Physics 206 March 11, 2005

First Midterm Exam
20 points total

9:00 am – 10:50 am

Calculators may be used, but no books or other written materials are permitted. Show the steps leading to your answers. Credit will be given based on evidence of your understanding of the material. Be sure to express all quantities used in the proper physical units.

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Possibly Useful Information

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\begin{aligned}
h &= 6.63\times 10^{-34} J\cdot s = 4.14\times 10^{-15} eV\cdot s \\
\hbar &= h/(2\pi) = 1.05\times 10^{-34} J\cdot s = 1.05\times 10^{-27} erg\cdot s \\
m_e &= 9.11\times 10^{-31} kg = 9.11\times 10^{-28} g \\
G &= 6.67\times 10^{-11} N\cdot m^2/kg^2 \\
e &= 1.602\times 10^{-19} C = 4.803\times 10^{-10} esu \\
\alpha &\equiv e^2/\hbar c \\
\gamma &= -\frac{e}{2mc} \\
\frac{1}{4\pi\varepsilon_0} &= 8.98\times 10^{9} N\cdot m^2/C^2 \\
c &= 2.998\times 10^{8} m/s \\
\mu_0 &= 4\pi\times 10^{-7} Wb/(A\cdot m) \\
1eV &= 1.602\times 10^{-19} J = 1.602\times 10^{-12} erg \\
1\text{Å} &= 10^{-10} m \\
1\ \text{tesla} &= 1\ N\cdot s/(C\cdot m) \\
\nabla^2 &= \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi^2} \\
\hat{L}_z &\rightarrow -i\hbar\frac{\partial}{\partial\phi} \\
\hat{L}^2 &\rightarrow -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial}{\partial\phi^2}\right)
\end{aligned}
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Read this: The grading of the problems will be based on (1) qualitative correctness, (2) quantitative correctness of your **numerical** answers (where required) and (3) your willingness to check the cancellation of units by explicit calculation (where required). Needless to say, show your work; all quantities must have units attached to them, unless they are dimensionless. Cancel out units to the full extent possible, ie. the combination $N/(kg\cdot m/s^2)$ is actually *dimensionless*. Hint: Ask yourself, "is my answer physically reasonable?"

1. (10 points total) An eigenstate of the hydrogen atom and the periodic table.

a) (1 points) Write down the Hamiltonian for the non-relativistic hydrogen atom. Don't worry about spin-orbit interactions, fine structure, etc.

b) (2 points) Show, by direct calculation, that the (non-normalized) wavefunction

$$\psi(r,\theta,\phi) = (r/a_0)e^{-r/2a_0}\sin\theta\; e^{i\phi}, \tag{1}$$

where $a_0 = \hbar^2/(me^2)$ is a simultaneous eigenstate of both $\hat{L}^2$ and $\hat{L}_z$. (Hint: Look at the equations on the first page of the exam.) What are the respective eigenvalues of these two angular momentum operators?

c) (3 points) Now show, by direct calculation (ie. $\hat{H}\psi = E\psi$), that the wavefunction is an energy eigenstate, and find its energy. What is the energy in electron volts (eV)? Show your work, using the values for e, m, etc. on the first page of the exam – I'm looking for a real number here with units (not an abstract algebraic expression) and an explicit check that the units work out properly.

d) (4 points) The electronic configurations for several elements are listed below. Use Hund's rules to determine ${}^{2S+1}L_J$ ground state quantum numbers. [Remember that 'L' = 'S', 'P', 'D', or 'F' for total orbital angular momenta 0, 1, 2, or 3. Here (Ar) and (Xe) are shorthand notation for the filled argon- and xenon-like cores.] Briefly explain your reasoning.

1. Hafnium (Hf): (Xe)$(6s)^2(5d)^2$
2. Iridium (Ir): (Xe)$(6s)^2(5d)^7$
3. Platinum (Pt): (Xe)$(6s)^1(5d)^9$
4. Ruthenium (Ru): (Ar)$(5s)^1(4d)^7$

2. (10 points total) Fun with angular momentum. Working in units where $\hbar = 1$, first consider some problems involving two particles of spin s_1 and s_2 (not necessarily spin-1/2). Thus, upon quantizing along the z-axis, for the first spin we have the usual relations $\vec{S}_1^2|s_1,m_1\rangle = s_1(s_1+1)|s_1,m_1\rangle$ and $S_1^z|s_1,m_1\rangle = m_1|s_1,m_1\rangle$. Similarly expressions hold for the second spin.

a) (2 points) Show that, for the special case $s_1 = s_2 = 1/2$, that the biquadratic operator $(\vec{S}_1\cdot\vec{S}_2)^2$ may be written as the sum of a bilinear operator and a constant, specfically $(\vec{S}_1\cdot\vec{S}_2)^2 = -\frac{1}{2}\,\vec{S}_1\cdot\vec{S}_2 + \frac{3}{16}$. Hint: Consider the possible eigenvalues of the operator $\vec{S}_1\cdot\vec{S}_2$.

b) (1 point) Derive a similar relationship for the biquadratic term in the case of $s_1 = 1$ and $s_2 = 1/2$.

c) (2 points) Now consider 3 particles. Following the notation $\frac{1}{2}\otimes\frac{1}{2} = 0\oplus 1$, what does $\frac{1}{2}\otimes 1\otimes\frac{3}{2} = ?$ Check that the total Hilbert space dimension is the same on both sides of the equation.

d) (2 points) Demonstrate (i) associativity: $(\frac{1}{2}\otimes 1)\otimes\frac{3}{2} = \frac{1}{2}\otimes(1\otimes\frac{3}{2})$ and commutivity: $(\frac{1}{2}\otimes 1)\otimes\frac{3}{2} = (\frac{3}{2}\otimes\frac{1}{2})\otimes 1$.

e) (3 points) Finally, consider the precession of an electron in a magnetic field directed along the x-axis, with magnitude $B = 1$ Tesla $= 10^4$ gauss. At time $t = 0$ the electron is described by a spinor in the $+z$ eigenstate:

$$\psi(t=0) = (1,0) \tag{2}$$

What is $\psi(t)$? What is the precession frequency in Hertz (cycles per second)? I want a real number here, not an abstract algebraic expression, with units worked out properly. Don't forget g!