Physics 206 April 15, 2002

Second Midterm Exam
20 points total

9:00 am – 11:00 am

Calculators may be used, but no books or other written materials are permitted. Show the steps leading to your answers. Credit will be given based on evidence of your understanding of the material. Be sure to express all quantities used in the proper physical units.

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Possibly Useful Information

$$\begin{aligned}
h &= 6.63\times10^{-34}J\cdot s = 4.14\times10^{-15}eV\cdot s\\
\hbar &= h/(2\pi) = 1.05\times10^{-34}J\cdot s = 1.05\times10^{-27}erg\cdot s\\
m &= 9.11\times10^{-31}kg = 9.11\times10^{-28}g\\
G &= 6.67\times10^{-11}N\cdot m^2/kg^2\\
e &= 1.602\times10^{-19}C = 4.803\times10^{-10}esu\\
\frac{1}{4\pi\varepsilon_0} &= 8.98\times10^{9}N\cdot m^2/C^2\\
c &= 2.998\times10^{8}m/s\\
\mu_0 &= 4\pi\times10^{-7}Wb/(A\cdot m)\\
1eV &= 1.602\times10^{-19}J = 1.602\times10^{-12}erg\\
1\mathring{A} &= 10^{-10}m\\
1\text{ tesla} &= 1\ N\cdot s/(C\cdot m)\\
\alpha &\equiv \frac{e^2}{\hbar c}\approx 1/137
\end{aligned}$$

Read this: The grading of the problems will be based on (1) qualitative correctness, (2) quantitative correctness of your **numerical** answers (where required) and (3) your willingness to check the cancellation of units by explicit calculation (where required). Needless to say, show your work; all quantities must have units attached to them, unless they are dimensionless. Cancel out units to the full extent possible, ie. the combination $N/(kg\cdot m/s^2)$ is actually *dimensionless*. Hint: Ask yourself, "is my answer physically reasonable?"

1. (10 points total) The Zeeman effect, re-examined. In class we studied the strong-field Zeeman effect in the hydrogen atom, where the word "strong" here meant that the coupling of the orbital and spin degrees of freedom to an external magnetic field,

$$\hat{H}_{Zeeman} = \frac{e}{2m}\,(\vec{L} + 2\vec{S}) \cdot \vec{B}\ , \tag{1}$$

was so much larger than the spin-orbit coupling,

$$\hat{H}_{so} = E_{so}\frac{\vec{L}\cdot\vec{S}}{\hbar^2}\ , \tag{2}$$

that we ignored the spin-orbit term.

In this problem you will analyze what happens instead when $|\vec{B}|$ is somewhat smaller, so that we should consider the effect of $\hat{H}_{so}$ perturbatively to first order.

a) (3 points) Begin by diagonalizing $\hat{H}_{Zeeman}$ which you may view as the unperturbed Hamiltonian. What good quantum numbers describe the eigenstates of $\hat{H}_{Zeeman}$? What are the corresponding eigenenergies? You may set $\vec{B} = B\hat{z}$ without loss of generality.

b) (4 points) Now consider the full problem $\hat{H} = \hat{H}_{Zeeman} + \hat{H}_{so}$ by treating $\hat{H}_{so}$ as a perturbation. What is the first-order correction to the eigenenergies that you found in part (a)? What serves as a small, dimensionless, parameter that controls the accuracy of your solution?

c) (3 points) Consider an electron in a 2p-orbital. You showed on the last midterm that the constant E_{so} was given by:

$$E_{so} = \frac{\alpha^4}{48}\,mc^2 \tag{3}$$

For what size of magnetic field does the spin-orbit splitting become comparable to the Zeeman splitting? In other words, when does your first order perturbative calculation break down? I'm looking for a real number here, not an abstract algebraic expression. Hint: $\mu_B \equiv e\hbar/(2m) = 5.788 \times 10^{-5}$ eV / T.

2. (10 points total) Go beyond the dipole approximation to study "forbiddden" transitions. Consider a particle of mass m moving in a one-dimensional simple harmonic oscillator (SHO) potential of oscillator frequency ω. It's unperturbed Hamiltonian is:

$$\hat{H}_0 = \frac{p^2}{2m} + \frac{m}{2}\,\omega^2 x^2\ . \tag{4}$$

In the presence of a *classical* longitudinal forcing field the system is perturbed by the addition of the term:

$$\hat{H}' = C\cos(kx - \omega_0 t)\ x \tag{5}$$

where C is a constant that has the physical units of force, ie. [C] = N in MKS units.

In the dipole approximation we ignore the spatial dependence of the longitudinal forcing field. This approximation is justified when the particle oscillates over a region of space much smaller than the wavelength $\lambda = 2\pi/k$ of the field. In this problem, however, you'll go beyond the dipole approximation.

a) (3 points) First, assume $|kx| \ll 1$. Show that to first order in kx the longitudinal forcing field $C\cos(kx - \omega_0 t)$ may be expanded as:

$$C * [\cos(\omega_0 t) + kx\sin(\omega_0 t)] + O[(kx)^2]\ . \tag{6}$$

b) (2 points) Now argue that, to first order in C, the term $C\cos(\omega_0 t)$ induces regular dipole transitions between SHO eigenstates $|n\rangle \leftrightarrow |n+1\rangle$. Likewise, argue that the term $C\ k\ x\ \sin(\omega_0 t)$ induces "forbidden" transitions of the type $|n\rangle \leftrightarrow |n+2\rangle$.

c) (3 points) Use Fermi's Golden Rule to find the transition rate for **both** the regular dipole, and the forbidden, transitions. Hints: Don't forget to include the Dirac δ-function. Remember that $\omega_0 = vk$ where v is the wave speed, and

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\ x + i\sqrt{\frac{1}{2m\omega\hbar}}\ p\ . \tag{7}$$

d) (1 point) **Important**: show that your expressions in part (c) for the transition rates are dimensionally correct.

e) (1 point) The forbidden transition rate is suppressed relative to dipole transitions by a dimensionless prefactor of $(\hbar\omega)/(mv^2)$. What is the physical meaning of this factor?