PH206

Second Midterm

Solution

April 15, 200?

1. (a) Eigenstates $\boxed{|l, l_z; s, s_z\rangle}$ 4 good quantum #'s

Eigenenergies $E = \frac{e\hbar}{2m}(l_z + 2s_z)B$

$= \boxed{\mu_B B (l_z + 2s_z)}$

where $l_z = -l, -l+1, \ldots 0, 1, \ldots l$

(b)

$$E = \mu_B B (l_z + 2s_z) + \frac{E_{SO}}{\hbar^2} \langle l, l_z; s, s_z | \vec{L}\cdot\vec{S} | l, l_z; s, s_z\rangle$$

But, $\langle l, l_z; s, s_z | \vec{L}\cdot\vec{S} | l, l_z; s, s_z\rangle$

$= \langle l, l_z | \vec{L} | l, l_z\rangle \cdot \langle s, s_z | \vec{S} | s, s_z\rangle$

$= l_z s_z \cdot \hbar^2$

So $E \cong \mu_B B (l_z + 2s_z) + E_{SO}\, l_z s_z$

NOTE: THERE IS NO NEED TO USE $\vec{J} \equiv \vec{L} + \vec{S}$!!

Small parameter : $\dfrac{E_{so}}{\mu_B B} = \dfrac{\alpha^4 mc^2}{48\,\mu_B B}$

Why don't we have to use degenerate perturbation theory? Because all off-diagonal matrix elements of $\hat{H}_{so}$ vanish! For example, state with $l_z = 1$ and $S_z = 1/2$ is degenerate at lowest order (just Zeeman energy) with state of $l_z = -1$, $S_z = +1/2$. But these states differ by $\Delta l_z = 2$. So,

$$\langle l, 1 | \vec{L} | l, -1 \rangle = 0$$

Therefore ~~all~~ our perturbation $\hat{H}_{so}$ is already diagonal in the $|l, l_z; s, S_z\rangle$ basis!

c) Breakdown occurs when small parameter $= O(1)$.

$$B = \frac{\alpha^4 mc^2}{48\mu_B} = \frac{(1/137)^4 \cdot (0.51 \times 10^6 \cancel{eV})}{48 \cdot (5.8 \times 10^{-5} \cancel{eV}/T)}$$

$$\boxed{= 0.5\,T = 5{,}000 \text{ gauss}}$$

2. a) $\cos(A-B) = \cos A \cos B + \sin A \sin B$

$$\Rightarrow \cos(kx - \omega_0 t) = \underbrace{\cancel{\cos(kx)}}_{\approx 1} \cos(\omega_0 t) + \underbrace{\cancel{\sin(kx)}}_{\approx kx} \sin(\omega_0 t)$$

$$\cong \cos(\omega_0 t) + kx \sin(\omega_0 t) + \mathcal{O}((kx)^2).$$

b) $\hat{x} \sim \hat{a} + \hat{a}^{+}$

So $\hat{H}'_{(0)}(t) = \bar{C} \cos(\omega_0 t)\, \hat{x}$ has non-zero matrix elements $\langle n | \hat{H}' | n+1 \rangle$

$\hat{H}'_{(2)}(t) = C k \hat{x}^2 \sin(\omega_0 t)$ has non-zero matrix elements $\langle n | \hat{H}' | n+2 \rangle$.

c) More precisels, $\hat{x} = \sqrt{\dfrac{\hbar}{2m\omega}}\,(\hat{a} + \hat{a}^{+})$.

+d)

$$R_{n \leftrightarrow n+1} = \frac{1}{4} \frac{2\pi}{\hbar} \left| \langle n+1 | \hat{H}' | n \rangle \right|^2 \delta(\hbar\omega - \hbar\omega_0)$$

$\cos(\omega_0 t)$ removed

due to $\cos(\omega_0 t) = \frac{1}{2}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right)$

So $R_{n\leftrightarrow n+1} = \frac{\pi}{2\hbar} \cdot \frac{\hbar}{2m\omega} \cdot c^2 \cdot \underbrace{|\langle n+1|\hat{a}^\dagger|n\rangle|^2}_{(\sqrt{n+1})^2} \delta(\hbar\omega-\hbar\omega_0)$

$$= \boxed{\frac{\pi}{4m\omega} c^2 (n+1)\, \delta(\hbar\omega - \hbar\omega_0)}$$

Check dimensions: $[R] = \frac{1}{kg/s} \cdot (kg\ m/s^2)^2 \cdot \underbrace{\frac{1}{kg\ m^2/s^2}}_{J^{-1}}$

$$= s^{-1} \checkmark.$$

$$R_{n\leftrightarrow n+2} = \frac{\pi}{2\hbar} \cdot c^2 K^2 \cdot \left(\frac{\hbar}{2m\omega}\right)^2 \cdot |\langle n+2|(\hat{a}^\dagger)^2|n\rangle|^2$$

$$\times\, \delta(2\hbar\omega - \hbar\omega_0)$$

$$= \boxed{\frac{\pi}{8} \frac{c^2 K^2 \hbar}{m^2\omega^2} (n+2)(n+1)\, \delta(2\hbar\omega - \hbar\omega_0)}$$

$$[R] = (s^{-1}) \cdot \frac{[K^2\hbar]}{[m\omega]} = \frac{\cancel{m^{-2}} \cdot \cancel{kg}\, \cancel{m^2}/s}{\cancel{kg}/s^2} = s^{-1}$$

e) $\frac{R_{n\leftrightarrow n+2}}{R_{n\leftrightarrow n+1}} \sim \frac{K^2\hbar}{m\omega} \sim \frac{(\omega_0/v)^2\hbar}{m\omega} \sim \boxed{\frac{\hbar\omega}{mv^2}}$

since $\omega_0 = (1 \text{ or } 2) \times \omega.$

Meanins of $\dfrac{\hbar\omega}{mv^2} \ll 1$:

Size of oscillations $\sim \sqrt{\langle \hat{x}^2 \rangle} \sim \sqrt{\dfrac{\hbar}{m\omega}}$

Wavelength of perturbing wave $\sim \dfrac{1}{k} \sim \dfrac{v}{\omega}$

$$\frac{\text{Size of oscillations}}{\text{Wavelength}} = \frac{\sqrt{\hbar / m\omega}}{v/\omega} \sim \sqrt{\frac{\hbar\omega}{mv^2}}$$

So transition rates for forbidden transition are

Suppressed by a factor of

$$\left(\frac{\text{Size of oscillation of SHO}}{\text{wavelength}} \right)^2 .$$