Physics 206 March 4, 2002

First Midterm Exam
20 points total

9:00 am – 11:00 am

Calculators may be used, but no books or other written materials are permitted. Show the steps leading to your answers. Credit will be given based on evidence of your understanding of the material. Be sure to express all quantities used in the proper physical units.

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Possibly Useful Information

$$\begin{aligned}
h &= 6.63 \times 10^{-34} J \cdot s = 4.14 \times 10^{-15} eV \cdot s \\
\hbar &= h/(2\pi) = 1.05 \times 10^{-34} J \cdot s = 1.05 \times 10^{-27} erg \cdot s \\
m_e &= 9.11 \times 10^{-31} kg = 9.11 \times 10^{-28} g \\
G &= 6.67 \times 10^{-11} N \cdot m^2/kg^2 \\
e &= 1.602 \times 10^{-19} C = 4.803 \times 10^{-10} esu \\
\frac{1}{4\pi\varepsilon_0} &= 8.98 \times 10^{9} N \cdot m^2/C^2 \\
c &= 2.998 \times 10^{8} m/s \\
\mu_0 &= 4\pi \times 10^{-7} Wb/(A \cdot m) \\
1eV &= 1.602 \times 10^{-19} J = 1.602 \times 10^{-12} erg \\
1\mathring{A} &= 10^{-10} m \\
1 \text{ tesla} &= 1\ N \cdot s/(C \cdot m)
\end{aligned}$$

Read this: The grading of the problems will be based on (1) qualitative correctness, (2) quantitative correctness of your **numerical** answers (where required) and (3) your willingness to check the cancellation of units by explicit calculation (where required). Needless to say, show your work; all quantities must have units attached to them, unless they are dimensionless. Cancel out units to the full extent possible, ie. the combination $N/(kg \cdot m/s^2)$ is actually *dimensionless*. Hint: Ask yourself, "is my answer physically reasonable?"

1. (10 points total) Use the variational principle to estimate the effect of the spin-orbit interaction on the $2p$ energy levels of the hydrogen atom. The spin-orbit interaction $\hat{H}_{so}$ of an electron in a hydrogen atom, in CGS units, is:

$$\hat{H}_{so} = \frac{e^2}{2m_e^2c^2}\,\frac{1}{r^3}\,\vec{L}\cdot\vec{S}\ . \tag{1}$$

Here $\vec{L}$ is the orbital angular momentum operator, and $\vec{S}$ is the electron spin operator. (We will derive $\hat{H}_{so}$ later in the course.) So the total Hamiltonian is $\hat{H} = \hat{H}_{Coulomb} + \hat{H}_{so}$. To solve for the energy levels variationally, you need to calculate

$$E_{njm} \approx \langle njm|\hat{H}|njm\rangle = -\frac{\mathcal{R}}{n^2} + \langle njm|\hat{H}_{so}|njm\rangle\ . \tag{2}$$

a) (3 points) To start, calculate the expectation value $\langle r^{-3}\rangle$ for the normalized $2p_z$ wavefunction:

$$\psi(r,\theta,\phi) = \frac{1}{\sqrt{32\pi a_0^3}}\,(r/a_0)e^{-r/2a_0}\cos\theta \tag{3}$$

where $a_0 = \hbar^2/(me^2)$. To check your answer, make sure that $\langle r^{-3}\rangle$ has the correct physical dimension of $(\text{length})^{-3}$.

b) (3 points) What are the possible eigenvalues of the operator $\vec{L}\cdot\vec{S}$ for an electron in a p-orbital? What are the allowed values of the total angular momentum j?

c) (2 points) Show that the spin-orbit splitting ΔE of the $2p$ levels is of order $\alpha^4 m_e c^2$ where α is the fine-structure constant. Pay attention to dimensional analysis.

d) (2 points) Finally, numerically evaluate the spin-orbit splitting ΔE of the $2p$ levels in electron-volts (eV), for hydrogen. I'm looking for a real number here, not an abstract algebraic expression. Show, by explicit cancellation, that the physical units work out properly.

2. (10 points total) Four short questions:

a) (3 points) Which (if any) of the following matrix elements could be non-zero? States $|n,\ell,m\rangle$ are eigenstates of the orbital angular momentum operators $\vec{L}^2$ and L_z, with eigenvalues $\ell(\ell+1)$ and m respectively. Briefly explain your reasoning, but do **not** attempt to evaluate the matrix elements quantitatively.

1. $\langle n,1,1|\hat{Z}|n,1,1\rangle$
2. $\langle n,1,1|\hat{Z}|n,1,0\rangle$
3. $\langle n,1,1|\hat{X}|n,1,1\rangle$
4. $\langle n,2,2|\hat{Y}|n,1,1\rangle$
5. $\langle n+2,1,0|\hat{Z}|n,1,0\rangle$
6. $\langle n,2,0|\hat{X}\hat{Z}|n,0,0\rangle$

b) (2 points) The electronic configurations for several elements are listed below. Use Hund's rules to determine ${}^{2S+1}L_J$ ground state quantum numbers. (Remember that 'L' = 'S', 'P', 'D', or 'F' for total orbital angular momenta 0, 1, 2, or 3.) Briefly explain your reasoning.

1. Phosphorus (P): (Ne)$(3s)^2(3p)^3$
2. Chlorine (Cl): (Ne)$(3s)^2(3p)^5$
3. Iron (Fe): (Ar)$(3d)^6(4s)^2$
4. Copper (Cu): (Ar)$(3d)^{10}(4s)^1$

c) (4 points) Work out the Bohr theory of the helium atom to estimate its ground state energy. To do this, put both electrons into the same circular orbit, and give each electron one unit of angular momentum, $\ell = 1\hbar$. Minimize the electrostatic repulsion between the two electrons by placing them on opposite sides of their common orbit, as shown in the figure. What is the ground state energy, in eV? I'm looking for a real number here, not an abstract algebraic expression. How does this compare with the exact result of $E_0 = -78.975$ eV? Hint: working this problem requires only *simple* algebra. You need to calculate the radius of the mutual orbit. Think about the Bohr theory of the hydrogen atom, and how it needs to be modified to fit this problem.

d) (1 point) Evaluate the inner product $\langle 1,1|(|3/2,1/2\rangle \otimes |1/2,1/2\rangle)$. Here bra state $\langle 1,1|$ has total angular momentum = 1, with z-component = 1.