Fiscal Expansions in Secular Stagnation:
What if it isn’t Secular? *

Vaishali Garga †

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Abstract

Low natural real interest rates limit the power of monetary policy to revive the economy due to the zero lower bound (ZLB) on the nominal interest rate. Fiscal stabilization via higher public debt is frequently recommended as a policy alternative at the ZLB as it trends to raise the natural real interest rate. This paper builds a non-Ricardian framework to study the costs and benefits of increasing public debt during a ZLB episode. It shows that the effect of debt is highly non-linear. Increasing debt raises the natural real rate of interest at low levels of debt, while at high levels it perversely decreases the natural real interest rate, thereby further complicating the use of monetary policy to fight recessions. The mechanism of the perverse effects relies on output distortions created by expectation of high debt burden in possible future states of the world in which the natural real interest rate has normalized and ZLB is no longer binding. These distortions arise from an increase in distortionary taxes, crowding out of productive capital, and the possibility of sovereign default. The threshold level of debt, beyond which the effect becomes perverse is a function of the duration of the ZLB episode. In a calibrated 60 period quantitative lifecycle model with aggregate uncertainty, if the ZLB episode is expected to last for 1.6 years, which corresponds to the average length of a recession in the US, this threshold level of debt is at 106% of the GDP. The insights from the paper are applicable to a low real interest rate environment even in a purely real model away from the ZLB.

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†Ph.D. Candidate, Department of Economics, Brown University. E-mail: vaishali_garga@brown.edu
1 Introduction

“There has been a significant decline in the long-run neutral real interest rate in the United States over the past few years. This decline in the long-run neutral real interest rate increases the future likelihood that the FOMC will be unable to achieve its objectives because of financial instability or because of a binding lower bound on the nominal interest rate. Plausible economic models imply that the fiscal authority can mitigate this problem by issuing more public debt, although such issuance is not without cost.”

- Narayana Kocherlakota (2015), Former President of the Federal Reserve Bank of Minneapolis

As shown in Figure 1a, there has been a persistent decline in the real interest rates in the US over the past three decades. A similar trend emerges when one looks at the yield on 10-year Treasury Inflation-Indexed Security of constant maturity. This has led to concerns that the economy may be in secular stagnation, defined as an era of low real interest rates with no force towards their recovery.

Low real rates complicate the use of monetary policy to fight recession due to the zero lower bound (ZLB) on the nominal interest rate. This is because the central bank’s main policy instrument is the nominal interest rate, which it cuts during recessions in order to stimulate the economy. The nominal interest rate is on average, equal to the real rate plus the central bank’s inflation target. With low real rates in normal times, the average nominal interest rate in normal times is lower, thereby giving monetary policy less room to reduce it in the event of a recession. Figure 1b shows the cut in the Federal Funds Rate (FFR) in response to the last four recessions as identified by the NBER (gray bars in the figure). In the early 1980’s, the Fed had room to cut the nominal rate from 18% to 9% (900 basis points), in early 1990’s from 10 to 3 (700 basis points), in early 2000’s from 6.5 to 1 (550 basis points) and in 2008 from 5.25 to 0 (525 basis points). Currently, the most the Fed could cut the FFR is 250 basis points.

The decline in the nominal rate over the first part of this period was due to declining inflation expectations. However, the most recent decline in the nominal rate is driven by the decline in the real rate, which even today, with the US arguably approaching full employment, remains around 0%.

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1Here, the real interest rate is calculated as the difference between Federal Funds Rate and expected inflation. Expected inflation is computed as a 4-quarter moving average of actual inflation.

2Note that for this paper, I take secular stagnation to refer simply to a period of persistently low real interest rates. Numerous other definitions have appeared in the literature. Hansen (1939) first coined the term to suggest that the slumping economies of the 1930’s were doomed to stagnation by poor growth prospects, a product of slowing innovation and aging populations. Summers (2013) resurrected the term to refer to protracted stretches of growth that are well short of previous trends or estimates of potential along with incompatibility between full employment and financial stability.
If low real rates represent a permanent change - a “new normal” - many suggest that fiscal policy is needed to restore the stabilizing capacity of monetary policy. One possibility is to use public debt policy to manipulate the real interest rate. Higher debt in most models, as long as they are not fully Ricardian, directly increases the real rate of interest which, in turn, gives monetary policy more room.\(^3\)

There are, however, two potential problems with the strategy of raising debt, raising the real rate and thus, getting out of the problem of secular stagnation. First, it may be that the secular stagnation hypothesis is incorrect - that the low real rates are not a permanent phenomenon but are just transitory - and that real rates will normalize in the future. Then if we issue all this debt now, we would find ourselves in a situation of having accumulated a lot of debt, which needs to be financed at positive real rates. The second problem is that if people begin to anticipate the first problem, then debt may not have a stimulative effect even today when real interest rates are low. This is because in the state of the world where real rates have normalized, accumulated debt will impose costs on the economy by triggering forces that lower output (distortionary taxes, for example). The expectation of these costs may then induce further savings against this potential state and reduce the expansionary effect of debt even today. If these costs are severe enough, then the precautionary savings generated by higher debt may be so large that it makes debt contractionary.

The paper focuses on analyzing the effect of debt in a model where the real interest rate in the equilibrium allocation with perfectly flexible wages and prices (called the natural real interest rate) is negative, with the understanding that in the presence of nominal frictions, this would imply that the ZLB on the nominal interest rate binds.\(^4\) With the

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\(^3\)An alternative is to increase the central bank’s inflation target, which reduces the chances of hitting the ZLB in the future. However, it is not without costs such as increased relative price dispersion and inflation volatility (Coibion, Gorodnichenko and Wieland (2012)), and also risks jeopardizing the credibility of monetary policy, which has worked hard to successfully establish a credible nominal anchor of 2%.

\(^4\)Following this, the terms ‘negative natural real interest rate in the absence of fiscal policy’ and ‘binding ZLB’ may be used interchangeably.
real interest rate unable to fall to the level of the market-clearing natural real interest rate, due to the binding ZLB, there will be an output contraction in the economy. The results of the paper, however, apply more generally to a low real interest rate environment (not necessarily negative) even in a purely real model away from the ZLB. In this sense, the paper answers a more general question: Does increasing debt increase the real interest rate in the economy? And the answer, for the reasons discussed above, is no, not always.

A case in point is the Japanese economy. It has experienced a seemingly secular decline in the real interest rate over the past three decades. Over the same period, the government has, in response, nearly quadrupled its debt to GDP ratio from 50% to 230% (Figure 2). But the economy continues to be in stagnation with low real rates. The hypothesis in this paper would suggest that this is because people are highly aware and worried about the potential burden of the extremely high debt, in the event that real rates rise in the future.

Survey evidence from Japan provides suggestive evidence in support of this hypothesis. According to a household survey conducted by the Japanese cabinet office in 2016, an increasing number of Japanese ‘feel worries or anxieties in everyday life’ and around a third of them report ‘fiscal balance’ as the reason for these anxieties (see Figure 13 in Appendix A). Further, there is evidence that high debt is often associated with expectation of future tax increases and potential unsustainability of debt. Figure 14 in Appendix A, taken from Kobayashi, Ueda et al. (2017), shows the number of occurrences of specific words in the morning and evening editions of the Nihon Keizai Shinbun, Japan’s financial newspaper, for the year 1981 to 2016. The count suggests that the frequency of usage of “fiscal failure (zaisei hatan)” or “fiscal crisis (zaisei kiki)”, words that reflect a fiscal imbalance, has increased continuously since 1981. Moreover, a combination of the words “tax increases (zozei)” with either “fiscal failure” or “default” has been used more
and more frequently too. Masayuki et al. (2017) provides further survey evidence which shows that there is considerable uncertainty over the future course of social security and tax system in Japan, and that this uncertainty suppresses consumption.

This paper shows that given the uncertainty regarding the path of future real rates emanating from unforeseeable forces, government debt may not be the silver bullet to solve the problem of persistently low real rates, as it is made out to be. In analyzing the effectiveness of public debt in raising low real interest rates, it is important to account for the trade-offs created by future contingencies where real rates are high and government debt is burdensome. The paper analyzes these trade-offs in a non-Ricardian framework of overlapping generations, both theoretically and quantitatively.

The main theoretical findings are the following. First, given the existing stock of debt, the effect of increasing government debt on the natural real rate of interest depends on the expected duration of the low state, defined as the state where natural real interest rate in the absence of any fiscal policy intervention is negative (and the ZLB binds). Second, conditional on this expected duration, the effect is highly non-linear - increasing debt raises the natural rate of interest at low levels of debt, while at high levels it perversely reduces the natural real rate of interest.

The mechanism relies on output distortions created by debt in the state of the world where the natural real interest rate in the absence of fiscal policy intervention becomes positive (and the ZLB no longer binds). Expectation of this future contingency where disposable income is lower feeds into the current savings decision of forward-looking agents and induces them save. Section 2 first builds an illustrative real model with perfectly flexible wages and prices and an exogenous output cost of debt to derive these results analytically and develop intuition. It then extends the model by adding nominal frictions. Section 3 extends the illustrative real model to allow for endogenous labor supply and capital accumulation decisions, which in turn endogenizes the distortionary effects of debt on output. Section 4 incorporates the possibility of sovereign default risk into the model, which makes debt even more distortionary in the future contingency.

In a calibrated 60-period quantitative lifecycle model, with aggregate uncertainty regarding the path of future natural real interest rates in the absence of fiscal policy intervention, presented in Section 5, I find that if the expected duration of the low state is 1.5 years (or less), which corresponds to the average length of a recession in the US, then increasing debt beyond its current level of 106% of GDP will - perversely - reduce the natural real interest rate. If the expected duration is greater than 1.5 years, then there is room to use debt policy to fight low natural real rates. How much room there is, depends on the expected duration of the low state. If the expected duration is 2 years, for example, then increasing debt upto 186% of GDP will increase the natural real interest rate, but further
increases in debt will be perverse.

**Related Literature**

This paper contributes to several existing literatures. First, it extends our understanding of effectiveness of public debt as a stabilization policy tool when real rates are seemingly permanently negative. A recent literature on secular stagnation has proposed an increase in public debt as a policy recommendation to the problem of persistently low real rates (Eggertsson and Mehrotra (2014), Summers (2015), Kocherlakota (2015)). In a quantitative lifecycle model, Eggertsson, Mehrotra and Robbins (2017) show that debt would need to double from 118% to 215% of GDP to increase the real rate from -1.47% to 1%. Such a large increase in debt, however presents risks to the economy, due to the possibility that real rates may rise in the future. Using data from 1870 for advanced economies, Mehrotra (2017) finds that despite current conditions of $r < g$, there is a moderate probability of reversion to conditions with $r > g$ over a 5 or 10 year horizon. A sharp rise in interest rates can quickly worsen debt dynamics and trigger the need for a fiscal consolidation. The model in this paper takes into account these risks and the trade-offs created by them, to show that higher public debt may not be stimulative even when real rates are negative.

Second, this paper contributes to the New Keynesian DSGE literature on the effectiveness of fiscal policy when the zero lower bound on the nominal interest rate binds. A large number of papers (Christiano, Eichenbaum and Rebelo (2011), Woodford (2011), Eggertsson (2011), among others) have documented that at zero rates, government spending is overexpansionary as in the absence of inflationary pressures, monetary policy does not undo the effect via increases in the nominal rate. Denes, Eggertsson and Gilbukh (2013) qualify this effect, and show that even at zero interest rates, running budget deficits may not necessarily be expansionary. The final effect depends on the interaction of current fiscal policy with expectations about long-run taxes and spending. In a similar spirit, Erceg and Lindé (2014) show that the size of the fiscal multiplier depends on the size of the fiscal stimulus, as the duration of the binding ZLB is endogenous to the stimulus size - while the multiplier is high for small increases in spending, it is substantially smaller at high spending levels. These papers, however, analyze a class of models where the steady state real rate is pinned down by the discount factor of the household/savers. As such, these models cannot accommodate a ZLB steady state with permanently negative real rates. By using an overlapping generations model, my paper allows for the possibility of secular stagnation to take this analysis further. I also incorporate default risk into my analysis.

Third, this paper contributes to the literature that shows that fiscal policy effects can depend on the initial level of public debt. Specifically, while at moderate levels of public
debt the effects of fiscal policy are of Keynesian style, they reverse into contractionary effects at extreme levels of public debt. The contribution of this article is to provide a rationale for this state-dependence, and show why it may be important to account for it, even at negative real rates. On the theoretical front, Sutherland (1997) has a similar finding in model with an assumed stochastic process of government debt, which creates uncertainty about future fiscal policy. On the empirical front, various authors have estimated a threshold debt-GDP ratio in the range of 80-100% (see Perotti (1999), Nickel and Tudyka (2013)). This paper provides quantitative estimates of this threshold, conditional on the expectations of agents regarding the future state of the economy.

2 Illustrative model: Exogenous output cost of debt

In this section, I build a theoretical model to illustrate the *perverse* effect of debt, that is, to show under what conditions, an increase in debt can reduce the real rate in a frictionless environment. At the end of the section, I will introduce nominal rigidity into the model and show how this perverse effect translates to a recession in output in the presence of nominal frictions.

Before delving into the details of the model, it is important to clarify some terminology that will be used in the rest of the paper.

**Definition 2.1 (Benchmark Real Rate).** The real interest rate in the frictionless (flexible price/wage) economy without fiscal policy intervention, that is, when fiscal policy is at its baseline steady state.

**Definition 2.2 (Natural Real Rate).** The real interest rate in the in the frictionless economy with fiscal policy intervention, that is, when fiscal policy deviates from its baseline steady state.

There are two key elements relative to the common exposition of OLG models: One, a stochastic process driving the underlying benchmark real rate in the economy, and two, a careful specification of fiscal policy. The stochastic process is such that the benchmark real rate is negative today, but reverts to positive territory with a positive probability - as seems plausible- in every period. Once it becomes positive, it stays at that level forever (assumption 1). On the fiscal policy front, the government chooses an exogenous level of purchases and stock of debt, while taxes are determined endogenously to balance the government’s budget constraint in every period (assumption 2).
2.1 Households

A household lives for two periods: middle age and old age, and maximizes utility from consumption of one aggregate good according to the following utility function.

\[
U_t(c_m^t, c_o^{t+1}) = \max_{c_m^t, c_o^{t+1}} E_t \left\{ \frac{(c_m^t)^{1-\gamma}}{1-\gamma} + \beta D_t \frac{(c_o^{t+1})^{1-\gamma}}{1-\gamma} \right\}
\]

s.t. \[c_m^t = w_t L_m^t - T_m^t - B^g_t + \theta Z_t\]
\[c_o^{t+1} = w_{t+1} L_o^{t+1} - T_o^{t+1} + (1 + r_t) B^g_t + (1 - \theta) Z_{t+1}\]

where, \(c_m^t\) is consumption when middle-aged, \(c_o^{t+1}\) is consumption when old, \(L_m^t\) & \(L_o^t\) are the labor endowments of the middle aged and old, respectively, \(\beta \in (0, 1)\) is the discount factor of the households, \(D_t\) is an inter-temporal wedge, \(\gamma > 0\) is the coefficient of relative risk aversion, \(B^g_t\) is one period risk-free government debt and \(Z_t\) are profits from firm-ownership. I assume that the total labor endowment is divided between the middle-aged and old according to \(L_m^t = \theta \bar{L}\) & \(L_o^t = (1 - \theta) \bar{L}\), and the inter-generational profit distribution is governed by \(\theta_z\).

The shock \(D_t\) plays a crucial role in the analysis. In particular, I am interested in studying the effect of government debt in an economy where the benchmark real rate is negative. In this model, a natural cause of the negative benchmark real rate is an increase in the inter-temporal preference wedge \(D_t\). A higher \(D_t\) implies that households will derive higher utility from each unit of future consumption, which will incentivize them to save more from the same income. This will reduce current aggregate demand, and hence, the real interest rate. If the shock to \(D_t\) is severe enough, the benchmark real rate would become negative.

The intertemporal decision of consumption versus saving for the middle aged is given by the Euler equation:

\[E_t c_o^{t+1} = [\beta D_t (1 + r_t)]^{\frac{1}{\gamma}} c_m^t\]

Households like to smooth consumption across their lifetime, and this smoothing motive depends on their discount factor \(\beta D_t\), the risk-free real rate \(r_t\) and on the coefficient of relative risk aversion \(\gamma\).

The consumption of the old is given by their budget constraint:

\[c_i^o = w_i L_i^o - T_i^o + (1 + r_{t-1}) B^g_{t-1}\]

Combining the Euler equation with the budget constraint of the middle aged and old
gives the following equation for the consumption of the middle aged:

\[
c^m_t = E_t w_{t+1}(1 - \theta) L_{t+1} + (1 - \theta z) E_t Z_{t+1} - E_t T^0_{t+1} + \frac{(1 + r_t) [w_t \theta L_t - T^m_{t+1} + \theta z Z_t]}{[\beta D_t (1 + r_t)]^{\frac{1}{\gamma}} + (1 + r_t)}.
\]

This is an important equation in the model, as expectations will matter for the economy through their effect on the consumption of the middle-aged. As shown in the equation above, the middle-aged consumption depends on both current as well as expected future wage and profit income net of taxes, along with the discount factor and the real rate. An increase in either current or expected future disposable income increases the consumption of the middle aged, while an increase in the discount factor reduces it.

2.2 Firms

Firms use labor to produce the final output in the economy according to the following production function:

\[
Y_t = L_t^{1-\alpha}.
\]

When \( \alpha = 0 \) there are constant returns to scale in production. In this case, profits are zero and the ownership structure of the firm is irrelevant for the results. However, if \( \alpha \in (0, 1) \), then there are decreasing returns to scale, and here I assume, as mentioned previously, that the division of firm ownership between the middle-aged and the old is governed by \( \theta_z \).

Firms choose labor to maximize profits, taking wages and prices as given:

\[
\max_{L_t} Z_t = P_t Y_t - W_t L_t.
\]

This gives the inverse demand function for labor in the economy:\(^5\)

\[
w_t \equiv \frac{W_t}{P_t} = (1 - \alpha) \frac{Y_t}{L_t}.
\]

\(^5\)An implicit assumption in the firm’s optimization problem is that the firm is not allowed to purchase consumption goods directly for the household that owns it. While this assumption is irrelevant under flexible prices, it will be important under sticky prices, as it prevents the firm from creating demand for its own product to resolve the inefficiencies that arise from price stickiness. This assumption arises endogenously in models of product differentiation as firms are highly specialized in the production of one variety of good. But since I do not model this explicitly in my analysis, it seems important to clarify this underlying assumption.
2.3 Government

The government issues one-period risk-free bonds and collects lumpsum taxes to finance its debt plus purchases. These taxes are determined endogenously to maintain a balanced budget in every period.\(^6\)

\[ B_t^g + T_t^m + T_t^o = (1 + r_{t-1})B_{t-1}^g + G_t \]

2.4 Equilibrium Under Flexible Prices

I want to use this set up to analyze the effect of government debt on the economy. A natural place to start this analysis is with the behavior of firms and households, as well as fiscal policy, for the case of flexible prices.

The economy is in equilibrium when the goods market clears, which is equivalent to the asset market clearing. This requires aggregate demand to equal aggregate supply. Aggregate demand is the sum total of consumption of the two generations and government purchases. Aggregate supply depends on labor supply. Since households do not obtain any disutility from working, they are willing to supply all of their labor endowment at any wage. Thus, under flexible prices the economy is always at full employment. Any disequilibrium created by exogenous shocks is equilibrated through adjustments in the real interest rate.

The flexible-price competitive equilibrium in this economy is defined as the sequence of endogenous variables \( \{Y_t, c_t^m, c_t^o, L_t, w_t, T_t^m, T_t^o, Z_t, r_t\} \) which satisfy the following equations, given exogenous fiscal policy variables \( \{B_t^g, G_t\} \) and parameters \( \{D_t, \bar{L}\} \):

Aggregate demand:

\[ Y_t = c_t^m + c_t^o + G_t \quad (1) \]

\[ c_t^m = \frac{\mathbb{E}_t w_{t+1}(1 - \theta)L_{t+1} - \mathbb{E}_t T_t^o + (1 - \theta_z)\mathbb{E}_t Z_{t+1} + (1 + r_t) [w_t \theta L_t - T_t^m + \theta_z Z_t]}{[\beta D_t (1 + r_t)]^{1/\gamma} + (1 + r_t)} \quad (2) \]

\[ c_t^o = w_t (1 - \theta)L_t - T_t^o + (1 - \theta_z)Z_t + (1 + r_{t-1})B_{t-1}^g \quad (3) \]

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\(^6\)Assuming proportional income tax would yield similar results due to the exogeneity of labor supply. This means income taxes will not be distortionary and act as lumpsum. The benefit of assuming lumpsum taxes is that it simplifies the algebra for the analytical results.
Aggregate supply:

\[ Y_t = L_t^{1-\alpha} \]  \hspace{1cm} (4)

\[ L_t = \bar{L} \]  \hspace{1cm} (5)

\[ w_t = (1-\alpha) \frac{Y_t}{L_t} \]  \hspace{1cm} (6)

\[ Z_t = Y_t - w_t L_t \]  \hspace{1cm} (7)

Fiscal Policy:

\[ T^m_t + T^q_t = (1 + r_{t-1}) B^S_{t-1} - B^S_t + G_t \]  \hspace{1cm} (8)

An equation describing the division of tax burden across the two generations closes the model. Before specifying this division, some assumptions need to be made explicit.

There are 2 possible states of the economy - “low” state and “business-as-usual” state, driven by the intertemporal wedge \( D_t \). This wedge takes on two values: \( D_l \) and \( D_b \). \( D_b \) is normalized to 1 while \( D_l > D_b \). A higher \( D_t \) implies that households derive a higher utility from the same amount of future consumption. This leads to higher savings and reduced benchmark real rate in the economy. This is the low state, which corresponds to \( D_l \). On the other hand, a low \( D_t \) reduces savings in the economy and increases the benchmark real rate. This is the business-as-usual state, corresponding to \( D_b \).

To solve the model, I make use of a simple assumption, now common in the New Keynesian literature, based on Eggertsson and Woodford (2003).\(^7\)

**Assumption 1** (Intertemporal wedge follows a 2-state Markov process). The economy is in a constant low state solution with \( D_t = D_l \). In every period, there is a probability \( 1 - \mu \) that \( D_t \) will revert to its business-as-usual state value \( D_b \). Once the economy is in the business-as-usual state, it stays there forever. The stochastic period in which the shock reverts to its business-as-usual state value is denoted by \( \Gamma \).

For fiscal policy, I assume that:

\(^7\)A useful extension of the current analysis, which I leave for future work, is to allow the probability of exit from the low state to be a function of the stock of debt. Relatedly, Erceg and Lindé (2014) model a New Keynesian environment where the duration of the liquidity trap is determined endogenously and depends on the size of the fiscal stimulus. In their setting, it becomes crucial to distinguish between the marginal and the average responses of output and government debt.
Assumption 2 (Fiscal Policy).

\[ W_t = (1 + r_t)B_t^g = W^* \quad \forall t \]

\[ G_t = 0 \quad \forall t \]

\[ T_t^m = \theta^s_\tau [(1 + r_{t-1})B_{t-1}^g - B_t^g + G_t], \quad s \in \{l, b\} \]

\[ T_t^o = (1 - \theta^s_\tau) [(1 + r_{t-1})B_{t-1}^g - B_t^g + G_t], \quad s \in \{l, b\} \]

In the absence of any shocks, the government maintains a constant stock of interest-inclusive debt in every period. Government purchases are zero.\(^8\) Lumpsum taxes are set so that the government budget constraint is satisfied. The tax burden sharing between the middle-aged and the old is governed by the parameter \(\theta^s_\tau\), which is allowed to depend on underlying state of the economy.\(^9\)

Characterization of equilibrium

Assuming log utility \((\gamma = 1)\) together with assumptions 2 gives a simplified equilibrium in the economy. It is formally defined as the sequence of endogenous variables \(\{Y_t, L_t, T_t^m, T_t^o, r_t, W_t\}\) which satisfy the following equations, given exogenous fiscal policy variable \(\{W^*\}\) and parameters \(\{D_t\}\):

\[
Y_t = \frac{[(1 - \theta)(1 - \alpha) + (1 - \theta_2)\alpha] E_t Y_{t+1} - E_t T_{t+1}^o + [(1 - \alpha)\theta + \theta_2\alpha] Y_t - T_t^m}{(\beta D_t + 1)(1 + r_t)} + \frac{[(1 - \alpha)(1 - \theta) + (1 - \theta_2)\alpha] Y_t - T_t^o + W_t}{(\beta D_t + 1)} \tag{9}
\]

\[ Y_t = L_t^{1-\alpha} \tag{10} \]

\[ L_t = \bar{L} \tag{11} \]

\[ T_t^m = \theta^s_\tau \left[ W_{t-1} - \frac{W_t}{1 + r_t} \right], \quad s \in \{l, b\} \tag{12} \]

\[ T_t^o = (1 - \theta^s_\tau) \left[ W_{t-1} - \frac{W_t}{1 + r_t} \right], \quad s \in \{l, b\} \tag{13} \]

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\(^8\)Having positive government purchases does not alter the qualitative results. In the quantitative model, government purchases will be calibrated to match moments in the data.

\(^9\)In the baseline, \(\theta^s_\tau\) will be set to 0.5, but alternative distributions of the tax burden between the two generations do not change the qualitative results.
\[ W_t = W^* \]  \hspace{1cm} (14)

Under flexible prices, monetary policy plays no role, so I will postpone its discussion until the next section where sticky prices are introduced.

Under assumption 1, the benchmark real rate is positive in the business-as-usual state while it is negative in the low state. This means that issuing government debt leads to tax cuts in the low state, but needs to be financed through higher taxes in the business-as-usual state. While I have assumed away any distortionary effects of taxation on output in the illustrative model with lumpsum taxes, this is not the case in the real world where service payments on debt are financed through distortionary taxes. This distortion will be modeled in detail in the next section, but for now, I assume a reduced form for it according to the following equation.

**Assumption 3 (Exogenous output cost of debt).** Government debt is costly in terms of output in the business-as-usual state.\(^{10}\) This cost is of the form:

\[ Y_t = L_t^{1-\alpha} - \nu_g W_t \]

where \( \nu_g \) determines the cost of output per unit debt.

This cost will arise endogenously in models with endogenous labor supply and capital via distortionary taxation and crowding out, and with sovereign default risk. A detailed discussion of these elements is presented in the later sections.

**2.5 Policy Experiment: Permanent increase in government debt**

The goal is to understand the effect of higher debt on the economy, when the benchmark real rate is currently low (negative) but may normalize (become positive) in the future. The policy experiment I contemplate is the following: Suppose the economy is initially in the low state with the stock of government debt \( W^*_{\text{orig}} \) and taxes determined endogenously according to assumption 2. Now suppose that the government in this economy unexpectedly announces an increase in its stock of debt to \( W^*_{\text{new}} \). It also announces that it will permanently maintain debt at this new high level. I will analyze the effect of this

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\(^{10}\)In the model where taxes are distortionary, higher debt will have output effects in the low state as well. In particular, the proceeds from higher debt will lead to lower tax rates which will increase output. Accounting for this positive effect does not change the qualitative results but complicates the analytical expressions. So, I do not account for it in the illustrative model, but it will be accounted for in the main model in the following sections.
policy change on the natural real rate in the frictionless economy, and on output in the economy with nominal frictions.

To begin with, it is helpful to abstract from the transition dynamics and focus on the constant solution to establish analytical results. The constant solution involves comparing the low-state natural real interest rate in the steady state of the economy with low ($W_{\text{orig}}^*$) versus high ($W_{\text{new}}^*$) stock of debt. In other words, it is equivalent to doing comparative statics with respect to the stock of debt inherited from the past.

The following Propositions 1 & 2 characterize this constant solution.

**Proposition 1.** Suppose assumptions 1, 2 and 3 hold.

If $D = D_b$, then the economy is in the business-as-usual state with a constant solution for the real rate given by:

$$r_b = f \left( W^*, \nu \right)$$

If $D = D_l$, then the economy is in the low state with a constant solution for the real rate given by:

$$r_l = g \left( W^*, \mu, r_b, \nu \right)$$

**Proof.** Proof along with closed-form expressions in Appendix B.1.

The real interest rate is determined in the goods market such that aggregate demand equals aggregate supply. In the business-as-usual state, aggregate demand depends on the stock of debt as it determines taxes, and asset income of the old, while aggregate supply depends on debt through its exogenous output cost $\nu_s$. In the low state, aggregate demand depends on the stock of debt for the same reasons, but it also depends now on the business-as-usual state solution. This is because the middle-aged households are forward looking and expect the economy to exit to the business-as-usual state with a positive probability.

Taking the partial derivative of the constant low state real interest rate ($r_l$) with respect to the stock of government debt ($W^*$) yields the next proposition, which characterizes the effect of debt on the natural real interest rate.

**Proposition 2.** Under assumptions 1, 2 and 3, the effect of increasing government debt on the
The low state constant real rate is given by:

\[
\frac{\partial r_l}{\partial W^*} = \mu \times \text{direct effect} + (1 - \mu) \times \text{indirect effect}
\]

where, \(d\) (\(r_l, W^*\)) \(\geq 0\)

and, indirect effect = \(i\left( r_l, \mu, r_b, W^*, \frac{\partial r_b}{\partial W^*}, \nu_g \right) \leq 0 \)

and, \(d\) and \(i\) are shown in the appendix.

**Proof.** Proof in Appendix B.2.

In the low state, the effect of debt on the real interest rate is driven by two opposing forces: a direct effect which increases the real rate and an indirect effect which reduces the real rate. The direct effect captures the effect of debt if the economy stayed in the low state forever, while the indirect effect captures the effect from possible exit into the business-as-usual state. The strength of the indirect effect depends on the probability with which the economy reverts to the business-as-usual state in the next period, \((1 - \mu)\).

Given the two opposing forces driving the overall effect, there emerges a threshold probability of reversion such that if the probability of reversion is higher than this threshold, then the indirect effect via expectations dominates, and increasing debt has a net negative (perverse) effect on the real interest rate. On the other hand, if the probability of reversion is lower than this threshold, then increasing debt increases the real interest rate. Proposition 3 characterizes this threshold.

**Proposition 3.** Suppose Assumptions 1, 2 and 3 hold and \(D = D_l\). Then:

\[
\exists \mu^* = \mu^* \left( r_l, \nu_g, \frac{\partial r_b}{\partial W^*}, W^* \right) \geq 0, \text{ such that } \left\{ \begin{array}{c}
\frac{\partial r_l}{\partial W^*} < 0, \text{ if } \mu \leq \mu^* \\
\frac{\partial r_l}{\partial W^*} \geq 0, \text{ if } \mu > \mu^*
\end{array} \right.
\]

**Proof.** Proof in Appendix B.3.

The logic of all the propositions will be discussed in the numerical examples in the next subsection, where I will also show an analog of Proposition 3 for \(W^*\). In particular, I will show that fixing \(\nu\), there exists a threshold \(W^*\) such that increasing debt up to that threshold has a positive effect on the real interest rate but further increases in debt beyond this threshold have a perverse effect.

A direct corollary of the preceding proposition is that holding other parameters fixed, the threshold \(\mu^*\) is an increasing function of \(\nu_g\). A sensitivity analysis with respect to \(\nu_g\) in the numerical examples will clarify the intuition.
Corollary 1. Under assumptions 1, 2 and 3, when \( D = D_t \):

\[
\frac{\partial \mu^*}{\partial \nu_s^*} \geq 0
\]


2.6 Numerical Example

It is useful to attach some illustrative parameters to the model to be able to visualize the closed-form results and understand the logic behind them. It will also enable an analysis of the transition solution, that is the solution which incorporates the transition dynamics in response to the policy experiment described in the subsection 2.5. The details of the parameterization are provided in in Table 7 in Appendix B.7.

Result 1: Effect of debt depends on \( \mu \)

The blue graph in Figure 3 shows the numerical analog of proposition 3. The effect of an increase in debt on the low state constant real interest rate depends on the probability of being in the low state tomorrow, \( \mu \). If this probability is high, the natural real rate increases, while if this probability is low, the natural real rate falls. In the extreme case where \( \mu = 1 \), that is the economy stays in the low state forever, debt has an unambiguous positive effect on the natural real rate. \( \mu = 1 \) is the implicit assumption of the “secular” stagnation literature, thereby rationalizing its policy implication of increasing government debt to revive the real rate in the economy.

Figure 3: Effect of debt on natural real rate in the low state
The mechanism for these results can be understood by contemplating the direct and indirect effects of debt on the low state aggregate demand, which determines the low state real interest rate, as shown in Figure 4. All the panels show the change in variables relative to the low state constant solution with $W^* = W_{\text{orig}}^*$, that is, before the policy change. Let us begin by focusing on the constant solution first.

Aggregate demand is made up of the consumption of the old and consumption of the middle aged (and government spending, which has been set to zero). In partial equilibrium, that is, holding fixed the prevailing real rate, increasing debt increases the consumption of the old through two channels: one, it increases their asset income (panel (a)), and two, it reduces their taxes (panel (b)). Higher debt also increases the consumption of the middle aged by reducing their partial equilibrium taxes (panel (c)). Together these constitute the direct positive effect of debt.

But recall that the consumption of the middle aged depends not just on their current income and taxes but also on their expected future income net of taxes. Thus, there is an additional indirect effect of debt on aggregate demand in the low state, which comes from its effect on the expected future disposable income. This expectation is a weighted average of the low state and business-as-usual state solution, with the weights given by the respective probability of being in each state in the next period. In the business-as-usual state, debt reduces aggregate demand through three channels: One, in partial equilibrium, that is, holding fixed the prevailing (positive) real rate, higher debt implies higher taxes on the old. Two, higher debt increases the real rate, which further increases the taxes on the old. Three, higher debt lowers output as debt is assumed to have exogenous output cost in this state. Together these imply that higher debt reduces the net disposable income in the business-as-usual state. To the extent that the current middle-aged agents expect the economy to be in this state tomorrow (probability $(1 - \mu)$), they expect lower future income, which induces them to save more while consuming less. This is the negative indirect expectation effect of debt (panel (d)).

The strength of the indirect effect depends inversely on the probability of staying in the low state $\mu$ (conditional on the existing stock of debt). When $\mu$ is low, the negative expectations channel is stronger. Thus, the net reduction in aggregate demand is greater when $\mu$ is lower. In the absence of adjustment on the supply side, to restore equilibrium in the market, aggregate demand must increase. This happens through a reduction in the real interest rate, which makes consumption today cheaper relative to tomorrow thereby spurring demand in the economy. If $\mu$ is low enough, then aggregate demand falls so much in response to debt, that the natural real interest rate falls.

The transition solution is given by the green graph in all the figures. As is evident from Figure 3, the perverse result is stronger for the transition solution. That is, the threshold
Figure 4: Mechanisms to explain the effect of debt on the natural real interest rate in the low state
(GE = general equilibrium, PE = partial equilibrium)

\( \mu \) below which increasing debt has a perverse effect on the economy, is higher for the transition solution relative to the constant solution. The plots in Figure 4 explain why. Under the transition solution experiment, higher debt leaves the asset income of the old unchanged, which weakens the direct positive effect of debt relative to the constant solution (panel (a)). However, higher debt also implies tax rebates to households, thereby lowering the middle-aged and old taxes by more than under constant solution (panels (b) and (c)). Since the asset income of the old leads to a one-to-one increase in demand in the economy, while the reduced taxes affect demand through a multiplier which is less than 1, the overall result is that the direct positive effect of debt is weaker in the transition solution relative to the constant solution. The indirect effect, on the other hand, remains unchanged (panel (d)). Thus, the perverse result is starker.

**Result 2: Effect of debt is non-linear with respect to initial stock of debt**

The response of the natural real rate to debt depends non-linearly on the initial stock of debt. Figure 5 illustrates this non-linearity for different \( \mu \)'s.\(^{11}\) I consider small changes

\(^{11}\)This figure shows the transition solution. The constant solution looks similar with the threshold
around the initial stock of debt to derive the effect.

![Figure 5: Non-linear w.r.t. the initial stock of debt](image)

The figure shows that when $\mu$ is low, that is, the probability of exit into the business-as-usual state is high, debt has an unambiguous perverse effect on the natural real interest rate, with the effect becoming more perverse as the initial debt-GDP ratio increases. On the other extreme, when $\mu$ is high, debt has an unambiguously positive effect on the natural real interest rate, although the effect becomes less positive as the initial debt-GDP ratio increases. For a range of $\mu$ in the middle, there is a threshold level of debt such that increasing debt up to this threshold level increases the natural real interest rate, but any further increase in debt beyond the threshold - perversely - reduces the natural real interest rate. This limit ranges between 250 to 300% of GDP for $\mu \in [0.4, 0.5]$.

### 2.7 Introducing Nominal Frictions: Low-State Sticky Price

In the analysis so far, we have seen that an increase in debt can perversely lower the equilibrium natural real rate, if it triggers expectations of sufficiently low income in business-as-usual state. In this section, I show how this perverseness translates to a contraction in output in an economy with nominal frictions.

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debt-GDP ratio slightly higher for each $\mu$. 

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I use the same two-state general equilibrium model with fiscal policy as introduced in the previous section. But instead of assuming that prices are flexible in both states, I assume that the price level in the low state is fixed at $\bar{P}$, while allowing prices in the business-as-usual state to remain flexible. In particular, the $\bar{P}$ is assumed to be fixed at a level which is too high to clear the market, which creates a shortfall in aggregate demand relative to the full-employment level of output. I assume that monetary policy cannot fully offset this inefficiency on account of sticky prices, as $D_l$ is very high and the inflation target $\Pi^*$ too low, so that the zero lower bound (ZLB) on the nominal interest rate becomes binding. Thus, the economy will be demand-determined and there will be rationing of labor. In this regard, the study of the sticky-price economy will build off of a general theme which is well-known in the New Keynesian literature that when nominal interest rates are zero, the economy becomes demand-determined.

The equations describing the sticky-price equilibrium closely resemble equations (9)-(11) from the flexible-price model. The full system of equations along with its characterization with $\gamma = 1$ is presented in Appendix B.5. Relative to the system under flexible prices, there are two main differences: one, labor market equilibrium is pinned down by the demand for labor which depends on the real wage, and two, there is an active role for monetary policy.

For monetary policy, I assume that:

**Assumption 4 (Monetary policy follows Taylor rule).** The central bank sets the nominal interest rate according to the standard Taylor rule:

$$1 + i_t = \max \left\{ 1, (1 + i^*) \left( \frac{\Pi_t}{\Pi^*} \right)^\phi_{\Pi} \right\}$$

where $\phi_{\Pi} > 1$ and $i^*$ and $\Pi^*$ are parameters of the policy rule that we hold constant.

This rule states that the central bank tries to stabilize inflation around an inflation target $\Pi^*$ unless it is constrained by the zero lower bound. This implies that in the business-as-usual state $\Pi = \Pi^*$.

The effect of debt on the low state constant solution is characterized by the following proposition.

**Proposition 4.** Suppose assumptions 1, 2, 3 and 4 hold and $D = D_l$. Then an increase in government debt will lower the low-state constant solution output if:

$$\mu < \mu^* \left( 0, \nu_s, \frac{\partial r^*_b}{\partial W^*}, W^* \right)$$

This is the same function as in Proposition 3 except with \( r_i \) set to 0. With nominal frictions, a binding ZLB and \( \Pi^* = 1 \) imply that the real rate is zero, according to the Fisher equation.

The blue graph in Figure 6 shows the numerical analog of the preceding proposition. When \( \mu \) is high, increasing debt is expansionary, while if \( \mu \) is low, increasing debt is perverse as it lowers output further.

Figure 6: Effect of debt on output in the low state

The logic of the theorem follows closely to that for the case of flexible prices (see Figure 15 in Appendix A). The effect of debt on aggregate demand can again be decomposed into a direct versus an indirect effect. The direct effect of debt is to increase the consumption of the old as it constitutes their asset income. With the real interest rates stuck at zero, however, increasing debt leaves the partial equilibrium taxes of both generations unchanged. The indirect effect of debt remains unchanged relative to the flexible price economy, as prices are allowed remain flexible in the business-as-usual state.

Overall, there is a direct positive effect of debt on aggregate demand, albeit weaker than that under flexible prices, and there is an indirect negative effect of debt on aggregate demand, working through expectations of lower future income. If the expectations channel is strong enough, which it is if the probability with which the economy exits to the business-as-usual state, \( (1 - \mu) \) is high, then debt reduces aggregate demand. As the economy is demand-determined in the low state, this leads to a contraction in output. For the same reasons as in the flexible price model, this perverse effect of debt is starker in the transition solution (green graph), that is it occurs for a larger range of \( \mu \)'s, relative to the constant solution.
In the analysis so far, I have made simplifying assumptions regarding the output cost or distortionary effect of debt for the purposes of illustration. In the following sections, I will relax these assumptions and endogenize these costs in extensions with endogenous labor and capital along with distortionary taxation, and following that, also with default risk. These extensions map more readily into the real world but come at the cost of analytical tractability. The results in the following sections will all be numerical, but the logic of the arguments will follow closely to what has been discussed so far in the illustrative model. The illustrative model, thus, can be regarded as reduced form for this more complicated model. I will only present the transition solution from here onwards.

3 Extension I: Endogenous Labor and Capital

So far we have seen that increasing debt can - perversely - lower the natural real rate in the economy if it triggers expectations of sufficiently low disposable income in the future. These negative expectations are built into the model via an assumption that debt is associated with an exogenous cost in terms of output in the business-as-usual state. In this section, I relax this assumption and instead, introduce endogenous labor supply and capital, with income taxation into the model. In this setting, the output cost of debt arises endogenously.

The complete model and equilibrium are presented in Appendix D. Here, I will highlight the main differences from the illustrative model which are relevant for the main results.

Household maximize utility according to the following function:

\[
U_t(c^m_t, c^o_{t+1}, h^m_t, h^o_{t+1}, K_t) = \max_{c^m_t, c^o_{t+1}, h^m_t, h^o_{t+1}, K_t} \left\{ \frac{1}{1 - \gamma} (c^m_t)^{1-\gamma} + \beta D_t \frac{1}{1 - \gamma} (c^o_{t+1})^{1-\gamma} - \lambda \left( \frac{(h^m_t)^{1+\frac{1}{\eta}}}{1 + \frac{1}{\eta}} - \lambda \beta D_t \left( \frac{(h^o_{t+1})^{1+\frac{1}{\eta}}}{1 + \frac{1}{\eta}} \right) \right) \right\}
\]

s.t. \[
c^m_t = w_t h^m_t (1 - \tau_t) - B^S_t [1 - r^k_t (1 - \tau_t)] K_t + \theta_z Z_t (1 - \tau_t)
\]
\[
c^o_{t+1} = w_{t+1} h^o_{t+1} (1 - \tau_{t+1}) + (1 - \theta_z) Z_{t+1} (1 - \tau_{t+1}) + (1 + r_t) B^S_t + (1 - \delta) K_t
\]

where, \( h^m_t \) & \( h^o_{t+1} \) is labor supply of the middle-aged and old, respectively, \( \eta \) is the Frisch elasticity of labor, \( \tau_t \) is the income tax rate, \( K_t \) is capital, \( r^k_t \) is the rental rate on capital and \( \delta \) is depreciation. For simplicity, I assume full depreciation. As in the previous model, a shock to the intertemporal wedge \( D_t \) will generate a negative benchmark real rate.
Under full depreciation, the no-arbitrage condition between government bonds and private capital implies:

$$r_t^k = \frac{1}{1 - \tau_t}$$

We can see right away that a higher tax rate increases the rental rate on capital, which in equilibrium, implies a higher marginal product of capital, lower stock of productive capital and lower output.

On the firm side, both capital and labor are now used in production according to the following Cobb-Douglas technology:

$$Y_t = K_t^\alpha L_t^{1-\alpha}$$

On the government side, revenue comes from income (rather than lumpsum) taxation on both labor wage as well as capital rental income.

$$B^{g}_t + \tau_t w_t L_t + \tau_t r_t K_t = (1 + r_{t-1}) B^{g}_t + G_t$$

$$B^{g}_t + \tau_t Y_t = (1 + r_{t-1}) B^{g}_t + G_t$$

Figure 7 is the analog of the transition solution in Figure 3. Similar to the illustrative model, the effect of debt on the low state natural real interest rate depends on $\mu$. In particular, there is a threshold $\mu^*$, such that for $\mu < \mu^*$ increasing debt perversely reduces the natural real interest rate, while for $\mu > \mu^*$, increasing debt increases the natural real interest rate.

Similarly, Figure 8 is the analog of Figure 5. Conditional on $\mu$, the effect of debt is highly non-linear with respect to the initial debt-GDP ratio.

The logic of these results is also analogous to that in the illustrative model, but no longer relies on an exogenously assumed output cost of debt in the business-as-usual state. The direct effect of debt is to lower the partial equilibrium tax rates of both generations (Figure 9a), which results in higher consumption (and labor supply). The indirect effect is to lower expected future disposable income. This is because higher debt implies higher real rate and correspondingly higher taxes in the business-as-usual state. This is distortionary on two accounts. One, Individuals responds to the higher taxes by reducing their labor supply, which lowers output. Two, higher taxes reduce the after-tax return on capital, which leads to a reduction in capital accumulation until the marginal product of capi-

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12 The new parameters relative to the illustrative model are $\eta$ set equal to 2 and $\delta$ set equal to 1.
Figure 7: Effect on debt on the natural real rate in the low state

Figure 8: Non-linearity w.r.t. the initial stock of debt

tal rises enough to restore the no-arbitrage condition. Both these forces lower expected wage as well as asset income of the current middle-aged (see Figure 9b), who in turn respond by increasing savings and reducing their consumption and labor supply. If the probability of exit into the business-as-usual state is high enough, then the indirect effect dominates and the natural real interest rate falls.
In the business-as-usual state, higher debt needs financing at positive rates which necessitates higher taxes. A crucial parameter which determines the responsiveness of labor supply to tax changes, and hence, the distortionary effect of higher debt on output in the business-as-usual state, is the Frisch elasticity of labor supply $\eta$. With a higher $\eta$, the same increase in tax rate corresponds to a larger drop in labor supply, hence, output.

This can be seen in Table 1, which shows that as $\eta$ increases, the implied cost of debt in the business-as-usual, computed as $\nu_g = -\frac{\partial Y_b}{\partial W_b}$, also increases. Thus, with higher $\eta$, the perverse expectations channel will be stronger for each $\mu$. This, in turn, implies that the threshold $\mu^*$ is (weakly) increasing in $\eta$.

<table>
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<tr>
<th>Frisch elasticity $\eta$</th>
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<th>7</th>
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<tbody>
<tr>
<td>Implied $\nu_g$</td>
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<td>1.53</td>
<td>1.57</td>
<td>1.60</td>
<td>1.62</td>
<td>1.64</td>
<td>1.65</td>
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<tr>
<td>$\mu^*$</td>
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<td>0.54</td>
<td>0.85</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
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Table 1: Sensitivity analysis
4 Extension II: Sovereign Default Risk

Another issue with running up massive stocks debt is its potential unsustainability in the state of the world where the benchmark real rates normalize and become high. The possibility of a sovereign debt crisis makes government debt risky, thereby increasing the real interest rate that needs to be paid on it to compensate investors for the risk of holding it. In a recent report from June 2018, the CBO argues that, assuming current policies and trends are not changed, “the likelihood of a fiscal crisis in the United States would increase. There would be a greater risk that investors would become unwilling to finance the government’s borrowing unless they were compensated with very high interest rates.”

Sovereign debt is a complex issue with several important considerations. In this section, I will incorporate it in a simple way into my model, and show how it creates an additional mechanism via which debt is costly in terms of output in the business-as-usual state.

Consider now that government debt is a risky asset which pays return $r^v_t$, but with probability $\rho_{t+1}$ it will not be repaid in the future. Thus, government debt will satisfy the asset-pricing equation:

$$E_t c^0_{t+1} = [\beta D_t (1 + r^v_t) E_t (1 - \rho_{t+1})]^{\frac{1}{\gamma}} c^m_t$$

This gives rise to a no-arbitrage condition between the risky rate and the risk-free rate:

$$1 + r_t = (1 + r^v_t) E_t (1 - \rho_{t+1})$$

I assume that monetary policy bases its decision on the risk-free rate. So the existence of sovereign default does not change anything else in the model, except the government budget constraint, where the relevant real interest rate is now the risky rate:

$$\tau_t Y_t = \frac{r_t + \rho_t}{(1 - \rho_t)(1 + r_t)} W^* + G_t$$

We can see right away that a higher default risk implies a higher risk premium, which in turn implies a higher tax rate for any given level of output.

**Assumption 5 (Default Risk).** The probability of non-repayment is given by:

$$\rho_t = \rho (W_t) = d_0 exp (d_1 W_t), \quad t \geq \Gamma$$
\[ \rho_t = 0, \quad t < \Gamma \]

where \(d_0\) is the probability of default when there is no government debt and \(d_1\) is the elasticity of the probability of default with respect to the change in the stock of government debt.

I assume that the probability of non-repayment is positive only in the business-as-usual state, because there is no reason for the government to default on its payment in the low state where real rates are negative. I also assume that the probability depends positively on the stock of government debt.\(^{13}\)

To complete the model, I need to specify what happens in the economy if default actually occurs. Following the literature on sovereign default, I assume that default is costly in terms of output.

**Assumption 6 (Occurrence of Default).** In the event of a default on government debt,

\[ Y_t = Y_t - d_g W^* \]

where \(d_g\) is the output cost of default.

\(d_g\) is a reduced form parameter for the output cost of sovereign default.\(^{14}\) There is a large literature that seeks to quantify this cost, and finds that it ranges from 5-10% of GDP per year depending on the severity of the default (De Paoli, Hoggarth and Saporta (2009), Trebesch and Zabel (2017)).

Comparing the model with and without default risk in Figure 10\(^{15}\), we can see that the effect of debt depends on \(\mu\) in a similar fashion in both models, but in the model with default risk, the threshold \(\mu^*\) is higher.

This is because while the positive direct effect of debt is unchanged relative to the model without default risk (Appendix A, Figure 16a), the negative indirect effect is stronger. The possibility of sovereign default introduces a risk premium on government debt and an output cost of a possible default. The risk premium, which co-moves positively with the stock of debt, increases the real rate in the business-as-usual state more than it would have increased in the absence of the premium. Further, the output cost lowers the expected income of the middle-aged more than it would have declined in the absence of this cost. Both of these imply that there is a larger reduction in expected future income (Appendix A, Figure 16b). This induces a larger precautionary savings trigger.

\(^{13}\)This is motivated from the literature on sovereign default. Reinhart and Rogoff (2010) show that a crisis is more likely to occur as GBO’s increase. D’Erasmo and Mendoza (2016) show theoretically that default on domestic government debt is more likely when debt is larger and tax revenue is smaller.

\(^{14}\)In an alternative with a fixed 2% output cost of default per year, as in Arellano (2008) and Aguiar and Gopinath (2007), the qualitative results remain unchanged.

\(^{15}\)The new parameters are set to \(d_0 = 0.05, d_1 = 1.5\) and \(d_g = 3\). Result of a sensitivity analysis with respect to \(d_1\) are provided in Table 5 Appendix A.
This is also shown in Table 2, where for every $\eta$, the implied cost of debt in the business-as-usual state is larger in the model with default risk than in the model without.

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<th>Frisch elasticity $\eta$</th>
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<td>Implied $v^*_g$: no default</td>
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Table 2: Implied output cost of debt in the model with and without default risk

5 Quantitative Lifecycle Model with Aggregate Uncertainty

In this section, I analyze a 60-period lifecycle model with aggregate uncertainty regarding the intertemporal wedge shock following closely on Heer and Maussner (2009) and in the spirit of Auerbach and Kotlikoff (1987) and Ríos-Rull (1996). The quantitative model is used to understand how increasing public debt affects the natural real interest rate when the benchmark real interest rate is currently negative, but may become positive with a fixed probability in every period. For reasons discussed in the preceding sections,
the effect will depend on both, the probability of benchmark rate normalization and the initial debt-GDP ratio. In what follows, I will outline the main elements of the model and leave its detailed presentation and a discussion of the solution algorithm to Appendix E.

5.1 Model Summary

Households live for 60 periods. In the first 40 periods of life, they work and endogenously determine how much labor to supply based on market prices. In the last 20 periods, they are retired and receive no labor income, but receive pensions. They die with certainty at age 60. They maximize expected lifetime utility at age 1. They are born and die without assets. As in the main model, the future stream of utility from consumption and disutility from labor is discounted using the discount factor $\beta$ and intertemporal wedge $D_t$.

Production is undertaken using a Cobb-Douglas production function. Firm’s profit maximization yields the inverse demand functions for labor and capital.

The government issues one-period risk free debt in every period and collects wage income taxes to finance its purchases, pension spending and payments on last period’s debt.

5.2 Calibration

I calibrate the quantitative model to the current US conditions and use the calibrated model to derive the effect of increasing public debt on the natural real rate in the low state. I also use the model to derive a threshold debt-GDP ratio beyond which increasing debt further has a perverse effect on the natural real interest rate in the economy. This threshold value is conditional on the underlying probability of benchmark real rate normalization.

The calibration strategy chooses a set of parameters directly from the literature and another set to match moments in the data. Panel A of Table 3 shows the parameters from the literature, while Panel B shows the parameters based on data. Fiscal variables are set to match their contemporary values. Government purchases are set to 18.2%, debt to GDP ratio is to 106.7% and the pension is set to 6.8% of GDP. The income tax rate is then determined endogenously from the government’s balanced budget constraint.

\footnote{Alternatively, I could introduce an age-specific survival probability along with perfect annuity markets.}
In their survey of the literature, Reichling and Whalen (2012) suggest a range of 2 to 4 for the macro Frisch elasticity. Prominent examples include Hall (2009) where calibration exercise produces a macro Frisch elasticity of labor supply in a sticky-wage model of 1.9, and Rogerson and Wallenius (2009) whose calibration exercise in an OLG model with labor taxation produced estimates from 2.3 to 3. Smets and Wouters (2007) employ parameter estimation in a DSGE model to obtain a point estimate of 1.9. I set the elasticity to 3, which is in the middle of this range.

<table>
<thead>
<tr>
<th>Panel A: Literature</th>
<th>Symbol</th>
<th>Value</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>household discount factor</td>
<td>$\beta$</td>
<td>0.99</td>
<td>Reichling and Whalen (2012)</td>
</tr>
<tr>
<td>coefficient of relative risk aversion</td>
<td>$\gamma$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Frisch elasticity of labor</td>
<td>$\eta$</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>disutility from working</td>
<td>$\chi$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>capital share</td>
<td>$\alpha$</td>
<td>0.3</td>
<td></td>
</tr>
<tr>
<td>annual depreciation rate</td>
<td>$\delta$</td>
<td>0.1</td>
<td></td>
</tr>
<tr>
<td>working years</td>
<td>$T$</td>
<td>40</td>
<td></td>
</tr>
<tr>
<td>retirement years</td>
<td>$R$</td>
<td>20</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: Data</th>
<th>Symbol</th>
<th>Value</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>debt (% of GDP)</td>
<td>$B^g$</td>
<td>106.7 %</td>
<td>FRED</td>
</tr>
<tr>
<td>government purchases (% of GDP)</td>
<td>$G$</td>
<td>18.2 %</td>
<td>FRED</td>
</tr>
<tr>
<td>pension (% of GDP)</td>
<td>$b$</td>
<td>6.8 %</td>
<td>OECD Data</td>
</tr>
</tbody>
</table>

Table 3: Calibration Parameters

Under this calibration, the steady state real interest rate is 4.2% and the wage tax rate is 33.84%.

### 5.3 Quantitative Findings

In all the following plots, while the x-axis still denotes the probability of staying in the low state, the y-axis now shows the change in the approximate natural real interest rate, that is, the natural real rate in deviation from its steady state, in response to an infinitesimal change in the approximate government debt, that is, government debt is deviation from its steady state.

---

17 The micro elasticity is based on the intensive margin of hours worked while the macro elasticity also incorporates the extensive margin. The estimates for the micro elasticity are much smaller and range from 0.2 to 0.54 (see MaCurdy (1981), Altonji (1986) and Chetty et al. (2011) for a literature survey.)
5.3.1 Effect of debt depends on probability of benchmark real rate normalization/expected duration of the low state

As shown in Figure 11, for a given debt-GDP ratio, the effect of increasing public debt on the natural real interest rate depends on the probability of staying in the low state ($\mu$) in the next period.\(^{18}\) On one extreme, when this probability is one, that is, the economy stays in the low state forever, higher debt increases the natural real rate. On the other extreme, when this probability is zero, that is, the economy exits into the business-as-usual state with certainty in the next period, higher debt has the perverse effect of reducing the natural real rate. The threshold probability at which the sign of the effect flips is at $\mu^* = 0.38$.\(^{19}\) This corresponds to an expected duration of the low state of 1.6 years.

![Figure 11: Effect conditional on probability of staying in the low state](image)

Thus, what this means is that if we start with debt at 106.7% of GDP, which is close to what we have in the US now, and a shock hits the economy such that the benchmark real rate becomes negative, increasing debt will be successful in raising the natural rate only if agents expect the low state to last for more than 1.6 years. If expectations are otherwise, then debt policy can not be used to stimulate the economy. Infact, using debt policy in this situation would worsen the problem of low real rates.

In more extreme conditions of secular stagnation, such as those that persisted during the 2008 financial crisis, the low state may be expected to last for longer than 6 quarters. In this scenario, there may still be room to use debt policy to stimulate the economy.

\(^{18}\)Note that under assumption 1, the expected duration of the low state is $\frac{1}{1-\mu}$. See Appendix E for the derivation.

\(^{19}\)This threshold $\mu^*$ depends crucially on the Frisch elasticity of labor supply, for reasons discussed in the preceding sections. A sensitivity analysis with respect to this elasticity is relegated to Table 6 in Appendix A.
However, due to the non-linear effects illustrated in the simple model, this is likely to work only up to a limit. In the next subsection, I explore this limit.

5.3.2 Effect is non-linear w.r.t. initial debt-GDP ratio

Conditional on $\mu$, the effect of higher public debt on the natural real rate depends also on initial debt-GDP ratio. In the baseline calibration, this ratio is set to $106.2\%$. Now, I consider an alternative exercise of fixing $\mu$ and analyzing the effect of increasing debt on the natural real rate for different levels of debt.$^{20}$

As shown in Figure 12, effect of debt on the natural rate is positive and increasing up to a limit (threshold 1), beyond which it begins to decline while remaining positive. As debt is increased even further, it reaches a limit (threshold 2), beyond which any further increases in debt have a perverse effect in the economy. Once debt crosses this second threshold, increasing debt reduces the natural real rate rather than increasing it. For $\mu = 0.5$ as shown in the figure, threshold 1 occurs at $110\%$ of GDP, while threshold 2 occurs at $186\%$ of GDP.

![Figure 12: Non-linear effect of debt (expected duration of low state = 2 years)](image)

Table 4 provides the values of these thresholds for different $\mu$’s. With higher $\mu$, the thresholds are correspondingly higher. This is because as $\mu$ increases, the weight of the direct positive effect relative to that of the indirect positive effect, in the agent’s expectations, becomes larger. As such, the threshold debt-GDP ratio for the effect to be perverse must also be greater.

$^{20}$In the approximate solution, this corresponds to changing the steady state around which the system is approximated.
Table 4: Non-linear effect of debt for different $\mu$

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>Expected duration of low state</th>
<th>Threshold 1</th>
<th>Threshold 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>1.3 years</td>
<td>23.5%</td>
<td>57.8%</td>
</tr>
<tr>
<td>0.5</td>
<td>2 years</td>
<td>110%</td>
<td>186%</td>
</tr>
</tbody>
</table>

6 Conclusion

This paper analyzes builds a non-Ricardian framework to analyze trade-offs in using debt policy to revive the natural real interest rate in the economy, when the benchmark real rate is negative. It shows that the effect of debt is highly non-linear - at low levels of debt, higher debt is stimulative, but at high levels of debt, debt has a perverse effect. This non-linearity is itself a function of the expected duration of the negative benchmark real rate. In the economy with frictions, the perverse effect translates to an output contraction. A calibrated 60-period quantitative lifecycle model with aggregate uncertainty provides policy-relevant threshold values of this debt-GDP ratio, conditional on the expected duration of the negative benchmark rates.

So far, the quantitative model is purely real and does not incorporate default risk. I would like to analyze these extensions in the near future. Additionally, the paper has focused on a deficit-financed increase in taxes, that is, proceeds from debt issuance are rebated to households in the form of lower taxes. Examples of this deficit-financed tax reduction in the US are the 2001 tax rebates, the Economic Stimulus Act of 2008, and most recently, the 2018 Trump tax cuts. The model, however, is general enough that it can be used to analyze different combinations of tax-spending policies in response to increase in debt. For example, the results on debt policy effectiveness could be quite different if the proceeds were used for meaningful purchases in the economy instead of sending checks to households. A few examples are health, education or research spending by the government. These purchases could directly increase aggregate supply in the economy, thereby reducing the need for lower real rates to equilibriate the market when aggregate demand increases. Of course, if government purchases or investment were perceived to be wasteful, then the results of the paper would continue to hold. Analysis of these alternative experiments is left for future work.

It would also be interesting to empirically analyze how current fiscal policy affects expectations of agents regarding future fiscal policy, conditional on the existing level of government debt. Literature in this area is still scant. In a marginally related study, Roth and Wohlfart (2018) examine how beliefs about the debt-to-GDP ratio affect people’s atti-
attitudes towards government spending and taxation. They find that people underestimate the debt-to-GDP ratio and favor a cut in government spending once they learn about the actual amount of debt, but do not alter their attitudes towards taxation. I would like to pursue this research agenda further in future work.
References


Kocherlakota, Narayana. 2015. “Public debt and the long-run neutral real interest rate.”


A Figures & Tables

Figure 13: Concerns about Government Debt

Figure 14: The Number of Occurrences of Specific Words in Newspaper
Figure 15: Mechanisms to explain the effect of debt on the output in the low state (illustrative model) (GE = general equilibrium, PE = partial equilibrium)

<table>
<thead>
<tr>
<th>Risk elasticity $d_1$</th>
<th>0</th>
<th>1.5</th>
<th>3</th>
<th>4.5</th>
<th>6</th>
<th>7.5</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Implied $\nu_g$</td>
<td>1.6</td>
<td>1.66</td>
<td>1.74</td>
<td>1.83</td>
<td>1.93</td>
<td>2.05</td>
<td>2.19</td>
</tr>
<tr>
<td>$\mu^*$</td>
<td>0.61</td>
<td>0.62</td>
<td>0.63</td>
<td>0.65</td>
<td>0.66</td>
<td>0.68</td>
<td>0.69</td>
</tr>
</tbody>
</table>

Table 5: Sensitivity analysis (model with default risk)
Figure 16: Mechanisms to explain the effect of debt on real rate in the low state (model with default risk)

<table>
<thead>
<tr>
<th>Frisch elasticity $\eta$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu^*$</td>
<td>0</td>
<td>0</td>
<td>0.4</td>
<td>0.6</td>
<td>0.6</td>
</tr>
</tbody>
</table>

Table 6: Sensitivity analysis w.r.t. $\eta$
(quantitative model)
B Appendix: Illustrative Model

B.1 Proof of Prop 1

Proof. Business-as-usual state constant solution:

Using

\[ Y_t = L_t^{1-a} - v_s W^*, \quad L_t = \bar{L} = 1, \quad Z_t = \alpha Y_t \]

\[ 1 - v_s W^* = \frac{[(1 - \alpha)(1 - \theta) + (1 - \theta_2)\alpha] (1 - v_s W^*) - (1 - \theta^b_l) W^* \left( \frac{r_b}{1+r_b} \right)}{(\beta D_b + 1)(1+r_b)} \]

\[ + \frac{[(1 - \alpha)\theta + \theta_2\alpha] (1 - v_s W^*) - \theta^b_l W^* \left( \frac{r_b}{1+r_b} \right)}{\beta D_b + 1} \]

\[ + [(1 - \alpha)(1 - \theta) + (1 - \theta_2)\alpha] (1 - v_s W^*) - (1 - \theta^b_l) W^* \left( \frac{r_b}{1+r_b} \right) + W^* \]

Low state constant solution:

Using

\[ Y_t = L_t^{1-a}, \quad L_t = L = 1, \quad Z_t = \alpha Y_t \]

\[ 1 = \frac{[(1 - \alpha)(1 - \theta) + (1 - \theta_2)\alpha] [\mu + (1 - \mu)(1 - v_s W^*)] - [\mu(1 - \theta^b_l) W^* \left( \frac{r_l}{1+r_l} \right) + (1 - \mu)(1 - \theta^b_l) W^* \left( \frac{r_l}{1+r_l} \right)]}{(\beta D_l + 1)(1+r_l)} \]

\[ + \frac{[(1 - \alpha)\theta + \theta_2\alpha] - \theta^b_l W^* \left( \frac{r_l}{1+r_l} \right)}{\beta D_l + 1} \]

\[ + [(1 - \alpha)(1 - \theta) + (1 - \theta_2)\alpha] - (1 - \theta^b_l) W^* \left( \frac{r_l}{1+r_l} \right) + W^* \]

B.2 Proof of Prop 2

Proof. Taking the derivative of the low state constant solution with respect to \( W^* \):

\[ 0 = \frac{[(1 - \alpha)(1 - \theta) + (1 - \theta_2)\alpha] (1 - \mu) (-v_s) - [\mu(1 - \theta^b_l) \left( \frac{r_l}{1+r_l} \right) + (1 - \mu)(1 - \theta^b_l) \left( \frac{r_l}{1+r_l} \right)]}{(\beta D_l + 1)(1+r_l)} \]

\[ - \frac{[(1 - \alpha)(1 - \theta) + (1 - \theta_2)\alpha] [\mu + (1 - \mu)(1 - v_s W^*)] - (1 - \mu)(1 - \theta^b_l) W^* \left( \frac{r_l}{1+r_l} \right)}{(\beta D_l + 1)(1+r_l)} \frac{\partial r_l}{\partial W^*} \]

\[ - \frac{\mu(1 - \theta_l) W^* \left[ (1+r_l)^2 - 2r_l(1+r_l) \right]}{\beta D_l + 1} \frac{\partial r_l}{\partial W^*} - \frac{(1 - \mu)(1 - \theta^b_l) W^* \left( \frac{r_l}{1+r_l} \right)}{(\beta D_l + 1)(1+r_l)} \frac{1 + r_b - r_b}{(1+r_b)^2} \frac{\partial r_b}{\partial W^*} \]

\[ - \frac{\theta^b_l \left( \frac{r_l}{1+r_l} \right)}{\beta D_l + 1} \frac{\partial r_l}{\partial W^*} - \frac{(1 - \theta^b_l) W^* \left( \frac{r_l}{1+r_l} \right)}{\beta D_l + 1} + 1 \]

\[ - \frac{\theta^b_l W^* \left[ 1 + r_l - r_l \right]}{\beta D_l + 1} \frac{\partial r_l}{\partial W^*} - \frac{(1 - \theta^b_l) W^* \left[ 1 + r_l - r_l \right]}{(1+r_l)^2} \frac{\partial r_l}{\partial W^*} \]
Combining and simplifying a few terms:

\[
\frac{\partial r_l}{\partial W^*} = \frac{\text{numerator}}{\text{denominator}}
\]

where

\[
\text{numerator} = 1 - \frac{[(1-\alpha)(1-\theta) + (1-\theta_2)\alpha](1-\mu)\nu_x + \left[\mu(1-\theta_T)\left(\frac{r_t}{1+r_t}\right) + (1-\mu)(1-\theta_T^b)\left(\frac{r_b}{1+r_b}\right)\right] + \theta_T^l r_l}{(\beta D_l + 1)(1+r_l)}
\]

\[
+ (1-\theta_T^l)\left(\frac{r_t}{1+r_t}\right) + (1-\mu)(1-\theta_T^b)W^*\left[\frac{1+r_b-r_b}{(1+r_b)^2}\right] \frac{\partial r_b}{\partial W^*}
\]

\[
\text{denominator} = \frac{[(1-\alpha)(1-\theta) + (1-\theta_2)\alpha]\left[\mu + (1-\mu)(1-\nu_g W^*)\right] - (1-\mu)(1-\theta_T^b)W^*\left(\frac{r_b}{1+r_b}\right)}{(\beta D_l + 1)(1+r_l)^2}
\]

\[
+ \frac{\mu(1-\theta_T^b)W^*}{\beta D_l + 1}\left[\frac{1-r_t^2}{(1+r_t)^4}\right] + \frac{\theta_T^l W^*}{(\beta D_l + 1)(1+r_l)^2} + \frac{(1-\theta_T^l)W^*}{(1+r_l)^2}
\]

In the equation of the derivative expression, setting \(\mu = 1\) gives the direct effect:

\[
\left(\frac{\partial r_l}{\partial W^*}\right)\text{direct} = \frac{1 - \left[\frac{(1-\theta_T^l)\left(\frac{r_t}{1+r_t}\right) + \theta_T^l r_l}{(\beta D_l + 1)(1+r_l)}\right] + (1-\theta_T^l)\left(\frac{r_b}{1+r_b}\right)}{(1-\alpha)(1-\theta) + (1-\theta_2)\alpha + (1-\theta_T^l)W^*\left[\frac{1-r_t^2}{(1+r_t)^4}\right] + \frac{\theta_T^l W^*}{(\beta D_l + 1)(1+r_l)^2} + \frac{(1-\theta_T^l)W^*}{(1+r_l)^2}}
\]

\[-1 \leq r_l < 0 \implies \left(\frac{\partial r_l}{\partial W^*}\right)\text{direct} \geq 0\]

Collecting the terms with coefficient \((1-\mu)\) and setting \(\mu = 0\) gives the indirect effect:

\[
\left(\frac{\partial r_l}{\partial W^*}\right)\text{indirect} = \frac{\left[\frac{[(1-\alpha)(1-\theta) + (1-\theta_2)\alpha]\nu_x + (1-\theta_T^b)\left(\frac{r_b}{1+r_b}\right)}{(\beta D_l + 1)(1+r_l)}\right] + (1-\theta_T^b)W^*\left[\frac{1}{(1+r_l)^2}\right] \frac{\partial r_b}{\partial W^*}}{[(1-\alpha)(1-\theta) + (1-\theta_2)\alpha]\left[1 - \nu_g W^*\right] - (1-\theta_T^b)W^*\left(\frac{r_b}{1+r_b}\right)}\]

\[0 < r_b \leq 1 \implies \left(\frac{\partial r_l}{\partial W^*}\right)\text{indirect} \leq 0\]
B.3 Proof of Prop 3

\[ 1 - \left\{ \frac{[(1 - \alpha)(1 - \theta) + (1 - \theta)\alpha] (1 - \mu)v_g + \left[ \mu(1 - \theta^b_t) \left( \frac{r_b}{1 + r_b} \right) + (1 - \mu)(1 - \theta^b_t) \left( \frac{r_b}{1 + r_b} \right) \right] + \theta_t^l r_l}{(\beta D_t + 1)(1 + r_l)} \right\} < 0 \]
\[ + (1 - \theta_t^l) \left( \frac{r_l}{1 + r_l} \right) + \left[ \frac{(1 - \mu)(1 - \theta^b_t)W^*}{(\beta D_t + 1)(1 + r_l)} \right] \left[ \frac{1}{(1 + r_b)^2} \right] \frac{\partial r_t}{\partial W^*} \]< 0

Without loss of generality, let’s set \( \theta^l_t = 1, \theta^b_t = 0 \) & \( \theta_z = 1 \) for analytical tractability:

\[ 1 - \frac{(1 - \alpha)(1 - \theta)(1 - \mu)v_g + \left[ (1 - \mu) \left( \frac{r_b}{1 + r_b} \right) \right] + r_l + \frac{(1 - \mu)W^*}{(1 + r_b)^2} \frac{\partial r_t}{\partial W^*} }{(\beta D_t + 1)(1 + r_l)} < 0 \]
\[ \Rightarrow ((\beta D_t + 1)(1 + r_l) - \left\{ (1 - \alpha)(1 - \theta)(1 - \mu)v_g + (1 - \mu) \left( \frac{r_b}{1 + r_b} \right) \right\} + r_l + \frac{(1 - \mu)W^*}{(1 + r_b)^2} \frac{\partial r_t}{\partial W^*} ) < 0 \]
\[ \Rightarrow (\beta D_t + 1)(1 + r_l) - r_l < (1 - \mu) \left\{ (1 - \alpha)(1 - \theta)v_g + \left( \frac{r_b}{1 + r_b} \right) + \frac{W^*}{(1 + r_b)^2} \frac{\partial r_t}{\partial W^*} \right\} \]
\[ \Rightarrow (\beta D_t + 1)(1 + r_l) - r_l < (1 - \mu) \left\{ (1 - \alpha)(1 - \theta)v_g + \left( \frac{r_b}{1 + r_b} \right) + \frac{W^*}{(1 + r_b)^2} \frac{\partial r_t}{\partial W^*} \right\} \]
\[ \Rightarrow (\beta D_t + 1)(1 + r_l) - r_l < \frac{(1 - \alpha)(1 - \theta)v_g + \left( \frac{r_b}{1 + r_b} \right) + \frac{W^*}{(1 + r_b)^2} \frac{\partial r_t}{\partial W^*} }{(1 - \alpha)(1 - \theta)v_g + \left( \frac{r_b}{1 + r_b} \right) + \frac{W^*}{(1 + r_b)^2} \frac{\partial r_t}{\partial W^*} } \]

B.4 Proof of Corollary 1

From the previous proof,

\[ \mu^* = 1 - \frac{(\beta D_t + 1)(1 + r_l) - r_l}{(1 - \alpha)(1 - \theta)v_g + \left( \frac{r_b}{1 + r_b} \right) + \frac{W^*}{(1 + r_b)^2} \frac{\partial r_t}{\partial W^*} } \]

Since \( v_g \) & \( \frac{\partial r_t}{\partial W^*} \) appear in the denominator of \( \mu^* \), this gives the result.

B.5 Equilibrium Under Low-State Sticky Prices

The sticky-price competitive equilibrium in the economy is defined as the sequence of endogenous quantities \( \{Y_t, c_t^m, c_t^e, L_t, w_t, T_t^r, T_t^v, Z_t, r_t\} \) and prices \( \{W_t, \Pi_t, i_t, P_t\} \) which satisfy the following equations, given exogenous fiscal policy variables \( \{B_t^f, G_t\} \) and parameters \( \{D_t, \bar{L}, \bar{P}, \Pi^*\} \):

Aggregate demand:

\[ Y_t = c_t^m + c_t^e + G_t \quad (17) \]

\[ c_t^m = \frac{E_t w_{t+1} (1 - \theta) L_{t+1} - E_t T_t^v + (1 - \theta) Z_{t+1} + (1 + r_t) [w_t \theta L_t - T_t^m + \theta Z_t]}{[\beta D_t (1 + r_l)]^{\frac{1}{2}} + (1 + r_l)} + \frac{(1 + r_t) [w_t \theta L_t - T_t^m + \theta Z_t]}{[\beta D_t (1 + r_l)]^{\frac{1}{2}} + (1 + r_l)} \quad (18) \]
\( c_t^i = w_t(1 - \theta)L_t - T^m_t + (1 - \theta_s)Z_t + (1 + r_{t-1})B^g_{t-1} \)  

Aggregate supply:

\[
Y_t = \begin{cases} 
L_t^{1-a}, & \text{if low state} \\
L_t^{1-a} - \nu_s W^*, & \text{if business-as-usual state} 
\end{cases}
\]

\( w_t = \frac{W_t}{P_t} \)  

\( w_t = (1 - \alpha) \frac{Y_t}{L_t} \)  

\[ Z_t = Y_t - w_t L_t \]

Fiscal Policy:

\[
T^m_t + T^o_t = (1 + r_{t-1})B^g_{t-1} - B^g_t + G_t
\]

Prices:

\[ P_t = P \]

\[ 1 + r_t = \mathbb{E}_t \frac{1 + i_t}{\Pi_{t+1}} \]

\[ 1 + i_t = \max \left\{ 1, (1 + i^*) \frac{\Pi_t}{\Pi^*} \phi(t) \right\} \]

Characterizing the equilibrium with \( \gamma = 1 \) yields the following system with the sequence of quantities \( \{Y_t, L_t, T^m_t, T^o_t, r_t, W_t\} \) and prices \( \{\Pi_t, i_t, W_t\} \) which satisfy the following equations, given exogenous \( \{D_t, W^*, \Pi_t\} \):

\[
Y_t = \frac{[(1 - \theta)(1 - \alpha) + (1 - \theta_s)\alpha] E_t Y_{t+1} - E_t T^p_{t+1}}{(\beta D_t + 1)(1 + r_t)} + \frac{[(1 - \alpha)\theta + \theta_s\alpha] Y_t - T^m_t}{(\beta D_t + 1)} + [(1 - \alpha)(1 - \theta) + (1 - \theta_s)] Y_t - T^o_t + W_{t-1}
\]

\[ Y_t = \begin{cases} 
L_t^{1-a}, & \text{if low state} \\
L_t^{1-a} - \nu_s W^*, & \text{if business-as-usual state} 
\end{cases} \]

\[
T^m_t = \theta_t^s \left[ W_{t-1} - \frac{W_t}{1 + r_t} \right], \quad s \in \{l, b\}
\]

\[
T^o_t = (1 - \theta_t^s) \left[ W_{t-1} - \frac{W_t}{1 + r_t} \right], \quad s \in \{l, b\}
\]

\[ W_t = W^* \]
\[
\frac{W_i}{\mathcal{P}} = (1 - \alpha) \frac{Y_i}{L_t} \\
(33)
\]

\[
\Pi_t = \Pi^* \\
(34)
\]

\[
1 + r_t = \mathbb{E}_t \frac{1 + i_t}{\Pi_{t+1}} \\
(35)
\]

\[
1 + i_t = \max \left\{ 1, (1 + i^*) \left( \frac{\Pi_t}{\Pi^*} \right)^{\phi t} \right\} \\
(36)
\]

B.6 Proof of Prop 4

Let’s call the constant business-as-usual state solution for output and taxes \(Y_b\) & \(T_b\). In the low state,

\[
\mathbb{E}_t Y_{t+1} = \mu Y_t + (1 - \mu) Y_b \\
\mathbb{E}_t T_{t+1} = \mu T_t + (1 - \mu) T_b
\]

Substituting these into the aggregate demand equation (28) implies:

\[
Y_t = \frac{[(1 - \alpha)(1 - \theta) + (1 - \theta_2)\alpha] \left[ \mu Y_t + (1 - \mu)(1 - v_g W^*) \right] - \left[ \mu(1 - \theta_1^*) W^* \left( \frac{r_t}{1 + r_t} \right) \right]}{(\beta D_t + 1)(1 + r_t)} + \frac{[(1 - \alpha)(1 - \theta) + (1 - \theta_2)\alpha] Y_t - (1 - \theta_1^*) W^* \left( \frac{r_t}{1 + r_t} \right) + W^*}{\beta D_t + 1}
\]

In the low state, \(r_t = 0\) (using \(i_t = 0\) & \(\Pi_t = 1\) in the Fisher equation). This implies:

\[
Y_t = \frac{[(1 - \alpha)(1 - \theta) + (1 - \theta_2)\alpha] \left[ \mu Y_t + (1 - \mu)(1 - v_g W^*) \right] - \left[ \mu(1 - \theta_1^*) W^* \left( \frac{r_t}{1 + r_t} \right) \right]}{(\beta D_t + 1)} + \frac{[(1 - \alpha)(1 - \theta) + (1 - \theta_2)\alpha] Y_t - (1 - \theta_1^*) W^* \left( \frac{r_t}{1 + r_t} \right) + W^*}{\beta D_t + 1}
\]

Taking the total derivative with respect to \(W^*\):

\[
\frac{\partial Y_t}{\partial W^*} = \frac{[(1 - \alpha)(1 - \theta) + (1 - \theta_2)\alpha] \left[ \mu \frac{\partial Y_t}{\partial W^*} + (1 - \mu)(-v_g) \right] - (1 - \mu)(1 - \theta_1^*) W^* \left( \frac{r_t}{1 + r_t} \right) - \left[ \mu(1 - \theta_1^*) W^* \left( \frac{r_t}{1 + r_t} \right) \right]}{(\beta D_t + 1)} + \frac{[(1 - \alpha)(1 - \theta) + (1 - \theta_2)\alpha] \frac{\partial Y_t}{\partial W^*} + \left[ (1 - \alpha)(1 - \theta) + (1 - \theta_2)\alpha \right] \frac{\partial Y_t}{\partial W^*} + 1}{\beta D_t + 1}
\]

\[
\Rightarrow \frac{\partial Y_t}{\partial W^*} = \frac{1}{\beta D_t + 1} \left[ (1 - \alpha)(1 - \theta) + (1 - \theta_2)\alpha \right] \mu \frac{\partial Y_t}{\partial W^*} - \left[ \frac{\partial Y_t}{\partial W^*} \right] \left[ (1 - \alpha)(1 - \theta) + (1 - \theta_2)\alpha \right] - \left[ \mu(1 - \theta_1^*) W^* \left( \frac{r_t}{1 + r_t} \right) \right]
\]
This implies:

\[
\frac{\partial Y_t}{\partial W^*} < 0
\]

\[\implies 1 < \frac{[(1 - \alpha)(1 - \theta) + (1 - \theta_z)\alpha] (1 - \mu)(v_g) + (1 - \mu)(1 - \theta^b_t) \left( \frac{r_b}{1 + r_b} \right) + (1 - \mu)(1 - \theta^l_t) \left( \frac{r_b}{1 + r_b} \right)}{\beta D_t + 1}
\]

\[\implies \beta D_t + 1 < [(1 - \alpha)(1 - \theta) + (1 - \theta_z)\alpha] (1 - \mu)(v_g) + (1 - \mu)(1 - \theta^b_t) \left( \frac{r_b}{1 + r_b} \right) + (1 - \mu)(1 - \theta^l_t) \left( \frac{r_b}{1 + r_b} \right)
\]

\[\implies (1 - \mu) > \frac{\beta D_t + 1}{[(1 - \alpha)(1 - \theta) + (1 - \theta_z)\alpha] v_g + (1 - \theta^b_t) \left( \frac{r_b}{1 + r_b} \right) + (1 - \theta^l_t) \left( \frac{r_b}{1 + r_b} \right)}
\]

\[\implies \mu < 1 - \frac{\beta D_t + 1}{[(1 - \alpha)(1 - \theta) + (1 - \theta_z)\alpha] v_g + (1 - \theta^b_t) \left( \frac{r_b}{1 + r_b} \right) + (1 - \theta^l_t) \left( \frac{r_b}{1 + r_b} \right)}
\]

Without loss of generality, let’s set \(\theta^l_t = 1, \theta^b_t = 0\) & \(\theta_z = 1\) to make the result comparable to that under flexible prices. This yields the expression in Proposition 4.

### B.7 Illustrative Parameterization

\(D_b\) is normalized to 1 and \(\beta\) is parameterized to target a steady state benchmark real rate of 10% per annum in the business-as-usual state. Given \(\beta, D_l\) is parameterized to target a steady state benchmark real rate of -2% in the low state. Given the stochastic nature of the intertemporal wedge driving the benchmark real rate, there will be a different \(D_l\) for each \(\mu\). I assume that \(1 + i^* = (1 + r^\mu) \Pi^*\). Finally, I also assume that profits and taxes are shared proportionally to labor income.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\alpha)</td>
<td>capital share</td>
<td>0.3</td>
</tr>
<tr>
<td>(\beta)</td>
<td>household discount factor</td>
<td>0.14</td>
</tr>
<tr>
<td>(r_b)</td>
<td>business-as-usual state constant real rate</td>
<td>10% p.a.</td>
</tr>
<tr>
<td>(r_l)</td>
<td>low state constant real rate</td>
<td>-2% p.a.</td>
</tr>
<tr>
<td>(\bar{L})</td>
<td>total labor endowment</td>
<td>1</td>
</tr>
<tr>
<td>(\theta)</td>
<td>middle-aged share of total labor endowment</td>
<td>0.5</td>
</tr>
<tr>
<td>(\theta_z)</td>
<td>middle-aged share of profits</td>
<td>0.5</td>
</tr>
<tr>
<td>(\theta^b_t, \theta^l_t)</td>
<td>middle-aged share of taxes</td>
<td>0.5</td>
</tr>
<tr>
<td>(\gamma)</td>
<td>coefficient of relative risk aversion</td>
<td>1</td>
</tr>
<tr>
<td>(v_g)</td>
<td>output cost of debt in the business-as-usual state</td>
<td>6</td>
</tr>
<tr>
<td>(W_{\text{orig}})</td>
<td>initial debt stock</td>
<td>0.1</td>
</tr>
<tr>
<td>(W_{\text{new}})</td>
<td>new debt stock</td>
<td>0.15</td>
</tr>
<tr>
<td>(G^*)</td>
<td>government purchases</td>
<td>0</td>
</tr>
<tr>
<td>(\phi_{\Pi})</td>
<td>weight on inflation targeting</td>
<td>2</td>
</tr>
<tr>
<td>(\bar{P})</td>
<td>fixed price</td>
<td>1</td>
</tr>
<tr>
<td>(\Pi^*)</td>
<td>inflation target</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 7: Parameters
B.8  Sensitivity Analysis

In the analysis so far, I have assumed an ad-hoc output cost of debt in the business-as-usual state given by $\nu_g$ without microfounding it. In the baseline, I have chosen $\nu_g = 6$. It therefore seems important to check how the results vary with this cost parameter. Table 8 shows the result for the frictionless model.

The higher the $\nu_g$, the higher is $\mu^*$ (as discussed in corollary 1). This is because for a given increase in government debt, aggregate demand in the business-as-usual state falls more if $\nu_g$ is higher. This strengthens the negative expectations channel for each $\mu$. With higher $\nu_g$ debt has a perverse effect on the low-state natural real rate for a larger range of $\mu$’s.

<table>
<thead>
<tr>
<th>Output cost of debt ($\nu_g$)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant solution $\mu^*$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.3</td>
<td>0.4</td>
</tr>
<tr>
<td>Transition solution $\mu^*$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.4</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Table 8: Sensitivity w.r.t. $\nu_g$ (frictionless model)

C  Appendix: Endogenous Labor

C.1  Households

On the household side, there is only one difference relative to the illustrative model: labor supply is endogenous. This makes the income tax rate have a distortionary effect on output.

$$U_t(c_t^m, c_{t+1}^o, h_t^m, h_{t+1}^o) = \max_{c_t^m, c_{t+1}^o} \left\{ \frac{1}{1 - \gamma} (c_t^m)^{1-\gamma} + \beta D_t E_t \frac{1}{1 - \gamma} (c_{t+1}^o)^{1-\gamma} - \lambda \left( \frac{h_t^m}{1 + \frac{1}{\eta}} - \lambda \beta D_t \frac{(h_{t+1}^o)^{1+\frac{1}{\eta}}}{1 + \frac{1}{\eta}} \right) \right\}$$

s.t. $c_t^m = w_t h_t^m (1 - \tau_t) - B_t^p + \theta_Z (1 - \tau_t)$
$c_{t+1}^o = w_{t+1} h_{t+1}^o (1 - \tau_{t+1}) + (1 - \theta_Z) Z_{t+1} (1 - \tau_{t+1}) + (1 + r_t) B_t^p$

(37)

where, $h_t^m$ & $h_{t+1}^o$ is labor supply of the middle-aged and old, respectively, $\eta$ is the Frisch elasticity of labor and $\tau_t$ is the income tax rate. As in the previous model, a shock to the intertemporal wedge $D_t$ will generate a negative benchmark rate (that is the frictionless real rate absent any fiscal policy intervention).

With endogenous labor, a crucial parameter in the analysis is $\eta$ as it determines the elasticity of labor supply to after-tax wages. In an (flexible-price) economy where service payments on debt are financed through taxes, an increase in debt implies a higher tax rate, which reduces labor supply and hence, output. This output distortion from higher taxes depends positively on $\eta$.

The reduction in future output will affect the consumption of the middle aged through expectations of future wage and profit income. It is therefore important to assume that the households work in both periods of life. If the old do not work, their income in the business-as-usual state would fall in response to
an increase in debt only to the extent that they receive lower profits. This would weaken the expectations channel. In the extreme case of $r_2 = 1$, where all profit income accrues to the middle-aged, the expectations channel would be completely eliminated and debt would have an unambiguous positive effect on output.

The intertemporal decision of consumption versus saving for the middle aged is given by the Euler equation:

$$\mathbb{E}_{t} c_{t+1}^o = \left[ \beta D_t (1 + r_t) \right]^\frac{1}{\gamma} c_t^m$$

The consumption of the old is given by their budget constraint:

$$c_t^o = w_t h_t^o (1 - \tau_t) + (1 - \theta_2) Z_t (1 - \tau_t) + (1 + r_{t-1}) B_t^p$$ (38)

Combining the budget constraint of the middle aged and old with the Euler equations yields the consumption of the middle aged:

$$c_t^m = \mathbb{E}_t \left[ w_{t+1} h_{t+1}^o (1 - \tau_{t+1}) + \mathbb{E}_t (1 - \theta_2) Z_{t+1} (1 - \tau_{t+1}) + (1 + r_t) B_t^p \right] = \left[ \beta D_t (1 + r_t) \right]^\frac{1}{\gamma} c_t^m$$

$$\mathbb{E}_t \left[ w_{t+1} h_{t+1}^o (1 - \tau_{t+1}) + \mathbb{E}_t (1 - \theta_2) Z_{t+1} (1 - \tau_{t+1}) + (1 + r_t) [w_t h_t^o (1 - \tau_t) + \theta_2 Z_t (1 - \tau_t)] \right] = \left[ \beta D_t (1 + r_t) \right]^\frac{1}{\gamma} c_t^m$$

$$\mathbb{E}_t \left[ w_{t+1} h_{t+1}^o (1 - \tau_{t+1}) + \mathbb{E}_t (1 - \theta_2) Z_{t+1} (1 - \tau_{t+1}) + (1 + r_t) [w_t h_t^o (1 - \tau_t) + \theta_2 Z_t (1 - \tau_t)] \right] = \left[ \beta D_t (1 + r_t) \right]^\frac{1}{\gamma} + (1 + r_t) \right] c_t^m$$

$$c_t^m = \mathbb{E}_t \left[ w_{t+1} h_{t+1}^o (1 - \tau_{t+1}) + \mathbb{E}_t (1 - \theta_2) Z_{t+1} (1 - \tau_{t+1}) + (1 + r_t) \right] [w_t h_t^o (1 - \tau_t) + \theta_2 Z_t (1 - \tau_t)] = \left[ \beta D_t (1 + r_t) \right]^\frac{1}{\gamma} + (1 + r_t) \right] c_t^m$$

The labor supply decision of the household is new and given by:

$$-\lambda (h_t^m)^{\frac{1}{\eta}} = \mu_t w_t (1 - \tau_t)$$

$$-\lambda \beta D_t (h_t^o)^{\frac{1}{\eta}} = \mu_{t+1} w_{t+1} (1 - \tau_{t+1})$$

$$\implies \frac{(h_t^m)^{\frac{1}{\eta}}}{(h_{t+1}^o)^{\frac{1}{\eta}}} = \beta D_t (1 + r_t) \frac{w_t (1 - \tau_t)}{w_{t+1} (1 - \tau_{t+1})}$$

$$\implies h_{t+1}^o = \left[ \frac{\beta D_t (1 + r_t) w_t (1 - \tau_t)}{w_{t+1} (1 - \tau_{t+1})} \right]^{-\eta} h_t^m$$

And,

$$\lambda (h_t^m)^{\frac{1}{\eta}} = (c_t^m)^{-\gamma} w_t (1 - \tau_t)$$

$$\implies h_t^m = \left[ \frac{w_t (1 - \tau_t)}{\lambda} \right]^{\eta} (c_t^m)^{-\gamma}$$

Similarly,

$$h_{t+1}^o = \left[ \frac{w_{t+1} (1 - \tau_{t+1})}{\lambda} \right]^{\eta} (c_{t+1}^o)^{-\gamma}$$

This final equation also comes from just combining the relation between $h_{t+1}^o$ & $h_t^m$ and $h_t^o$ & $c_t^m$ with the Euler equation of the household.
C.2 Firms

The production side of the economy is exactly the same as in the illustrative model. Firms are owned by the middle-aged. They produce the final output using labor as the sole input in a decreasing returns to scale technology production function:

\[ Y_t = L_t^{1-\alpha}, \quad \alpha \in (0,1) \] (40)

\[ Z_t = \max_{L_t} P_t L_t^{1-\alpha} - w_t L_t \] (41)

This gives the following inverse demand functions for labor:

\[ w_t = (1-\alpha) \frac{Y_t}{L_t} \] (42)

C.3 Government

The government’s problem is the same as before except taxes are proportional instead of lumpsum. The government chooses taxes, purchases and debt in every period to balance its budget constraint. The tax rate is determined endogenously in every period.

\[ B_g^t + \tau_t w_t (h^m_t + h^o_t) + \tau_t Z_t = (1 + r_{t-1}) B_{g-1}^t + G_t \]

C.4 Equilibrium Under Flexible Prices

The competitive equilibrium in this economy is defined as a sequence of variables \( \{Y_t, L_t, w_t, Z_t, C^m_t, C^o_t, h^m_t, h^o_t, r_t, \tau_t, B_g^t, W_t, G_t\} \) which satisfy the following equations, for a given sequence of exogenous fiscal policy parameters \( \{W^*, G^*\} \) and preference parameter \( \{D_t\} \).

\[ Y_t = L_t^{1-\alpha} \] (43)

\[ w_t = (1-\alpha) \frac{Y_t}{L_t} \] (44)

\[ Z_t = Y_t - w_t L_t \] (45)

\[ Y_t = C^m_t + C^o_t + G_t \] (46)

\[ C^m_t = \frac{\mathbb{E}_t w_{t+1} h^m_{t+1} (1 - \tau_{t+1}) + \mathbb{E}_t (1 - \theta_z) Z_{t+1} (1 - \tau_{t+1})}{\beta D_t (1 + r_t)} + \frac{(1 + r_t) [w_t h^m_t (1 - \tau_t) + \theta_z Z_t (1 - \tau_t)]}{\beta D_t (1 + r_t)} \] (47)

\[ C^o_t = w_t h^o_t (1 - \tau_t) + (1 - \theta_z) Z_t (1 - \tau_t) + (1 + r_{t-1}) B_{g-1}^t \] (48)
Characterization of Equilibrium Under Flexible Prices

The equilibrium system is similar to that of the illustrative model except that labor supply is endogenously chosen by the households. This introduces two new equations into the model, one each for the labor supply of the middle-aged and the old, and changes the labor market clearing condition. Additionally, lumpsum taxes everywhere are replaced with proportional income taxes.

Formally, the flexible price competitive equilibrium of the economy with endogenous labor is defined as a sequence of variables \( \{Y_t, L_t, w_t, Z_t, h_t^m, h_t^o, r_t, \tau_t, \mathbb{W}_t\} \) which satisfy the following equations, given \( \{D_t, \mathbb{W}^*\} \).

\[
Y_t = L_t^{1-\alpha} \tag{56}
\]

\[
w_t = (1 - \alpha) \frac{Y_t}{L_t} \tag{57}
\]

\[
Z_t = Y_t - w_t L_t \tag{58}
\]

\[
Y_t = \frac{\mathbb{E}_t w_{t+1} h_{t+1}^o (1 - \tau_{t+1}) + \mathbb{E}_t (1 - \theta_z) Z_{t+1} (1 - \tau_{t+1}) + \left[ w_t h_t^m (1 - \tau_t) + \theta_z Z_t (1 - \tau_t) \right]}{(\beta D_t + 1)(1 + r_t)} + \frac{w_t h_t^m (1 - \tau_t)}{\beta D_t + 1} \tag{59}
\]

\[
h_t^m = \left[ \frac{w_t (1 - \tau_t)}{\lambda} \right]^{\eta} (\epsilon_t^m)^{-\eta} \tag{60}
\]

\[
h_t^o = \left[ \frac{w_t (1 - \tau_t)}{\lambda} \right]^{\eta} (\epsilon_t^o)^{-\eta} \tag{61}
\]

\[\mathbb{W}_t = (1 + r_t) \mathbb{B}_t^g \tag{51}\]

\[\mathbb{W}_t = \mathbb{W}^* \tag{52}\]

\[G_t = G^* \tag{53}\]

\[\tau_t Y_t = \mathbb{W}_{t-1} - \frac{\mathbb{W}_t}{1 + r_t} + G_t \tag{54}\]

\[L_t = h_t^m + h_t^o \tag{55}\]
\[ W_t = W^* \] (62)

\[ \tau_t Y_t = W_{t-1} - \frac{W_t}{1 + r_t} \] (63)

\[ L_t = h_t^n + h_t^q \] (64)

### C.5 Policy Experiment & Numerical Results

The goal is to find out the effect of a permanent increase in the stock of government debt on the economy under the same policy experiment as described in the previous section. For the parameters common with the illustrative model, the calibration remains the same. The new parameter \( \eta \) is calibrated to 3. This is in the middle of the range of the empirical macroeconomics estimates of Frisch elasticity of labor (see Chetty et al. (2011) for a meta-analysis of the existing literature).

In what follows, I will present only the transition solution. The constant solution was discussed in the illustrative model for the purpose of analytical tractability.

**Result: Higher debt can lower the natural real rate**

The main result is that the effect of debt on the low-state real rate depends on \( \mu \), that is, the probability with which households expect the economy to be in the low state in the next period. As shown in Figure 17, if this probability is low, then increasing debt will lead to a further reduction in the real rate.

![Figure 17: Effect of debt on real rate in the low state](image)

The logic of this result is similar to that in the illustrative model, with the main difference being that the expectations channel no longer relies on an ad hoc exogenous output cost of debt. Instead, the expectations channel arises endogenously from distortionary income taxation.
Increasing debt has a direct effect of reducing low state tax rates in partial equilibrium, that is, holding current output fixed (see Figure 18a). Lower taxes result in higher consumption (labor supply) by the old and by the middle aged.

Debt has an additional indirect effect through its effect on the expectations of net future disposable income, which affects the consumption, and hence, the labor supply of the middle aged. Higher debt implies higher real rate and higher taxes in the business-as-usual state. Individuals respond to the higher taxes by reducing their labor supply, which lowers output. Thus, higher tax rates lead to a reduction in wage and profit income of both the middle-aged and old. Together this means that higher debt lowers disposable income of both generations in the business-as-usual state. Expectations of lower income in the future (see Figure 18b) reduce consumption and hence, labor supply of the middle aged today.

In summary, while increasing debt has an unambiguous positive effect on the consumption of the old, its effect on the consumption of the middle-aged is determined by two opposing forces: a direct effect which increases it, and an indirect expectations effect which reduces it. If the probability of exit into the business-as-usual state is high, then the expectations effect dominates and real interest rate falls.

**Sensitivity Analysis wrt $\eta$**

Thus, the threshold $\mu^*$, below which the perverse effect of debt exists, increases with $\eta$. In the benchmark simulation, for $\eta = 0$, debt has an unambiguously positive effect on the real rate while for $\eta > 6$, debt has an unambiguously negative effect on the real rate.

<table>
<thead>
<tr>
<th>Frisch elasticity $\eta$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Implied $\nu_g$</td>
<td>0.3</td>
<td>0.42</td>
<td>0.48</td>
<td>0.52</td>
<td>0.55</td>
<td>0.56</td>
<td>0.58</td>
<td>0.59</td>
<td>0.6</td>
<td>0.61</td>
</tr>
<tr>
<td>$\mu^*$</td>
<td>0</td>
<td>0.06</td>
<td>0.41</td>
<td>0.64</td>
<td>0.8</td>
<td>0.92</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 9: Sensitivity analysis
D Appendix: Adding Capital to the Model with Endogenous Labor

D.1 Households

With the addition of capital, households can now save in government bonds as well as capital, which is rented out to the firms in the same period. I assume full depreciation. This is a reasonable assumption as every period in the model corresponds to 20 years in the real world. Capital is unlikely to serve as a store of wealth of such a long period of time.

\[
U_t(c_t^m, c_t^o, h_t^m, h_{t+1}^o) = \max_{c_t^m, c_t^o} \left\{ \frac{1}{1-\gamma} (c_t^m)^{1-\gamma} + \beta D E_t \frac{1}{1-\gamma} (c_{t+1}^o)^{1-\gamma} - \lambda \left( \frac{h_t^m)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} - \lambda \beta D_t \left( h_{t+1}^o \right)^{1+\frac{1}{\eta}} \right) \right\}
\]

\[
\text{s.t.} \quad c_t^m = w_t h_t^m (1 - \tau_t) - B_t^m \left[ 1 - r_t^k (1 - \tau_t) \right] K_t + Z_t (1 - \tau_t)
\]

\[
c_{t+1}^o = w_{t+1} h_{t+1}^o (1 - \tau_{t+1}) + (1 + r_t) B_t^m + (1 - \delta) K_t
\]

where, \( h_t^m \) & \( h_{t+1}^o \) is labor supply of the middle-aged and old, respectively, \( \eta \) is the Frisch elasticity of labor and \( \tau_t \) is the income tax rate. I assume that when households are middle-aged, they convert consumption goods into productive capital at a fixed 1:1 rate. They rent the capital to firms in the same period and receive rental income. In the next period of life when they are old, they convert the undepreciated capital back into consumption goods at the same fixed rate, to use for consumption. As in the previous model, a shock to the intertemporal wedge \( D_t \) will generate a negative efficient rate (that is the frictionless real rate absent any fiscal policy intervention).

The intertemporal decision of consumption versus saving for the middle aged is given by the Euler equation:

\[
E_t c_{t+1}^o = [\beta D_t (1 + r_t)]^{\frac{1}{\gamma}} c_t^m
\]

The consumption of the old is given by their budget constraint:

\[
c_t^o = w_t h_t^o (1 - \tau_t) + (1 + r_{t-1}) B_{t-1}^m
\]

The first order condition with respect to capital yields:

\[
1 + r_t = \frac{1 - \delta}{1 - r_t^k (1 - \tau_t)}
\]

Under full depreciation:

\[
r_t^k = \frac{1}{1 - \tau_t}
\]

Combining the budget constraint of the middle aged and old with the Euler equations yields the consumption of the middle aged:

\[
E_t \left[ w_{t+1} h_{t+1}^o (1 - \tau_{t+1}) + (1 + r_t) B_t^m \right] = [\beta D_t (1 + r_t)]^{\frac{1}{\gamma}} c_t^m
\]

\[
E_t \left[ w_{t+1} h_{t+1}^o (1 - \tau_{t+1}) \right] + (1 + r_t) \left[ w_t h_t^m (1 - \tau_t) - \left[ 1 - r_t^k (1 - \tau_t) \right] K_t + Z_t (1 - \tau_t) - c_t^m \right] = [\beta D_t (1 + r_t)]^{\frac{1}{\gamma}} c_t^m
\]

\[
E_t \left[ w_{t+1} h_{t+1}^o (1 - \tau_{t+1}) \right] + (1 + r_t) \left[ w_t h_t^m (1 - \tau_t) + Z_t (1 - \tau_t) \right] = \left\{ [\beta D_t (1 + r_t)]^{\frac{1}{\gamma}} + (1 + r_t) \right\} c_t^m
\]
\[ c_t^m = \frac{E_t \left[ w_{t+1} h_{t+1}^m (1 - \tau_t) \right]}{\beta D_t (1 + r_t)} + \frac{(1 + r_t) \left[ w_t h_t^m (1 - \tau_t) + Z_t (1 - \tau_t) \right]}{\beta D_t (1 + r_t)} \]  

(67)

As before, the labor supply decision of the household is given by:

\[ \implies h_t^m = \left[ \frac{w_t (1 - \tau_t)}{\lambda} \right]^{\frac{\eta}{\gamma}} \left( c_t^m \right)^{-\gamma \eta} \]

\[ h_{t+1}^o = \left[ \frac{w_{t+1} (1 - \tau_{t+1})}{\lambda} \right]^{\frac{\eta}{\gamma}} \left( c_{t+1}^o \right)^{-\gamma \eta} \]

D.2 Firms

The production side of the economy now includes capital in the firm’s production function. Thus, there are constant returns to scale in production. Firm ownership is irrelevant now as profits are zero.

\[ Y_t = L_t^{1-\alpha} K_t^\alpha, \quad \alpha \in (0, 1) \]  

(68)

\[ Z_t = \max_{l_t} P_t L_t^{1-\alpha} - w_t L_t - r_t K_t \]  

(69)

This gives the following inverse demand functions for labor and capital:

\[ w_t = (1 - \alpha) \frac{Y_t r_t}{L_t} = \alpha \frac{Y_t}{K_t} \]  

(70)

D.3 Government

The government’s problem is the same as before except tax revenues are now derived from capital income as well. The government chooses taxes, purchases and debt in every period to balance its budget constraint. The tax rate is determined endogenously in every period.

\[ B_t^g + \tau_t w_t (h_t^m + h_t^o) + \tau_t Z_t + \tau_t r_t K_t = (1 + r_{t-1}) B_{t-1}^g + G_t \]

\[ B_t^g + \tau_t Y_t = (1 + r_{t-1}) B_{t-1}^g + G_t \]

D.4 Equilibrium Under Flexible Prices

The competitive equilibrium in this economy is defined as a sequence of variables \{Y_t, L_t, w_t, r_t^k, Z_t, K_t, C_t^m, C_t^o, I_t, h_t^m, h_t^o, r_t, \tau_t, B_t^g, W_t, G_t\} which satisfy the following equations, for a given sequence of exogenous fiscal policy parameters \{W^*, G^*\} and preference shock \{D_t\}.

\[ Y_t = L_t^{1-\alpha} K_t^\alpha \]  

(71)

\[ w_t = (1 - \alpha) \frac{Y_t}{L_t} \]  

(72)
\[ r_t^k = \alpha \frac{Y_t}{K_t} \]  
\[ r_j^k = \frac{1}{1 - \tau_t} \]  
\[ Z_t = Y_t - w_t L_t - r_t^k K_t \]  
\[ Y_t = C_t^m + C_t^o + I_t + G_t \]  
\[ C_t^m = \frac{E_t w_{t+1} h_{t+1}^m (1 - \tau_{t+1})}{[\beta D_t (1 + r_t)]^{\frac{3}{4}} + (1 + r_t)} + \frac{(1 + r_t) [w_t h_{t}^m (1 - \tau_t) + Z_t (1 - \tau_t)]}{[\beta D_t (1 + r_t)]^{\frac{1}{4}} + (1 + r_t)} \]  
\[ C_t^o = w_t h_t^o (1 - \tau_t) + (1 + r_{t-1}) B_{t-1}^g \]  
\[ I_t = K_t \]  
\[ h_t^m = \left[ \frac{w_t (1 - \tau_t)}{\lambda} \right]^\eta (c_t^m)^{-\gamma \eta} \]  
\[ h_t^o = \left[ \frac{w_t (1 - \tau_t)}{\lambda} \right]^\eta (c_t^o)^{-\gamma \eta} \]  
\[ \mathbb{W}_t = (1 + r_t) B_t^g \]  
\[ \mathbb{W}_t = \mathbb{W}^* \]  
\[ G_t = G^* \]  
\[ \tau_t Y_t = \mathbb{W}_{t-1} - \frac{\mathbb{W}_t}{1 + r_t} + G_t \]  
\[ L_t = h_t^m + h_t^o \]  

This can be further simplified to a sequence of variables \( \{Y_t, L_t, w_t, r_t^k, K_t, h_t^m, h_t^o, r_t, \tau_t, \mathbb{W}_t\} \) which satisfy the following equations, given exogenous \( \{D_t, \mathbb{W}^*\} \).

\[ Y_t = L_t^{1-a} K_t^a \]  

xviii
\[ w_t = (1 - \alpha) \frac{Y_t}{L_t} \]  

\[ r^k_t = \frac{Y_t}{K_t} \]  

\[ r^k_t = \frac{1}{1 - \tau_t} \]  

\[ Y_t = \frac{\mathbb{E}_t w_{t+1} h^m_{t+1}(1 - \tau_{t+1})}{[\beta D_t(1 + r_t)]^{\frac{1}{\gamma}} + (1 + r_t)} + \frac{(1 + r_t) [w_t h^m_t(1 - \tau_t)]}{[\beta D_t(1 + r_t)]^{\frac{1}{\gamma}} + (1 + r_t)} + w_t h^o_t (1 - \tau_t) + W_{t-1} + K_t \]  

\[ h^m_t = \left[ \frac{w_t (1 - \tau_t)}{\lambda} \right]^{\eta} (c^{m}_t)^{\gamma \eta} \]  

\[ h^o_t = \left[ \frac{w_t (1 - \tau_t)}{\lambda} \right]^{\eta} (c^{o}_t)^{\gamma \eta} \]  

\[ W_t = W^* \]  

\[ \tau_t Y_t = W_{t-1} - \frac{W_t}{1 + r_t} \]  

\[ L_t = h^m_t + h^o_t \]  

### Appendix: Quantitative Lifecycle Model

#### Households

Households live for 60 periods. Each generation is of measure 1/60. The first 40 periods, they are working, while in the last 20 periods, they are retired and receive pensions. Households maximize expected lifetime utility at age 1 in period \( t \):

\[ \mathbb{E}_t \sum_{s=1}^{60} \beta^{s-1} D_{t+s-1} u \left( c^{s}_{t+s-1}, n^{s}_{t+s-1} \right) \]

Instantaneous utility is a function of both consumption and leisure:

\[ u(c, n) = \frac{c^{1-\gamma}}{1 - \gamma} - \lambda^{\eta \gamma} \frac{n^{1+\frac{1}{\eta}}}{1 + \frac{1}{\eta}} \]

The working agent of age \( s \) faces the following budget constraint in period \( t \):

\[ a^{t+1}_{t+1} = (1 + r_t) a^s_t + w_t n^s_t (1 - \tau^s_t) - c^s_t, \quad s = 1, \ldots, 40 \]
The budget constraint of the retired worker is given by:

\[ a_{t+1}^s = (1 + r_t) a_t^s + b - c_t^s, \quad s = 41, \ldots, 60 \]

with \( a_{t}^{61} = 0 \) and \( n_{t}^{41} = n_{t}^{52} = \ldots = n_{t}^{60} = 0 \). Pensions \( b \) are constant.

The first-order conditions of the households for \( s = 1, \ldots, 60 \) in period \( t \) for labor supply and next-period assets, respectively:

\[
\begin{align*}
    u_n(c_{t+s-1}^s, n_{t+s-1}^s) &= D_t(-\chi) \left( n_{t+s-1}^s \right)^{\frac{1}{\eta}} = -\lambda_t^s (1 - \tau_t) w_t \\
    u_c(c_{t+s-1}^s, n_{t+s-1}^s) &= D_{t+s-1} (c_{t+s-1}^s)^{-\gamma} = \lambda_t^s \\
    \beta \mathbb{E}_t \frac{\lambda_{t+1}^{s+1}}{\lambda_t^s} &= \beta \mathbb{E}_t \frac{D_{t+1}^{s+1}}{D_t} (1 + r_{t+1})
\end{align*}
\]

**Firms**

Production \( Y_t \) is characterized by constant returns to scale and assumed to be Cobb-Douglas:

\[ Y_t = N_t^{1-\alpha} K_t^\alpha \]

In a factor market equilibrium, factors are rewarded with their marginal product:

\[
\begin{align*}
    w_t &= (1 - \alpha) N_t^{-\alpha} K_t^\alpha \\
    r_t &= \alpha N_t^{1-\alpha} K_t^{\alpha-1} - \delta
\end{align*}
\]

**Government**

The government budget is balanced in every period \( t \):

\[ \tau_t w_t N_t + B_{t+1}^s = (1 + r_t) B_t^s + G_t + \frac{20}{60} b \]

**Equilibrium**

In equilibrium, individual and aggregate behavior are consistent:

\[ N_t = \sum_{s=1}^{60} \frac{n_t^s}{60} \]

\[ K_{t+1} + B_{t+1}^s = \sum_{s=1}^{60} \frac{a_{t+1}^s}{60} \]

and the goods market clears:

\[ N_t^{1-\alpha} K_t^\alpha = \sum_{s=1}^{60} \frac{c_t^s}{60} + K_{t+1} - (1 - \delta) K_t + G_t \]

xx
**E.1 Non-stochastic steady state**

For $s = 1, \ldots, T - 1$:

$$
(c^s)^\gamma \chi(n^s)^\frac{1}{\eta} = (1 - \tau)w
$$

(97)

$$
\left(\frac{(1 + r)a^s + (1 - \tau)wn^s - a^{s+1}}{(1 + r)a^{s+1} + (1 - \tau)wn^{s+1} - a^{s+2}}\right)^{-\gamma} = \beta(1 + r)
$$

(98)

For $s = T$:

$$
(c^T)^\gamma \chi(n^T)^\frac{1}{\eta} = (1 - \tau)w
$$

(99)

$$
\left(\frac{(1 + r)a^T + (1 - \tau)wn^T - a^{T+1}}{(1 + r)a^{T+1} + b - a^{T+2}}\right)^{-\gamma} = \beta(1 + r)
$$

(100)

For $s = T + 1, \ldots, T + R - 1$:

$$
\left(\frac{(1 + r)a^s + b - a^{s+1}}{(1 + r)a^{s+1} + b - a^{s+2}}\right)^{-\gamma} = \beta(1 + r)
$$

(101)

Remember that households are born and die without wealth hence, $a^{T+R+1} = 0$. The equations (97)-(101) constitute a system of $T + R - 1 + T = 2T + R - 1$ equations in the $T + R - 1 + T = 2T + R - 1$ unknowns $\{a^s\}_{s=2}^{T+R}$ and $\{n^s\}_{s=1}^{T}$.

**E.2 Linear Approximation**

Log-linearization of the household first-order conditions around the business-as-usual steady state:

$$
\frac{1}{\eta} \dot{n}^s_i = \lambda^s_i - \frac{\tau}{1 - \tau} \dot{h}^s_i + \dot{\omega}_i, \quad s = 1, \ldots, 40
$$

$$
-\gamma \dot{c}^s_i = \lambda^s_i, \quad s = 1, \ldots, 60
$$

$$
\dot{\lambda}^s_i + \dot{D}_i - E \dot{D}_{i+1} = E \dot{\lambda}^{s+1} + \frac{r}{1 + r} E \dot{h}_{i+1}, \quad s = 1, \ldots, 59
$$

Log-linearization of the working household’s budget constraint around the steady state for one-year old with $a_1^s \equiv 0$:

$$
a^2 \dot{a}_{i+1} = -\tau wn^1 \dot{h}_i + (1 - \tau)wn^1 \dot{\omega}_i + (1 - \tau)wn^1 \dot{h}_i^1 - c^1 \dot{c}^1_i
$$

and for $s = 2, \ldots, 40$:

$$
a^s \dot{a}_{i+1} = (1 + r)a^s \dot{a}_i + ra^s \dot{h}_i - \tau wn^s \dot{h}_i + (1 - \tau)wn^s \dot{\omega}_i + (1 - \tau)wn^s \dot{h}_i^s - c^s \dot{c}^s_i
$$

Log-linearization of the retired household’s budget constraint around the steady state for $s = 41, \ldots, 59$: 
\[ a^{s+1} = (1 + r) a^s + r a^s \hat{t} - c^s \hat{c}_t \]

and for \( s = 60: \)

\[ c^{60} = (1 + r) c_60 + r c_60 \hat{t} \]

Thus, we have 60 controls \( \hat{c}_t, s = 1, \ldots, 60 \), 40 controls \( \hat{a}_t, s = 1, \ldots, 40 \), 60 costates \( \lambda_t, s = 1, \ldots, 60 \) and 59 predetermined variables \( \hat{a}_t, s = 2, \ldots, 60 \). We also have \( 60 + 40 + 60 + 59 = 219 \) equations.

We have three further endogenous variables \( \hat{w}_t, \hat{r}_t, \hat{\tau}_t \). Log-linearization of the reward to each factor along with the market-clearing equation yields:

\[ \hat{w}_t = -\alpha \Sigma_{s=1}^{40} \frac{n^s}{N} \hat{a}_t + \alpha \left[ A \Sigma_{s=1}^{60} \frac{a^s}{A} \hat{a}_t - B^s \hat{B}_t \right] \]

\[ \hat{r}_t = (1 - \alpha) \Sigma_{s=1}^{40} \frac{n^s}{N} \hat{a}_t - (1 - \alpha) \left[ A \Sigma_{s=1}^{60} \frac{a^s}{A} \hat{a}_t - B^s \hat{B}_t \right] \]

Log-linearization of the government budget constraint:

\[ \hat{\tau}_t = -\hat{w}_t - \Sigma_{s=1}^{40} \frac{n^s}{N} \hat{a}_t - B^s \hat{B}_t + \frac{(1 + r) B^s}{\tau w n} \hat{B}_t + \frac{r B^s}{\tau w n} \hat{r}_t + \frac{G}{\tau w n} \hat{G}_t \]

These constitute three further equations in three endogenous variables \( \hat{w}_t, \hat{r}_t, \hat{\tau}_t \).

\( \hat{B}_t, \hat{G}_t \) and \( \hat{D}_t \) are exogenous variables with the law of motion:

\[ \hat{B}_t = \hat{B}_{t-1}, \quad \hat{D}_t = \hat{D}_{t-1}, \quad \hat{G}_t = \hat{G}_{t-1} \]

### E.3 Solution Algorithm

**Step 1: Express as a Linear Rational Expectations Model**

It is convenient to express this system of equations in the form:

\[ C_u u_t = C_{x \lambda} x_t + C_z z_t \]

\[ D_{x \lambda} E_t \left[ x_{t+1} \right] + F_{x \lambda} \left[ x_t \right] = D_{a \lambda} E_t u_{t+1} + F_{a} u_t + D_{z} E_t z_{t+1} + F_z z_t \]

where, I define:

\[ u_t = \left[ c_1^t, c_2^t, \ldots, c_{60}^t, a_1^t, a_2^t, \ldots, a_{60}^t, \hat{t}, \hat{w}_t, \hat{\tau}_t \right]' \]

\[ x_t = \left[ \hat{a}_1^t, \hat{a}_2^t, \ldots, \hat{a}_{60}^t \right]' \]
\[ \lambda_t = \left[ \hat{\lambda}^1_t, \hat{\lambda}^2_t, \ldots, \hat{\lambda}^{59}_t \right]' \]
\[ z_t = \left[ D_t, \hat{G}_t, B^8_t \right]' \]

**Step 2: Solve the business-as-usual state solution**

Solving this system of equations yields a solution of the form:

\[ x_{t+1} = L_x^1 x_t + L_x^2 z_t \]
\[ u_t = L_u^1 u_t + L_u^2 z_t \]
\[ \lambda_t = L_l^1 x_t + L_l^2 z_t \]

**Step 3: Solve the low state solution**

In the low state, the equations are the same as in the business-as-usual state. The only difference is that the expectation of the variable which are not pre-determined at time \( t \) depends on the probability of benchmark real rate normalization. In particular,

\[ E_t u_{t+1} = \mu E_t u_{t+1} \text{low state} + (1 - \mu) E_t u_{t+1} \text{business-as-usual state} \]
\[ E_t \lambda_{t+1} = \mu E_t \lambda_{t+1} \text{low state} + (1 - \mu) E_t \lambda_{t+1} \text{business-as-usual state} \]

This requires modifying the input matrices slightly and resolving the entire system of equations using the new modified matrix system.

Again, the solution will be of the form:

\[ x_{t+1} = L_x^1 x_t + L_x^2 z_t \]
\[ u_t = L_u^1 u_t + L_u^2 z_t \]
\[ \lambda_t = L_l^1 x_t + L_l^2 z_t \]

From this, we can extract the coefficient of the shocks to derive the effect of public debt on the natural real interest rate and other variables in the economy.

### E.3.1 Input matrices for numerical solution

Contemporary equations system comprises of:

\[ \frac{1 - \eta^s_t}{\eta^s_t} + \frac{\tau}{1 - \tau} \hat{e}_t - \hat{w}_t = \hat{\lambda}^s_t, \quad s = 1, \ldots, 40 \]

\[ \hat{\lambda}^s_t = -\gamma^s_t \hat{y}_t, \quad s = 1, \ldots, 59 \]

\[ c_t^{60} a_t^{60} - r a_t^{60} \hat{y}_t = (1 + r) a_t^{60} \hat{a}_t^{60} \]
\[
\dot{\omega}_t + a \sum_{s=1}^{40} \frac{n^s}{N} \delta_t^s = a \left[ \sum_{s=2}^{60} \frac{a^s}{K} \delta_t^s - \frac{B^s}{K} \delta_t^1 \right]
\]

\[
\dot{\lambda}_t - (1 - a) \sum_{s=1}^{40} \frac{n^s}{N} \delta_t^s = -(1 - a) \left[ \sum_{s=2}^{60} \frac{a^s}{K} \delta_t^s - \frac{B^s}{K} \delta_t^1 \right]
\]

\[
\dot{\tau}_t + \omega_t + \sum_{s=1}^{40} \frac{n^s}{N} \delta_t^s - \frac{r B^s}{\tau w n} \dot{\tau}_t = \frac{-B^s}{\tau w n} \delta_{t+1}^s + \frac{(1 + r) B^s}{\tau w n} \delta_t^s + \frac{G}{\tau w n} \dot{\gamma}_t
\]

Dynamic equations system comprises of:

\[
\lambda_s^t - \mathbb{E}_t \lambda_{s+1}^{t+1} = \frac{r}{1 + r} \mathbb{E}_t \dot{\lambda}_{t+1} - \hat{D}_t + \mathbb{E}_t \dot{D}_{t+1}, \quad s = 1, \ldots, 58
\]

\[
\lambda_t^{59} = -\gamma \mathbb{E}_t c_{t+1}^{60} + \frac{r}{1 + r} \mathbb{E}_t \dot{\lambda}_{t+1} - \hat{D}_t + \mathbb{E}_t \dot{D}_{t+1}
\]

\[
a^{s+1} \dot{a}_t^{s+1} = \tau w n^s \dot{\delta}_t^s + (1 - \tau) w n^s \dot{\omega}_t + (1 - \tau) w n^s \dot{\theta}_t - c^1 \dot{e}_t^1
\]

\[
a^{s+1} \dot{a}_t^{s+1} = ra^s \dot{\delta}_t^s - \tau w n^s \dot{\delta}_t^s + (1 - \tau) w n^s \dot{\omega}_t + (1 - \tau) w n^s \dot{\theta}_t - c^s \dot{e}_t^s,
\]

\[
ap = 2, \ldots, 40
\]

\[
a^{s+1} \dot{a}_t^{s+1} = (1 + r) a^s \dot{\delta}_t^s + c^s \dot{e}_t^s,
\]

\[
s = 41, \ldots, 59
\]

So far, this is a system of 222 equations in 225 unknowns. Need 3 more equations for the exogenous variables that constitute \(z_t\):

\[
z_t = \Pi z_{t-1} + \epsilon_t, \quad \epsilon_t \sim N(0, \Sigma)
\]

\[
\Pi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

Stability requires that the eigenvalues of the matrix \(\Pi\) lie within the unit circle.

### E.3.2 Business-as-usual state:

The contemporary equation system is characterized by matrices \(C_{it}\), \(C_{i\lambda}\) & \(C_z\) defined as follows:
\[ C_U = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \]

with the 40 \times 60 submatrix \(A_{11}\), 40 \times 40 submatrix \(A_{12}\) and 40 \times 3 submatrix \(A_{13}\) as follows:

\[ A_{11} = A_{22} = A_{31} = 0 \]

\[ A_{12} = \begin{bmatrix} \frac{1}{\eta} & 0 & \cdots & 0 \\ 0 & \frac{1}{\eta} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\eta} \end{bmatrix} \]

\[ A_{13} = \begin{bmatrix} 0 & -1 & \frac{1}{\tau} \\ 0 & -1 & \frac{1}{\tau} \\ \vdots & \vdots & \ddots \\ 0 & 0 & \cdots \end{bmatrix} \]

and the 60 \times 60 submatrix \(A_{21}\), 60 \times 40 submatrix \(A_{22}\) and 60 \times 3 submatrix \(A_{23}\) as follows:

\[ A_{21} = \begin{bmatrix} -\gamma & 0 & \cdots & 0 & 0 \\ 0 & -\gamma & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\gamma & 0 \\ 0 & 0 & \cdots & 0 & e^{60} \end{bmatrix} \]

\[ A_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ -r_\delta^{60} & 0 & 0 \end{bmatrix} \]

and the 3 \times 60 submatrix \(A_{31}\), 3 \times 40 submatrix \(A_{32}\) and 3 \times 3 submatrix \(A_{33}\) as follows:

\[ A_{32} = \begin{bmatrix} \frac{\alpha^1 N^1}{\delta^1 \delta_0} & \frac{\alpha^2 N^1}{\delta^2 \delta_0} & \cdots & \frac{\alpha^{60} N^1}{\delta^{60} \delta_0} \\ -(1 - \alpha) \frac{\alpha^1 N^1}{\delta^1 \delta_0} & -(1 - \alpha) \frac{\alpha^2 N^1}{\delta^2 \delta_0} & \cdots & -(1 - \alpha) \frac{\alpha^{60} N^1}{\delta^{60} \delta_0} \\ \frac{\alpha^1 N^2}{\delta^2 \delta_0} & \frac{\alpha^2 N^2}{\delta^2 \delta_0} & \cdots & \frac{\alpha^{60} N^2}{\delta^{60} \delta_0} \\ \frac{\alpha^1 N^3}{\delta^3 \delta_0} & \frac{\alpha^2 N^3}{\delta^3 \delta_0} & \cdots & \frac{\alpha^{60} N^3}{\delta^{60} \delta_0} \end{bmatrix} \]

\[ A_{33} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -r_\delta^{60} N/\delta_0 & 1 & 1 \end{bmatrix} \]
The top right sub-matrix is $[I_{40 \times 40}|0_{40 \times 19}]$.

Here I am assuming that $\hat{B}_{t} = \hat{B}_{t+1}$ in the linearized government budget constraint.

The dynamic equation system is characterized by matrices $D_{x\lambda}$, $F_{x\lambda}$, $D_{u}$, $F_{u}$, $D_{z}$ and $F_{z}$ defined as follows:

\[
D_{x\lambda} = \begin{bmatrix}
D_{x\lambda,11} & D_{x\lambda,12} \\
D_{x\lambda,21} & D_{x\lambda,22}
\end{bmatrix}
\]

where $D_{x\lambda,ij}$ are $59 \times 59$ submatrices defined as:

\[
D_{x\lambda,11} = D_{x\lambda,22} = 0
\]

\[
D_{x\lambda,12} = \begin{bmatrix}
0 & -1 & 0 & \cdots & 0 & 0 \\
0 & 0 & -1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & -1 \\
0 & 0 & 0 & \cdots & 0 & 0
\end{bmatrix}
\]

\[
D_{x\lambda,21} = \begin{bmatrix}
a^2 & 0 & \cdots & 0 \\
0 & a^3 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a^{60}
\end{bmatrix}
\]
\[ F_{\lambda} = \begin{bmatrix} F_{\lambda,11} & F_{\lambda,12} \\ F_{\lambda,21} & F_{\lambda,22} \end{bmatrix} \]

where \( F_{\lambda,ij} \) are \( 59 \times 59 \) submatrices defined as:

\[ F_{\lambda,11} = F_{\lambda,22} = 0 \]

\[ F_{\lambda,12} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \]

\[ F_{\lambda,21} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ -(1 + r)a^2 & 0 & \cdots & 0 & 0 \\ 0 & -(1 + r)a^3 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -(1 + r)a^{59} & 0 \end{bmatrix} \]

\[ D_u = \begin{bmatrix} D_{u,11} & D_{u,12} & D_{u,13} \\ D_{u,21} & D_{u,22} & D_{u,23} \end{bmatrix} \]

where \( D_{u,11}, D_{u,21} \) are \( 59 \times 60 \) submatrices, \( D_{u,12}, D_{u,22} \) are \( 59 \times 40 \) submatrices, and \( D_{u,13}, D_{u,23} \) are \( 59 \times 3 \) submatrices defined as:

\[ D_{u,12} = D_{u,21} = D_{u,22} = D_{u,23} = 0 \]

\[ D_{u,11} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\gamma \end{bmatrix} \]

\[ D_{u,13} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \frac{r}{1+t} & 0 & \cdots & 0 \\ \frac{r}{1+t} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{r}{1+t} & 0 & \cdots & 0 \end{bmatrix} \]

\[ F_u = \begin{bmatrix} F_{u,11} & F_{u,12} & F_{u,13} \\ F_{u,21} & F_{u,22} & F_{u,23} \end{bmatrix} \]

where \( F_{u,11}, F_{u,21} \) are \( 59 \times 60 \) submatrices, \( F_{u,12}, F_{u,22} \) are \( 59 \times 40 \) submatrices, and \( F_{u,13}, F_{u,23} \) are \( 59 \times 3 \) submatrices defined as:

\[ F_{u,11} = F_{u,12} = F_{u,13} = 0 \]
E.3.3 Low state:

The contemporary equation system is again characterized by matrices $C_u$, $C_x\lambda$ & $C_z$ defined above.
The dynamic equation system is characterized by matrices $D_{x\lambda}, F_{x\lambda}, D_u, F_u, D_z$ and $F_z$ as follows:

$$
\begin{bmatrix}
D_{11}^{x\lambda} + D_{12}^{x\lambda} (1 - \mu) L_{x}^{\lambda} & D_{12}^{x\lambda} \mu \\
D_{21}^{x\lambda} + D_{22}^{x\lambda} (1 - \mu) L_{x}^{\lambda} & D_{22}^{x\lambda} \mu
\end{bmatrix}
E_t \begin{bmatrix} x_{t+1} \\ \lambda_{t+1} \end{bmatrix} + F_{x\lambda} \begin{bmatrix} x_t \\ \lambda_t \end{bmatrix} = D_u \mu E_{t} u_{t+1} + D_u (1 - \mu) L_{x}^{u} x_{t+1} + F_u u_t + D_z \mu E_t z_{t+1} + F_z z_t
$$

### E.4 Derivation: Expected duration of low state

Given the 2-state Markov process for the shock driving the benchmark real rate:

Expected duration of low state = $(1 - \mu) + \mu (1 - \mu) 2 + \mu^2 (1 - \mu) 3 + \ldots$

= $(1 - \mu) \left[ 1 + 2\mu + 3\mu^2 + \ldots \right]$

= $(1 - \mu) \left[ \frac{1}{1 - \mu} + \frac{\mu}{(1 - \mu)^2} \right]$

= $(1 - \mu) \left[ (1 - \mu) + \mu \right]$

= $\frac{1}{1 - \mu}$