PH 206 March 11, 2005

Mid Term #1

1. a) $$\hat{H} = \frac{\vec{P}^2}{2m} - \frac{e^2}{r} \qquad \text{(CGS units)}$$

b) $$-i\hbar \frac{\partial}{\partial \phi} e^{i\phi} = \hbar e^{i\phi} \Rightarrow \boxed{l_z = \hbar}$$

$$-\hbar^2 \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right) \sin\theta \, e^{i\phi}$$

$$= -\hbar^2 e^{i\phi} \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} (\cos\theta \sin\theta) - \frac{\sin\theta}{\sin^2\theta} \right)$$

$$= -\hbar^2 \sin\theta \, e^{i\phi} \left(\frac{\cos^2\theta - \sin^2\theta - 1}{\sin^2\theta} \right) = 2\hbar^2 \sin\theta \, e^{i\phi}$$

$$\Downarrow$$

$$\boxed{l^2 = 2\hbar^2}$$

due to $\vec{L}^2$

c) $$\left[\frac{-\hbar^2}{2m} \left(\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{2}{r^2} \right) - \frac{e^2}{r} \right] r e^{-r/2a_0}$$

$$= E \, r e^{-r/2a_0}. \qquad a_0 \equiv \frac{\hbar^2}{me^2}$$

$$\left(\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\right) r e^{-r/2a_0} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 - \frac{r^3}{2a_0}\right) e^{-r/2a_0}$$

$$= \frac{1}{r^2}\left(2r - \frac{3r^2}{2a_0} - \frac{r^2}{2a_0} + \frac{r^3}{4a_0^2}\right) e^{-r/2a_0}$$

$$= \left(\frac{2}{r} - \frac{4}{2a_0} + \frac{r}{a_0^2}\right) e^{-r/2a_0}.$$

$$= \left(\frac{2}{r^2} - \frac{\cancel{4}2}{a_0 r} + \frac{1}{4a_0^2}\right) r e^{-r/2a_0}$$

Cancels with $\vec{L}^2$ Cancels with $\frac{-e^2}{r}$ Energy!

$$\Rightarrow \quad E = -\frac{\hbar^2}{2m} \cdot \frac{1}{4a_0^2} = -\frac{me^4}{8\hbar^2} = -\frac{1}{4} R_y$$

$$= -\frac{(9.11\times10^{-28}\,g)(4.803\times10^{-10}\,esu)^4}{8\cdot(1.05\times10^{-27}\,erg\cdot sec)^2}$$

$$= -5.5\times10^{-12}\ \frac{gm\cdot(erg\cdot cm)^2}{\cancel{erg}^2\ sec^2} = -5.5\times10^{-12}\,erg$$

$1\,esu^2 = erg\cdot cm$

$$= -5.5\times10^{-12}\,erg \times \frac{1\,eV}{1.6\times10^{-12}\,erg} = \boxed{-3.43\,eV}$$

d) See Baym, page 452 (Table 20-1)

2) a) $$\vec{S}_1 \cdot \vec{S}_2 = \begin{cases} 1/4, & S_{TOT} = 1 \\ -3/4, & S_{TOT} = 0 \end{cases}$$

$$\Rightarrow \left(\vec{S}_1 \cdot \vec{S}_2\right)^2 = \begin{cases} 1/16, & S_{TOT} = 1 \\ 9/16, & S_{TOT} = 0 \end{cases}$$

But $$-\frac{1}{2}\vec{S}_1 \cdot \vec{S}_2 + \frac{3}{16} = \begin{cases} \frac{1}{16}, & S_{TOT} = 1 \\ \frac{9}{16}, & S_{TOT} = 0 \end{cases}$$

Since the two operators are diagonal in the same eigenbasis, with the same eigenvalues, they are <u>exactly equal</u>.

b) $$\vec{S}_1 \cdot \vec{S}_2 = \begin{cases} 1/2 & (S_{Total} = 3/2) \quad (dgen = 4) \\ -1 & (S_{Total} = 1/2) \quad (dgen = 2) \end{cases}$$

Check: $\left(\frac{1}{2}\right) \times (dgen = 4) + (-1) \times (dgen = 2) = 0$ ✓

Thus, $\left(\vec{S}_1 \cdot \vec{S}_2\right)^2 = \frac{1}{2} - \frac{1}{2}\vec{S}_1 \cdot \vec{S}_2$.

c) $\left(\frac{1}{2} \otimes 1\right) \otimes \frac{3}{2} = \left(\frac{1}{2} \oplus \frac{3}{2}\right) \otimes \frac{3}{2} = \left(\frac{1}{2} \otimes \frac{3}{2}\right) \oplus \left(\frac{3}{2} \otimes \frac{3}{2}\right)$

$= (1 \oplus 2) \oplus (0 \oplus 1 \oplus 2 \oplus 3) = 0 \oplus 1 \oplus 1 \oplus 2 \oplus 2 \oplus 3$

$\dim(\text{LHS}) = 2 \times 3 \times 4 = 24$

$\dim(\text{RHS}) = 1 + 3 + 3 + 5 + 5 + 7 = 24$ ✓

d) i) $\frac{1}{2} \otimes \left(1 \otimes \frac{3}{2}\right) = \frac{1}{2} \otimes \left(\frac{1}{2} \oplus \frac{3}{2} \oplus \frac{5}{2}\right)$

$= \left(\frac{1}{2} \otimes \frac{1}{2}\right) \oplus \left(\frac{1}{2} \otimes \frac{3}{2}\right) \oplus \left(\frac{1}{2} \otimes \frac{5}{2}\right)$

$= 0 \oplus 1 \oplus 1 \oplus 2 \oplus 2 \oplus 3$ = part (c) above.

ii) $\left(\frac{3}{2} \otimes \frac{1}{2}\right) \otimes 1 = (2 \oplus 1) \otimes 1 = (2 \otimes 1) \oplus (1 \otimes 1)$

$= 1 \oplus 2 \oplus 3 \oplus 0 \oplus 1 \oplus 2$. ✓ (Same as part (c))

e) $\hat{H} = g \frac{eB}{2mc} \cdot \hat{S}_x \qquad \omega = \frac{geB}{2mc}$ (CGS)

$\hat{U}(t) = e^{-i\hat{H}t/\hbar} = e^{\frac{i\omega t}{2}\sigma_x}$

$$U(t) = \begin{pmatrix} \cos\left(\frac{\omega T}{2}\right) & -i\sin\left(\frac{\omega T}{2}\right) \\ -i\sin\left(\frac{\omega T}{2}\right) & \cos\left(\frac{\omega T}{2}\right) \end{pmatrix}$$

$$\Psi(t) = U(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\omega T}{2}\right) \\ -i\sin\left(\frac{\omega T}{2}\right) \end{pmatrix}$$

Spin points in $+z$ direction when

$$\frac{\omega T}{2} = n\pi \qquad n = 0, \pm 1, \pm 2, \ldots \qquad \text{Why?}$$

$$\Rightarrow \quad f = \frac{\omega}{2\pi} \cong \frac{eB}{2\pi m c} \quad (\text{CGS})$$

$$= \frac{eB}{2\pi m} \quad (\text{MKS}) \qquad \frac{\cancel{\text{Kg}}\cdot \text{m}/\text{s}^2}{\cancel{\text{J}/\text{m}}}$$

$$= \frac{(1.6\times 10^{-19}\ \cancel{\text{Coul}})\left(1\text{T} = 1\,\frac{\text{N}\cdot\text{s}}{\cancel{\text{C}\cdot\text{m}}}\right)}{2\pi\cdot(9.11\times 10^{-31}\ \cancel{\text{Kg}})}$$

$$= 28\ \text{GHz}$$