PH 206 Final Exam 2003

Solutions

1. a) $H(t) = -\vec{\mu} \cdot \vec{B}(t)$

$\vec{\mu} = \gamma \vec{S}$, $\gamma = \frac{-e}{m_e c}$, $\vec{S} = \frac{\hbar}{2} \vec{\sigma}$

$$\Rightarrow H(t) = \frac{e \hbar B}{2 m_e c} \cos(2\pi f t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

b) $$U(t) = e^{-\frac{i}{\hbar} \int_0^t H(t')\, dt'}$$

$$= \begin{pmatrix} e^{-\frac{ieB}{4\pi f m_e c} \sin(2\pi f t)} & 0 \\ 0 & e^{+\frac{ieB}{4\pi f m_e c} \sin(2\pi f t)} \end{pmatrix}$$

c) $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \chi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

points in $+x$ direction.

$$\chi(t) = U(t)\, \chi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i(\ )} \\ e^{+i(\ )} \end{pmatrix}$$

d) $$\text{Prob}(t) = |\langle \chi_- | \chi(t) \rangle|^2$$ where

$$\chi_- = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$ points in $-x$ direction

$$\Rightarrow \text{Prob}(t) = \sin^2\left(\frac{eB}{4\pi m_e c f}\sin(2\pi f t)\right)$$

e). Spin flips from $+x$ to $-x$ direction if

$$\text{Prob}(t) = 1 \Rightarrow \frac{eB}{4\pi m_e c f}\sin(2\pi f t) = \pi/2$$

Minimum B satisfies $\dfrac{eB_{min}}{4\pi m_e c f} = \dfrac{\pi}{2}$

$$\Rightarrow B_{min} = \frac{2\pi^2 m_e c f}{e} = 1{,}100 \text{ Gauss} = 0.11\,T$$

Unit Check (CGS) $[F] = \left[\dfrac{q v B}{c}\right]$

$$\Rightarrow [B] = \frac{[F]}{[q]} = \frac{\text{gm cm/s}^2}{\text{esu}}.$$

$$[B_{min}] = \frac{\text{gm}\cdot(\text{cm/s})\cdot s^{-1}}{\text{esu}} \checkmark.$$

A reasonable number.

2. a) As $\alpha \to \infty$ recover hard-sphere result $\sigma = 4\pi a^2$

b) $h_0^{(1)}(r)$ blows-up as $r \to 0$, so Eq. (2) doesn't apply for $r<a$. In general, for $r<a$, have:

$$\psi_<(r,\theta) = \sum_{l=0}^{\infty} B_l \, j_l(kr) \, P_l(\cos\theta)$$

c) Retain only $l=0$ terms and take $ka \to 0$ limit for simplicity.

Continuity: $\psi_<(r=a,\theta) = \psi_>(r=a,\theta) \Rightarrow B_0 = A\left[1 + \frac{C_0}{\sqrt{4\pi}}\left(\frac{-i}{ka}\right)\right]$

Discontinuity of radial derivative:

$$\int_{a-\epsilon}^{a+\epsilon} dr \left\{ \frac{-\hbar^2}{2m} \frac{d^2u}{dr^2} + \alpha\,\delta(r-a)\,u \right\} = 0$$

$$\Rightarrow \left.\frac{du}{dr}\right|_{a+\epsilon} - \left.\frac{du}{dr}\right|_{a-\epsilon} = \frac{2m\alpha}{\hbar^2} u(a).$$

$$\Rightarrow +i\frac{AC_0}{\sqrt{4\pi}} \frac{1}{ka^2} = \frac{2m\alpha}{\hbar^2} B_0$$

Thus we have, from the 2 matching equations,

$$\begin{cases} B_0 = A\left[1 - \dfrac{i\,C_0}{\sqrt{4\pi}}\,\dfrac{1}{ka}\right] \\[2ex] B_0 = \dfrac{i\,A\,C_0}{\sqrt{4\pi}\,\phi}\,\dfrac{1}{ka} \end{cases} \qquad \text{where } \boxed{\phi \equiv \frac{2m\alpha a}{\hbar^2}}$$

Eliminate B_0 to obtain:

$$C_0 = -i\sqrt{4\pi}\,ka\,\phi\left(1 - \frac{i\,C_0}{\sqrt{4\pi}\,ka}\right)$$

$$= -i\sqrt{4\pi}\,ka\,\phi - \phi\,C_0$$

$$\Longrightarrow \quad \boxed{C_0 = \frac{-i\sqrt{4\pi}\,ka\,\phi}{1+\phi}}$$

d) $$f(\theta) = \frac{-i\,C_0}{\sqrt{4\pi}\,k} = \frac{-a\phi}{1+\phi}$$

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{a^2\phi^2}{(1+\phi)^2}$$

$$\sigma = \int \frac{d\sigma}{d\Omega}\,d\Omega = \frac{4\pi a^2\phi^2}{(1+\phi)^2}$$

Question: Since $f(\theta)$ is real, how is optical theorem obeyed? Hint: consider $ka \neq 0$

Check. $\alpha \to \infty \Longrightarrow \sigma = 4\pi a^2$. YES!

3. a) Only the 5^{th} term is non-Hermitian.

$$\left(\vec{P}\cdot\left[\vec{P}, V(r)\right]\right)^{\dagger} = \left(\left[\vec{P}, V(r)\right]\right)^{\dagger}\cdot\vec{P}$$

$$= -\left[\vec{P}, V(r)\right]\cdot\vec{P} \neq \vec{P}\cdot\left[\vec{P}, V(r)\right].$$

Note: $\left(\left[A, B\right]\right)^{\dagger} = -\left[A, B\right].$

b) Correct normalization for the Dirac spinor include the "small" component:

$$\mathcal{N} = \int d^3x \left\{ |\chi(\vec{r})|^2 + |\Phi(\vec{r})|^2 \right\}.$$