dagger(\vec{x},t)\,\psi(\vec{x},t)$. Must have same sign everywhere!

$Q = \int d^3x\, \rho(\vec{x},t) = e$ due to normalization of $|\psi\rangle$.