Nash Implementation and Tie-Breaking Rules

Mert Kimya †

Brown University, Department of Economics, 64 Waterman St, Providence RI 02912 USA.

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Abstract

I study Nash implementation when agents might use a tie-breaking rule to choose among the messages they are materially indifferent between. If the planner is endowed with the knowledge of the rule, this might expand or shrink the set of implementable social choice correspondences (SCC) depending on the particular rule used by the agents. The effect might be considerable. For instance, there exists a tie-breaking rule under which any SCC is implementable in the presence of three or more agents. If the planner is not endowed with the knowledge of the rule, then the problem of implementation is almost equivalent to double implementation in Nash and strict Nash equilibrium. A characterization is provided and it is shown that this severely limits the set of implementable SCCs.

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1 Introduction

Both through introspection and empirical evidence we know that individuals are not solely interested in their welfare. They also have considerations such as caring for the welfare of others, an inclination to conform or a tendency to tell the truth. In a nutshell, this paper investigates the problem of Nash implementation when such considerations can lexicographically affect the preference of individuals as tie-breakers in case of indifference.

It turns out even such a minor departure from sole self-interest can have a major impact on the problem of Nash implementation. This impact might be permissive, to the extent that any SCC can be implementable in the presence of more than three individuals, or limiting, to the extent that it could exclude many of the reasonable SCCs that are Nash implementable.

To understand the idea better, suppose agent $i$ has lexicographic preference for Pareto efficiency. The Pareto concern of the individual will only kick in when he is materially indifferent between sending two messages. Suppose agent $i$ believes that the other agents will send the message profile $m_{-i}$. If he is indifferent between the outcome he receives

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†E-mail Address: mert_kimya@brown.edu
when he announces the message $m_i$ and the message $m'_i$, but if the outcome corresponding to $(m_i, m_{-i})$ Pareto dominates the outcome corresponding to $(m'_i, m_{-i})$ then the agent will strictly prefer to send the message $m_i$. This is what is meant by lexicographic preference for ‘Pareto efficiency’ or any other concern the individual might have.

The following examples demonstrate the wide range of tie-breaking rules agents might use.

**Example 1. (Partial Honesty (Dutta and Sen (2012), Matsushima (2008)))** An individual is said to be partially honest if he prefers to announce the true state of the world whenever lying does not strictly improve his welfare.

**Example 2. (Pareto)** An individual with Pareto concerns breaks indifference in favor of Pareto dominating outcomes.

**Example 3. (Dissent)** An individual is said to be a dissident if he breaks indifference against the objectives of the planner.\(^1\)

**Example 4. (Fairness)** The agent has a strict, transitive and possibly incomplete ranking of alternatives $\succ^\theta$ such that if $a \succ^\theta b$ then the agent believes that the outcome $a$ is fairer than outcome $b$ in state $\theta$. The agent breaks indifference in favor of outcomes that he perceives as fairer.

**Example 5. (Hurt me, I’ll hurt you)** Agent $i$ views some of the messages sent by agent $j$ as hostile. If agent $i$ believes that agent $j$ will send one of these messages, then he breaks indifference in favor of the messages that decrease the welfare of agent $j$.

**Example 6. (Help me, I’ll help you)** Same as above, except now the agent views some messages as friendly and retaliates by trying to increase the welfare of the corresponding agent.

**Example 7. (Conformity)** The individual views the set $S$ of the agents as his social circle. He prefers his message (or the outcome he receives) to be similar to what the agents in his social circle sends (or receives).

Throughout, it is assumed that there is complete information and the focus is on Nash implementation. The paper is divided into two main sections. In one of those sections I will assume that the designer is aware of the particular rule used by the agents to break indifference. I will demonstrate that depending on the rule used this might expand or shrink the set of implementable SCCs. The effect is considerable. For instance there exists a tie-breaking rule under which any SCC is implementable in the presence of three or more agents.

The permissive results in this section and the availability of a number of reasonable tie-breaking rules imply that it might be a strong assumption to endow the designer with the knowledge of the tie-breaking rule. In the light of this concern, in the second main section of the paper I will assume that the designer is not aware of the tie-breaking rule used by the agents and he wants to come up with a mechanism that implements the SCC no matter what tie-breaking rule agents might be using.

It is shown that this leads to a rather restrictive result. A condition, which I call strong quasi-monotonicity (SQM) is necessary for implementation. This condition is a non-trivial strengthening of Maskin monotonicity. Furthermore robust implementation in this

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\(^1\)Formal definition of dissent is provided in Section 2.
sense is almost equivalent to double implementation in Nash and strict Nash equilibrium. An almost complete characterization of double implementation is also obtained as a byproduct of this section.

I will start with the literature review in Section 1.1. In Section 2, I will introduce the model. In Section 3, I will assume that the designer is aware of the particular tie-breaking rule used by the agents. I will characterize implementation under three reasonable rules, which are ‘partial honesty’, ‘pareto’ and ‘dissent’. This section will be concluded by mentioning some of the pitfalls of the awareness assumption. In Section 4, I will consider the case where the designer is not aware of the tie-breaking rule, hence wants to come up with a mechanism that implements the SCC no matter what tie-breaking rule the agents might be using. It is shown that Nash implementation in this sense is almost equivalent to double implementation in Nash and strict Nash equilibrium and I characterize the set of implementable SCCs under this assumption.

1.1 Relationship to the Literature

The paper is most closely related to the literature on implementation when agents have lexicographic preferences for honesty (partial honesty). Dutta and Sen (2012) has provided an almost complete characterization of Nash implementation under this assumption. They show that if the planner knows that at least one agent is partially honest then any SCC satisfying no veto power (NVP) can be Nash implemented. Proposition 1 is a slight extension of their result.

Matsushima (2008a) shows that if there are at least three agents and if the planner can impose fines, then any social choice function can be implemented through iterative elimination of dominated strategies if the agents are partially honest. Whereas Matsushima (2008b) studies the question in a Bayesian environment. Kartik et al. (2014) demonstrate that when agents have small preferences for honesty, under a condition called ‘separability’ any social choice function can be implemented through two rounds of iterative elimination of strictly dominated strategies. Ortner (2015) shows that under partial honesty, if there are at least 5 individuals then any social choice function is implementable in fault tolerant equilibrium (closely related to Eliaz (2002)) and stochastically stable equilibrium (see Kandori et al. (1993), Young (1993)).

Only Section 3.1 of this paper studies the assumption of partial honesty, the rest of the paper either studies other considerations or robust implementation. Furthermore all of the above papers present very permissive results, whereas the current paper shows that agents having concerns that affect their preferences lexicographically might indeed make the implementation problem considerably harder.

Aside from these, there is a growing literature on behavioral implementation. De Clippel (2014) generalizes implementation theory to the case where the agents’ choice need not be consistent with the maximization of a preference relation. In his work, the choices of the agents depend only on the set of options available to them, hence it does not contain my work as a special case. Eliaz (2002) studies Nash implementation when agents might be faulty, in the sense that they may fail to act optimally. In Cabrales and

\[\text{Among these, partial honesty has already been extensively studied in the literature. See the Literature Review for details.}\]

\[\text{For more on implementation under the assumption of partial honesty, see Korpela (2014) and Lombardi and Yoshihara (2011).}\]
Serrano (2011) agents myopically adjust their actions in the direction of best responses.4

The paper is also related to the literature on double implementation, which seeks to find mechanisms that implement a given SCC in two different solution concepts. Yamato (1993) studies double implementation in Nash and undominated Nash equilibrium, Saijo et al. (2007) in Nash and dominant strategy equilibrium and Schmeidler (1980) and Suh (1997) in Nash and strong Nash equilibrium.5

Finally, I show that robust implementation is almost equivalent to double implementation in Nash and strict Nash equilibrium. To my knowledge there does not exist a characterization of double implementation in Nash and strict Nash equilibrium, hence the current paper is also the first to provide this. Nash implementation has been characterized by Maskin (1999) and strict Nash implementation has been characterized by Cabrales and Serrano (2011).

2 Preliminaries

The environment is \{N,Z,\Theta\}, where N is the finite set of agents, Z is the set of alternatives and \Theta is the set of states. Each agent \(i \in N\) has a complete, transitive and reflexive preference ordering \(R^\theta_i\) over \(Z\) for all \(\theta \in \Theta\). \(P^\theta_i\) and \(I^\theta_i\) denote the asymmetric and symmetric components of \(R^\theta_i\). I assume that there is complete information.

A social choice correspondence (SCC) is a mapping \(f\) that assigns a nonempty set of alternatives to each state, i.e. \(f : \Theta \rightarrow 2^Z \setminus \emptyset\). A mechanism \(g = (M, \pi) = \{(M_i)_{i \in N}, \pi\}\) consists of a message set \(M_i\) for each agent \(i\) and an outcome function \(\pi : \prod_{i \in N} M_i \rightarrow Z\). Let \(G\) denote the set of all mechanisms. In each state \(\theta\) a mechanism \(g\) induces a game. Let \(N(g, R^\theta)\) denote the set of Nash equilibrium outcomes of the corresponding game in state \(\theta\). A SCC \(f\) is implementable in Nash equilibrium if there exists a mechanism \(g \in G\) such that for all \(\theta \in \Theta\), \(f(\theta) = N(g, R^\theta)\).

**Definition 1. Tie-breaking Rule**

\(T\) is a tie-breaking rule if for any \(g = (M, \pi) \in G\) and \(\theta \in \Theta\), \(T(g, \theta)\) is an asymmetric and transitive relation on \(M \times M\). Let \(T\) denote the set of all tie-breaking rules.

We can think of the tie-breaking rule as a second strict and transitive preference relation the agent consults only when he is indifferent between sending two different messages. The rule may depend on the mechanism and the state of the world. It might be incomplete. For example it might be empty, in which case we would be back to Nash implementation.

The idea is that if an individual is using the tie-breaking rule \(T\), then given a mechanism \(g\) and a state of the world \(\theta\) if he is indifferent between sending the message \(m_i\) and \(m'_i\) when the message profile announced by other agents is \(m^*_\cdot\), then he will prefer to send \(m_i\) over \(m'_i\) if \(((m_i, m^*_\cdot), (m'_i, m^*_\cdot)) \in T(g, \theta)\). The definition of extended preference below formalizes this notion.

**Definition 2. Extended Preference \((\succeq_{T,\theta})\)**

Given a tie-breaking rule \(T\), a state \(\theta \in \Theta\) and a mechanism \(g = (M, \pi)\), the extended preference of agent \(i\), \(\succeq_{i, T, \theta}\), is defined as:

- If \(\pi(m)S^\theta_i \pi(m')\) then \(m \succeq_{i, T, \theta} m'\).

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4 Also see Bierbauer and Netzer (2014), Glazer and Rubinstein (2012) and Korpela (2012)
5 For more on double implementation see Peleg (1996a, 1996b) and Corchon and Wilkie (1996).
• If $\pi(m) I^\theta_i \pi(m')$ and $(m, m') \in T(g, \theta)$ then $m \succ_i^{T, \theta} m'$.

• If $\pi(m) I^\theta_i \pi(m')$ and $(m, m') \notin T(g, \theta)$ then $m \sim_i^{T, \theta} m'$.

Consistent with the paragraph above, the only place in which the extended preference would not agree with the underlying preference is when there is indifference in the underlying preference. Note that although $T(g, \theta)$ is assumed to be transitive, the extended preference may fail to be transitive. For an example see the Pareto rule and Footnote 8. Nevertheless, the asymmetric part of the extended preference, i.e. $\succ_i^{T, \theta}$, will always be transitive (see Lemma 8 in Appendix A). Below, I explicitly write down some of the tie-breaking rules discussed in the Introduction.

Partial Honesty: An individual is said to be partially honest if he prefers to announce the true state of the world whenever lying does not strictly improve his welfare. Formally, a partially honest individual $i$ uses the following tie-breaking rule. $T(g, \theta) = \{(m, m') | m^1 = \theta, m'^1 \neq \theta\}$ if $g$ is such that $M_i = \Theta \times S_i$ for some set $S_i$, otherwise $T(g, \theta) = \emptyset$. Note that partial honesty would only have a bite in direct mechanisms or mechanisms where the message space is augmented with the state space.

Pareto: An individual with Pareto concerns breaks indifference in favor of Pareto dominating outcomes, i.e. for any $g$ and $\theta$ we have $(m, m') \in T(g, \theta)$ iff $\pi(m) \succ^P \pi(m')$, where for $a, b \in Z$, $a \succ^P b$ iff $aR^i_\theta b$ for all $i \in N$ and $aS^j_i b$ for some $j \in N$.

Dissent: An individual is said to be a dissident if he breaks indifference against the objectives of the planner. In particular for any $g$ and $\theta$ we have $(m, m') \in T(g, \theta)$ iff $\pi(m) \notin f(\theta)$ and $\pi(m') \in f(\theta)$.

3 With the Knowledge of the Tie-Breaking Rule

In this section I will assume that the planner is endowed with the knowledge of the tie-breaking rules the agents use. I will dispense with this assumption in the next section. I will start by characterizing Nash implementation under several intuitive tie-breaking rules. The rules have been chosen to show that the knowledge of the tie-breaking rule might expand (partial honesty), shrink (dissent) or neither expand nor shrink (Pareto) the set of implementable SCCs.

The following definition assumes that every agent uses the same rule to break indifference and the planner knows this rule. It is basically the definition of Nash implementation with the extended preferences.

Definition 3. $T$-Implementation

Given $T \in \mathcal{T}$, we say that a SCC $f$ is $T$-Implementable in Nash equilibrium if there exists a mechanism $g = (M, \pi)$ such that for all $\theta \in \Theta$, $f(\theta) = N(g, \succ_i^{T, \theta})$.

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6 As usual, $\sim_i^{T, \theta}$ and $\succ_i^{T, \theta}$ denote the symmetric and asymmetric components of $\succ_i^{T, \theta}$.

7 $m^1_i$ denotes the first component of agent $i$’s message.

8 The extended preferences corresponding to this rule may fail to be transitive. For example let $Z = \{a, b, c\}$, $N = \{1, 2, 3\}$, let the underlying preferences in state $\theta$ be $aR_1 \theta b$ and $cS_2 \theta b$ and $aR_3 \theta bS_3 \theta c$. Let $g = (M, \pi)$ be a mechanism that includes the messages $m_1$, $m_2$ and $m_3$, where $\pi(m_1) = a$, $\pi(m_2) = b$ and $\pi(m_3) = c$. Then if agent 1 is using the Pareto rule to break indifference, his extended preference will be the following which is intransitive: $m_1 \succ_1^\theta m_2 \sim_1^\theta m_3 \sim_1^\theta m_1$.

9 For any $\theta \in \Theta$ we have $a \in N(g, \succ_i^{T, \theta})$ if there exists a message profile $m^*$ with $\pi(m^*) = a$ such that for all $i \in N$, $m^* \succ_i^{T, \theta} (m_i, m^*_{-i})$ for all $m_i \in M_i$
3.1 Partial Honesty

If a SCC is implementable when the agents use this rule, I will say that the SCC is honesty-implementable.

Dutta and Sen (2012) showed that if the planner knows that at least one of the agents is partially honest and if \( n \geq 3 \) then any SCC satisfying no veto power (NVP) is honesty-implementable.\(^ {10} \) The following proposition extends their result by assuming that all individuals are partially honest. Note that the proposition is still valid if the planner knows that there are at least 2 partially honest individuals, but for consistency with the coming sections I will assume that all individuals are partially honest.

**Proposition 1.** Suppose \( n \geq 3 \) and \( f \) satisfies unanimity then \( f \) is honesty-implementable.\(^ {11} \)

The proof of the proposition, as well as all other proofs, is in the Appendix. With a little more work we can show that the assumption of partial honesty greatly expands the set of implementable SCCs.

**Corollary 1.** If a SCC \( f \) is Nash implementable then it is honesty-implementable. A SCC might be honesty-implementable, but not Nash implementable.

3.2 Pareto

If a SCC is implementable when the agents use this rule, I will say that the SCC is Pareto-implementable. The following modification of Maskin monotonicity is a necessary condition for Pareto-implementation.

**Definition 4.** P-Monotonicity (P-Mon)

A SCC \( f \) satisfies P-Mon if for all \( \theta, \theta' \in \Theta \) whenever \( a \in f(\theta) \) and \( a \notin f(\theta') \), there exists \( i \in N \) and \( b \in Z \) such that \( a \succ_{\theta}^P b \), \( b \nsucc_{\theta}^P a \) and either \( b \succ_{\theta'}^P a \) or \( b \nsucc_{\theta'}^P a \).

**Lemma 1.** If \( f \) is Pareto-implementable then \( f \) satisfies P-Mon.

P-Mon neither implies nor is implied by Maskin monotonicity. The next two examples demonstrate this point using the strong and weak Pareto correspondences.

**Example 8.** Weak Pareto correspondence \( (f^{WP}) \) selects all weakly Pareto optimal outcomes, i.e. for all \( \theta \in \Theta \),

\[
 f^{WP}(\theta) = \{ z \in Z | \text{there exists no } z' \in Z \text{ with } z' \succ_{\theta}^S z \text{ for all } i \in N \} 
\]

It is well-known that the weak Pareto correspondence satisfies monotonicity (see Maskin and Sjostrom (2002)). But it violates P-Mon. To see this, let \( N = \{1, 2\} \), \( Z = \{x, y\} \), \( \Theta = \{\theta, \theta'\} \) and let the preferences be \( xI_1^\theta y \), \( yS_2^\theta x \), \( yS_1^\theta x \) and \( yS_2^\theta x \). Then \( f^{WP}(\theta) = \{x, y\} \) and \( f^{WP}(\theta') = \{y\} \). Observe that \( x \in f^{WP}(\theta) \) and \( x \notin f^{WP}(\theta') \), but there does not exist an alternative \( b \neq x \) such that \( b \nsucc_{\theta'}^P x \). Hence, P-Mon is violated.

\(^ {10} \)Where \( n = |N| \) and a SCC \( f \) satisfies NVP if, for all \( a \in Z, \theta \in \Theta \) and \( j \in N \) whenever \( aR_i^\theta b \) for all \( b \in Z \) and \( i \neq j \) then \( a \notin f(\theta) \).

\(^ {11} \)A SCC \( f \) satisfies unanimity if for all \( \theta \in \Theta \) and \( a \in Z \), if \( aR_i^\theta b \) for all \( b \in Z \) and \( i \in N \) then \( a \in f(\theta) \).
Example 9. Strong Pareto correspondence \((f^{SP})\) selects all Pareto optimal outcomes, i.e. for all \(\theta \in \Theta\),

\[
f^{SP}(\theta) = \{ z \in Z | \text{there exists no } z' \in Z \text{ with } z' P^{PD}_\theta z \}\]

It is well-known that the strong Pareto correspondence violates monotonicity (see Maskin and Sjostrom (2002)). But, it satisfies P-Mon. To see this suppose \(a \in f^{SP}(\theta)\) and \(a \notin f^{SP}(\theta')\). But then there exists \(b \in Z\) such that \(b \not P^{PD}_\theta a\) and \(b P^{PD}_{\theta'} a\), furthermore since \(b \not P^{PD}_\theta a\), there exists \(i \in N\) with \(a R^b_i b\) and hence P-Mon is satisfied.

It turns out that P-Mon is almost sufficient for Pareto-implementation. In particular with the following weakening of NVP it is sufficient.

**Definition 5. Weak No Veto Power (WNVP)**

A SCC satisfies WNVP if for all \(\theta \in \Theta\), \(a \in Z\) and \(i \in N\) whenever \(a\) is Pareto efficient in \(\theta\) and there exists \(i \in N\) such that \(a R^b_i b\) for all \(j \neq i\) and for all \(b \in Z\) then \(a \in f(\theta)\).

**Proposition 2.** If \(n \geq 3\) and if \(f\) satisfies P-Mon and WNVP then \(f\) is Pareto-implementable.

Finally, the set of Nash implementable SCCs and the set of Pareto-implementable SCCs do not contain each other. We have already seen that the strong Pareto correspondence is not Nash implementable, but it satisfies P-Mon. It also trivially satisfies WNVP and hence it is Pareto-implementable. On the other hand, weak Pareto correspondence is not Pareto-implentable by Example 8, but it satisfies both NVP and Maskin monotonicity, hence it is Nash implementable.

**Corollary 2.** A SCC might be Nash implementable, but not Pareto-implementable. A SCC might be Pareto-implementable, but not Nash-implementable.

### 3.3 Dissent

If a SCC is implementable when the agents use this rule, I will simply say that the SCC is dissent-implementable. The following modification of Maskin monotonicity is a necessary condition for dissent-implementation.

**Definition 6. D-Monotonicity (D-Mon)**

A SCC \(f\) satisfies D-Mon if for all \(\theta, \theta' \in \Theta\) whenever \(a \in f(\theta)\) and \(a \notin f(\theta')\), there exists \(i \in N\) and \(b \in Z\) such that \(b S^{\theta'}_i a\) and either \(a S^b_i b\) or \(a I^b_i b\) and \(b \in f(\theta)\).

**Lemma 2.** If \(f\) is dissent-implementable then \(f\) satisfies D-Mon.

It is immediate that D-Mon is a stronger condition than Maskin monotonicity. D-Mon and NVP is sufficient for dissent-implementation.

**Proposition 3.** If \(n \geq 3\) and if \(f\) satisfies D-Mon and NVP then \(f\) is dissent-implementable.

Finally, the set of SCCs that are dissent-implementable is a strict subset of the set of SCCs that are Nash-implementable.

**Corollary 3.** If a SCC \(f\) is dissent-implementable then it is Nash-implementable. A SCC might be Nash-implementable, but not dissent-implementable.
3.4 Further Results and Critique

Perhaps, the most striking result of the previous section is that under the seemingly innocuous assumption of partial honesty, any SCC satisfying unanimity can be implemented. One interesting question is how much further can we take this result? That is, is there a rule that could expand the set of implementable SCCs even further? The answer is yes, and indeed we can go all the way. The proposition below shows that there exists a rule that is the ‘easiest to implement’ and furthermore any SCC is implementable under this rule when there are three or more agents.

**Proposition 4.** There exists a rule $\hat{T}$ such that if $n \geq 3$ then

- If a SCC $f$ is $T$-implementable for some $T \in \mathcal{T}$ then it is $\hat{T}$-implementable.
- Any SCC $f$ is $\hat{T}$-implementable.

The proof is by construction. In particular, I construct a tie-breaking rule under which agents prefer to announce the true state of the world. If they are announcing the true state of the world and if an integer game is being played then they prefer to be the ‘winner’ of the integer game. With such a rule it is easy to get rid of the ‘bad’ equilibria using an integer game. See the Appendix for details.

**Critique**

The permissive results of the previous sections and the availability of a number of intuitive tie-breaking rules imply that in some situations endowing the designer with the knowledge of the tie-breaking rule might be a strong assumption to make. The example below substantiates this concern. As partial honesty is the rule that has been used in the literature before, I am going to assume that individuals are partially honest and show that this assumption might lead a quite perverse and unreasonable SCC to be implemented.

**Example 10.** An Exchange Economy

There are 3 agents and 3 goods ($x, y$ and $z$), the total endowment of each good is 1. There are two states $\theta$ and $\theta'$, the preferences are represented by the following utility functions: $u^\theta_1 = x, u^\theta_2 = y, u^\theta_3 = z$ and $u'^\theta_1 = z, u'^\theta_2 = x, u'^\theta_3 = y$. The SCC is defined by $f(\theta) = \{(0,0,1), (1,0,0), (0,1,0)\}$ and $f(\theta') = \{(1,0,0), (0,1,0), (0,0,1)\}$, where the first entry denotes the bundle assigned to agent 1. The planner is malevolent, he assigns the most preferred bundles in state $\theta$ to state $\theta'$ and vice versa, by doing so he is also assigning the least preferred bundles to each state. The SCC is always going against the interests of every agent, but under the assumption of partial honesty it can be implemented with the following simple mechanism. Let $M_i = \{\theta, \theta'\}$ for $i = 1, 2, 3$ (note that the mechanism is the same one that is used by Dutta and Sen (2012) to prove Theorem 4 in their paper):

- **Rule 1:** If everyone announces the same state $\theta^*$, then $f(\theta^*)$ is implemented.
- **Rule 2:** If everyone except agent $j$ announces the same state $\theta^*$, while agent $j$ announces $\theta_1 \neq \theta^*$, then $j$ gets $f(\theta^*)_j$ (that is what he gets under $f(\theta^*)$), while all other agents get $(0,0,0)$.

Observe that when all of the agents are partially honest, it is a strictly dominant strategy to announce the true state of the world. Hence, the mechanism honesty-implements the SCC. Although the agents can get their most preferred alternative in every state, they do not, but get the least preferred instead.
This example also shows that given a certain mechanism several intuitive and reasonable rules might be contradicting each other. In this example, when everybody is lying, an agent deviates in order to be truthful. But through this deviation he is hurting all of the other agents, while keeping his welfare constant. Hence, in this mechanism the assumption of partial honesty contradicts with the assumption of Pareto concerns, although both assumptions seem reasonable at first sight.

Therefore, we might want our mechanism to be robust in the sense that we might want it to implement a certain SCC no matter what rule the agents might be using. The following section analyses this problem.

4 Without the Knowledge of the Rule

The following definition assumes that the designer is not endowed with the knowledge of the rule agents might be using. Therefore, he has to design a mechanism that implements the SCC for every possible rule agents might use.

**Definition 7. Robust T-Implementation (RT-implementation)**

A SCC $f$ is RT-Implementable in Nash equilibrium if there exists a mechanism $g$ such that for all $\theta \in \Theta$ and for all $T \in T$, $f(\theta) = N(g, \succeq^T, \theta)$.

The following lemma directly follows from Corollary 3.

**Lemma 3.** If $f$ is RT-implementable then $f$ is Nash implementable. Therefore Maskin monotonicity is a necessary condition for RT-implementation. Furthermore an SCC might be Nash implementable, but not RT-implementable.

A strengthening of monotonicity, which I call strong quasi-monotonicity is a necessary condition for RT-implementation.\(^{12}\)

**Definition 8. Strong Quasi-monotonicity (SQM)**

A SCC $f$ satisfies SQM if for all $\theta, \theta' \in \Theta$ whenever $a \in f(\theta)$ and $a \notin f(\theta')$, there exists $i \in N$ and $b \in Z$ such that $a \succeq^\theta_i b$ and $b \succeq^\theta_i a$.

**Lemma 4.** If $f$ is RT-implementable then $f$ satisfies SQM.

SQM is a nontrivial strengthening of Maskin monotonicity and it shrinks the set of implementable SCCs. For example weak Pareto correspondence and dictatorship satisfy Maskin monotonicity but neither of these satisfy SQM.\(^{13}\)

We say that a mechanism $g$ is participative if it allows every agent to participate in the mechanism, i.e. for each $i \in N$, $|M_i| > 1$. The following condition was defined in Cabrales and Serrano (2011) and it is a necessary condition for strict Nash implementation by means of a participative mechanism.\(^{14}\) From now on, if a condition is necessary

\(^{12}\)The name refers to the condition of quasimonotonicity defined in Cabrales and Serrano (2011). Quasimonotonicity is a necessary condition for strict Nash implementation, which is closely related to RT-implementation (see the next section), furthermore SQM is a strengthening of quasimonotonicity.

\(^{13}\)To see this take $N = \{1\}$, $Z = \{a, b\}$, $\Theta = \{\theta, \theta'\}$, $a \not\succeq^\theta_i b$ and $b \succeq^\theta_i a$. In this example both the dictatorship and the weak Pareto correspondence is $f(\theta) = \{a, b\}$ and $f(\theta') = \{b\}$. It violates SQM.

\(^{14}\)Cabrales and Serrano (2011) show that NWA is necessary for strict Nash implementation by means of a ‘non-imposing’ mechanism. A non-imposing mechanism allows every agent to meaningfully participate in the game, i.e. for every $i \in N$ there exists $m_i, m'_i \in M_i$ such that for some $m_{-i}$, $\pi(m_i, m_{-i}) \neq \pi(m'_i, m_{-i})$. Hence every non-imposing mechanism is a participative mechanism, but not the other way around. Nevertheless it is easy to see that NWA is a necessary condition for strict Nash implementation by means of a participative mechanism.
for implementation through a participative mechanism, I will say that the condition is essentially necessary.

**Definition 9. No Worst Alternative (NWA)**

A SCC $f$ satisfies NWA whenever for every $i \in N$, $\theta \in \Theta$ and $a \in f(\theta)$ there exists an outcome $z_i^{a,\theta}$ such that $a S_i^{\theta} z_i^{a,\theta}$.

NWA is not a necessary condition for RT-implementation even through a participative mechanism. The example in the proof of Lemma 6 shows this. Although NWA is not essentially necessary, a slight weakening of NWA is essentially necessary.

**Definition 10. Weak NWA (WNWA)**

A SCC $f$ satisfies WNWA whenever for every $\theta \in \Theta$ and $a \in f(\theta)$ there exists $j \in N$ such that for every $i \neq j$, there exists an outcome $z_i^{a,\theta}$ such that $a S_i^{\theta} z_i^{a,\theta}$.

WNWA requires that $a \in f(\theta)$ is not the worst alternative for every agent, whereas WNWA requires that $a$ is not the worst alternative for $n - 1$ agents. Hence it is a slight weakening of NWA and it is essentially a necessary condition.

**Lemma 5.** If a SCC $f$ is RT-implementable by means of a participative mechanism then $f$ satisfies WNWA.

Finally, SQM and WNWA almost completely characterize RT-implementation.

**Proposition 5.** If $n \geq 3$ then $f$ is RT-implementable if $f$ satisfies SQM, NVP and NWA.

**Relation to Double Implementation in Nash and Strict Nash Equilibrium**

One interesting property of RT-implementation is that it is almost equivalent to double implementation in Nash and strict Nash equilibrium and in this section I will show this. Let $SN(g, R^\theta)$ denote the set of strict Nash equilibrium outcomes of mechanism $g$ in state $\theta$.

**Definition 11. Double Implementation**

A SCC $f$ is doubly implementable in Nash equilibrium and strict Nash equilibrium if there exists a mechanism $g$ such that for all $\theta \in \Theta$, $f(\theta) = N(g, R^\theta) = SN(g, R^\theta)$.

The following is a strengthening of RT-implementation. It will be useful, as it is equivalent to double implementation and almost equivalent to RT-implementation. Just like RT-implementation, it requires that for all $\theta \in \Theta$ and for all $T \in T$, $f(\theta) = N(g, \succeq^{T,\theta})$. But on top of this, it also requires that for each $\theta$ and $a \in f(\theta)$, there is a rule independent equilibrium of the mechanism $g$ in state $\theta$ that implements $a$. Let $\hat{N}(g, \succeq^{T,\theta})$ denote the set of Nash equilibria of $g$ with the extended preferences corresponding to $T$.

**Definition 12. Strong Robust T-implementation (SRT-implementation)**

A SCC $f$ is SRT-implementable if there exists a mechanism $g$ such that

- For all $\theta$ and $a \in f(\theta)$ there exists a message profile $m^*$ such that for all $T \in T$ we have $m^* \in \hat{N}(g, \succeq^{T,\theta})$ and $\pi(m^*) = a$. 

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• If \( a \in N(g, \succeq^T, \theta) \) for some \( T \in \mathcal{T} \) then \( a \in f(\theta) \).

The definition directly implies RT-implementation. However, the reverse is not true. The following lemma states this.

**Lemma 6.** If a SCC \( f \) is SRT-implementable then \( f \) is RT-implementable. A SCC \( f \) might be RT-implementable, but not SRT-implementable.

Although SRT-implementation does not imply RT-implementation, they are still almost equivalent. In particular, in the example used to prove Lemma 6 (see Appendix) NWA is violated, which is not an essentially necessary condition for RT-implementation. But as we have seen a very slight weakening of NWA, which I called WNWA, is necessary for RT-implementation.

The proposition below establishes the equivalence between double implementation and SRT-implementation.

**Proposition 6.** A SCC \( f \) is SRT-implementable iff \( f \) is doubly implementable in strict Nash and Nash equilibrium.

NWA is an essentially necessary condition for strict Nash implementation (see Cabrales and Serrano (2011)), hence by the proposition above it is essentially necessary for SRT-implementation. Furthermore, since SRT-implementation is a strengthening of RT-implementation SQM is a necessary condition for SRT-implementation and by the proposition above for double implementation. The corollary below states this.

**Corollary 4.** SQM is necessary for both SRT-implementation and double implementation in Nash and strict Nash equilibrium. And NWA is an essentially necessary condition.

Finally, combined with NVP, these conditions are also sufficient. This proposition also shows that RT-implementation and SRT-implementation (and hence double implementation) are almost equivalent.

**Proposition 7.** If \( n \geq 3 \) and if a SCC \( f \) satisfies SQM, NWA and NVP then it is SRT-implementable (hence, doubly implementable in Nash and strict Nash).

## 5 Concluding Remarks

**Remark 1.** In Section 3, I have characterized implementation with only three tie-breaking rules. In general, given a tie-breaking rule it is quite straightforward to characterize implementation under the rule. All we have to do is to figure out how Maskin monotonicity and NVP under the extended preferences translate into a condition under the fundamental preferences. Therefore, general rule-dependent-conditions of Maskin monotonicity and NVP can be written. However, as the rule may depend on the particular mechanism used, I did not find this particularly useful. This is why such a general condition does not appear in the paper. Nevertheless, Appendix-A contains a general characterization of the rules that do not depend on the particular mechanism used.

**Remark 2.** The tie-breaking rule is defined as a deterministic rule, but it can also be defined as a probabilistic rule in which the agents use a certain tie-breaking rule with a certain probability. As, under robust implementation the mechanism implements the SCC for any rule, the results on robust implementation would not change even if we include probabilistic tie-breaking rules in \( \mathcal{T} \).
Remark 3. The reader might wonder how the implementation problem would be affected if the agents’ concerns had stronger effects on their actions. In particular, an agent with a particular concern might act upon it even if acting upon it might reduce his welfare. Although this is an interesting question, I do not attempt to answer this question in this paper. The reason is that I think this question can most naturally be answered in a cardinal framework. Whereas I have chosen to stay in an ordinal framework for the sake of simplicity and generality.

Appendix A.

Extended Preferences

Here I will develop some of the properties of the extended preferences. First note that extended preferences are necessarily complete and reflexive, but as we have already noted they need not be transitive.

Lemma 7. Extended preferences need not be transitive.

Proof. See the Pareto rule and Footnote 8.

Nevertheless, this intransitivity is mild in the sense that any preference cycle should include at least one indifference.

Lemma 8. For any tie-breaking rule $T$, mechanism $g = (M, \pi)$ and $\theta \in \Theta$ if for some $i \in N$ we have $m_1 \succeq_i T, \theta m_2 \succeq_i T, \theta m_3 \ldots \succeq_i T, \theta m_n \succeq_i T, \theta m_{n+1}$, where $m_{n+1} = m_1$ then $m_j \sim_i T, \theta m_{j+1}$ for at least one $j = 1, \ldots, n$.

Proof. Assume to the contrary we have $m_1 \succ_i T, \theta m_2 \succ_i T, \theta m_3 \ldots \succ_i T, \theta m_n \succ_i T, \theta m_{n+1}$, if for at least one $j = 1, \ldots, n$ we have $m_j S_i^\theta m_{j+1}$ then $m_1 S_i^\theta m_{n+1} = m_1$, a contradiction. Hence, $m_1 I_i^\theta m_2 I_i^\theta \ldots m_n I_i^\theta m_{n+1}$. But then by the definition of extended preferences $(m_j, m_{j+1}) \in T(g, \theta)$ for all $j = 1, \ldots, n$. By the transitivity of $T(g, \theta)$ we have $(m_1, m_1) \in T(g, \theta)$. A contradiction to the asymmetry of $T(g, \theta)$.

The following corollary directly follows from the lemma. It will be used in the proof of Lemma 6.

Corollary 5. Let $T$ be a tie-breaking rule and $g = (M, \pi)$ be a finite mechanism. For any $m^* \in M$, $\theta \in \Theta$ and $i \in N$, let $C_i(m^*, \theta) = \{m \in M_i | (m, m^*_{-i}) \succeq_i T, \theta (m', m^*_{-i})$ for all $m' \in M_i\}$. Then $C_i(m^*, \theta) \neq \emptyset$ for all $i \in N$ and $\theta \in \Theta$.

Outcome-based Rules

An outcome-based rule is a rule that does not depend on the particular mechanism used by the designer. More specifically, the agent has an ordering of alternatives at each state that he consults when he is indifferent. Pareto and dissent are examples of outcome-based rules.

Definition 13. Outcome-based Rule

$T \in T$ is an outcome based rule if there exists an asymmetric and transitive relation $\succ_i^\theta$ for each $\theta$ such that for each $g \in G$ and $\theta \in \Theta$ we have $(m, m') \in T(g, \theta)$ iff $\pi(m) \succ_i^\theta \pi(m')$. 


Since the actions of the agents do not depend on the particular mechanism (messages) used, the implementation under outcome-based rules reduces down to implementation under a reflexive and complete (but possibly intransitive) preference relation $\succeq^\theta_i$ on the outcomes which agrees with the extended preferences $\succeq^{T,\theta}_i$.

Hurwicz (1986) has shown that Maskin’s theorem is still valid even when the preferences of the agents fail to be transitive. Hence, all we need to do is to translate Maskin monotonicity and NVP from the extended preferences to the underlying preferences.

The definition of T-Monotonicity below corresponds to Maskin monotonicity when the agents are using an outcome-based rule. By the argument above, it is a necessary condition for T-implementation for outcome-based rules.

**Definition 14. T-Monotonicity (T-Mon)**

Let $T$ be an outcome-based rule. A SCC $f$ satisfies T-Mon if for all $\theta, \theta' \in \Theta$ whenever $a \in f(\theta)$ and $a \notin f(\theta')$, there exists $i \in N$ and $b \in Z$ such that

- $a \succeq^\theta_i b$ and either $a \succ^\theta_i b$ or $b \nprec^\theta_i a$
- And $b \succeq^\theta' i a$ and either $b \succ^\theta' i a$ or $b \nprec^\theta' i a$

T-Mon will be sufficient for implementation with the following modification of NVP.

**Definition 15. T-NVP**

Let $T$ be an outcome-based rule. A SCC satisfies T-NVP if for all $\theta \in \Theta$, $a \in Z$ if there exists $i \in N$ such that either $a \succeq^\theta Jb$ or $a \nprec^\theta J a$ for all $j \neq i$ and for all $b \in Z$ then $a \in f(\theta)$.

**Proposition 8.** Let $T$ be an outcome-based rule. If $f$ is $T$-implementable then $f$ satisfies T-Mon. Furthermore, if $n \geq 3$ and if $f$ satisfies T-Mon and T-NVP then $f$ is $T$-implementable.

**Proof.** As mentioned above, the result follows from Hurwicz (1986) and Definitions 2 and 13.

Finally note that T-NVP is a weakening of NVP, that is if $f$ satisfies NVP then it satisfies T-NVP. So, we have the following corollary.

**Corollary 6.** Let $T$ be an outcome-based rule. If $n \geq 3$ and if $f$ satisfies T-Mon and NVP then $f$ is $T$-implementable.

**Appendix B. Proofs**

**Proof of Proposition 1.** I will use exactly the same mechanism used by Dutta and Sen (2012). Let $M_i = \Theta \times Z \times N$ for every $i \in N$. The outcome function is defined by the following rules:

**Rule 1:** If $n-1$ individuals announce the same state $\theta$ with the same alternative $a \in f(\theta)$, then the outcome is $a$.

**Rule 2:** In all other cases the outcome is the one announced by the agent announcing the highest integer. (Ties are broken in favor of the agent with the lowest index).

\[15\] That means, for any $\theta \in \Theta$ and $g = (M, \pi) \in G$ we have $m \succeq^{T,\theta}_i m'$ iff $\pi(m) \succeq^\theta_i \pi(m')$
Step 1: For all $\theta \in \Theta$, we have $f(\theta) \subseteq N(g, \succeq^{T,\theta})$

Take any $\theta \in \Theta$ and $a \in f(\theta)$. Then $m_i = (\theta, a, 0)$ for all $i$ is a Nash equilibrium with the extended preferences, since no agent can change the outcome by deviating and since everyone is already truthful.

Step 2: For all $\theta \in \Theta$, we have $N(g, \succeq^{T,\theta}) \subseteq f(\theta)$

Observe that in any equilibrium everyone is truthful, as otherwise they can deviate to the truthful announcement without changing the outcome, which would be a profitable deviation. Hence if $m^*$ is an equilibrium in Rule 1 in state $\theta$ then $\pi(m^*) \in f(\theta)$.

Suppose there is an equilibrium under Rule 2 in state $\theta$, but this means that the outcome implemented, say $a$, is the most preferred alternative for everyone. By unanimity $a \in f(\theta)$. □

Proof of Corollary 1. For the second claim, observe that many SCCs such as the strong Pareto correspondence satisfy unanimity, but do not satisfy Maskin monotonicity, which is a necessary condition for Nash implementation (see Maskin and Sjostrom (2002)).

For the first claim, suppose that $f$ is Nash-implementable with the mechanism $g' = (M, \pi)$. Let $Z' = \{z \in Z | \pi(m) = z \text{ for some } m \in M\}$. I will use the mechanism in the proof of Proposition 1 by restricting the outcome announcement to $Z'$. Let $M_i = \Theta \times Z' \times N$ for every $i \in N$. The outcome function is defined by the following rules:

Rule 1: If $n-1$ individuals announce the same state $\theta$ with the same alternative $a \in f(\theta)$, then the outcome is $a$.

Rule 2: In all other cases the outcome is the one announced by the agent announcing the highest integer. (Ties are broken in favor of the agent with the lowest index).

Step 1: For all $\theta \in \Theta$, we have $f(\theta) \subseteq N(g, \succeq^{T,\theta})$

Take any $\theta \in \Theta$ and $a \in f(\theta)$. Then $m_i = (\theta, a, 0)$ for all $i$ is a Nash equilibrium with the extended preferences, since no agent can change the outcome by deviating and since everyone is already truthful.\(^{16}\)

Step 2: For all $\theta \in \Theta$, we have $N(g, \succeq^{T,\theta}) \subseteq f(\theta)$

Observe that in any equilibrium everyone is truthful, as otherwise they can deviate to the truthful announcement without changing the outcome, which would be a profitable deviation. Hence if $m^*$ is an equilibrium in Rule 1 in state $\theta$ then $\pi(m^*) \in f(\theta)$.

Suppose there is an equilibrium under Rule 2 in state $\theta$, but this means that the outcome implemented, say $a$, is the most preferred alternative in set $Z'$ for everyone in state $\theta$. But, in mechanism $g'$ there exists a message profile $m^*$ such that $\pi(m^*) = a$ and since $a$ is the most preferred alternative for everyone in state $\theta$, $m^*$ is an equilibrium of $g'$ in state $\theta$. Since $g'$ Nash implements $f$ we have $a \in f(\theta)$. □

Proof of Lemma 1. The rule is an outcome based rule and $(m, m') \in T(g, \theta)$ iff $\pi(m) \succeq^{T,\theta} \pi(m')$. Hence the result follows from Proposition 8. □

Proof of Proposition 2. Follows from Proposition 8. □

Proof of Lemma 2. The rule is an outcome based rule and $(m, m') \in T(g, \theta)$ iff $\pi(m) \notin f(\theta)$, but $\pi(m') \in f(\theta)$. Hence the result follows from Proposition 8. □

Proof of Proposition 3. Follows from Corollary 6. □

\(^{16}\)Note that $a \in Z'$, as otherwise $g'$ would not be implementing $f$ in Nash equilibrium.
Table 1: $f(\theta) = a$ and $f(\theta') = b$

Proof of Corollary 3. For the first part, suppose that $f$ is dissent-implementable by a mechanism $g$. I will show that $g$ Nash-implements $f$. As for all $\theta$ we have $N(g, \geq_{T, \theta}) \subseteq N(g, \geq_{T, \theta})$ and $f(\theta) \subseteq N(g, \geq_{T, \theta})$ we have that $f(\theta) \subseteq N(g, R^\theta)$. So, we only need to show that $N(g, R^\theta) \subseteq f(\theta)$. Towards a contradiction assume that for some $\theta$, $a \in N(g, R^\theta)$, but $a \notin f(\theta)$. Since $a \notin f(\theta)$, when agents are using the rule dissent, they do not have any incentive to deviate to an outcome that is indifferent to $a$ and since it is a Nash equilibrium they also do not have a deviation to a strictly preferred outcome. But then $a \in N(g, \geq_{T, \theta})$, a contradiction.

For the second part consider the following example. $N = \{1, 2, 3, 4\}$, $\Theta = \{\theta, \theta'\}$, $Z = \{a, b\}$, the preferences and the SCC is given in Table 1. $f$ satisfies Maskin monotonicity and no veto power, hence it is Nash implementable, but $f$ does not satisfy D-Mon, therefore it is not dissent-implementable. \hfill $\square$

Proof of Proposition 4. I will show the result by constructing such a $\hat{T}$ and then I will show that any SCC is $\hat{T}$-implementable, which implies the first part. In words, individuals using $\hat{T}$ break indifference in favor of the truth and if they are announcing the true state and if an integer game is being played then they break indifference in favor of being the winner of the game. First, I will define the mechanism and then I will define the rule. Let $M_i = \Theta \times \mathbb{N} \times Z$ for every $i \in N$. The outcome function is defined by the following rules:

Rule 1: If $n-1$ individuals announce the same state $\theta$ with the same alternative $a \in f(\theta)$, then the outcome is $a$.

Rule 2: In all other cases the outcome is the one announced by the agent announcing the highest integer. (Ties are broken in favor of the agent with the lowest index).

For all $i \in N$ let $T_i(g, \theta) = \{(m, m') | m_i^1 = \theta, m_i^2 \neq \theta \text{ or } m_i^1 = \theta = m_i^2, m' \in \text{Rule 2}, m_i^2 \leq m_i^2 \text{ for some } j \neq i \text{ and } m_i^2 > m_j^2 \forall j \neq i\}$.

Step 1: For all $\theta \in \Theta$, we have $f(\theta) \subseteq N(g, \geq_{T, \theta})$

Take any $\theta \in \Theta$, let $a \in f(\theta)$. Then $m_i = (\theta, a, 0)$ for all $i$ is a Nash equilibrium, since no agent can change the outcome by deviating and since everyone is already truthful and no one is playing the integer game.

Step 2: For all $\theta \in \Theta$, we have $N(g, \geq_{T, \theta}) \subseteq f(\theta)$

First note that in any equilibrium everybody is making a truthful announcement of the state, because otherwise any lying agent can deviate to a truthful announcement without changing the outcome. Hence if the equilibrium is under Rule 1 then we are done. But there cannot be a truthful equilibrium under Rule 2 as an integer game is being played and in any strategy profile anybody can deviate and become the winner by announcing the highest integer. \hfill $\square$

Proof of Lemma 4. Suppose $f$ is RT-implementable and $g$ implements $f$. Take any $\theta, \theta' \in \Theta$, $a \in Z$ such that $a \in f(\theta)$ and $a \notin f(\theta')$. As $g$ implements $f$ for every possible rule, it also implements $f$ when each agent uses the following rule: $T(g, \theta) = \{(m, m') | \pi(m) \neq a, \pi(m') = a\}$ and $T(g, \theta') = \{(m, m') | \pi(m) = a, \pi(m') \neq a\}$. $a$ is
an equilibrium outcome of \( g \) under the extended preferences in state \( \theta \) and there exists a deviation from the corresponding equilibrium in \( \theta' \) to an alternative \( b \). But with the extended preferences every agent strictly prefers to stay with \( a \) in \( \theta' \) instead of getting an alternative indifferent to \( a \), hence we should have that there exists \( b \in Z \) and \( i \) such that \( bS_i^\theta a \). And with the extended preferences every agent strictly prefers to deviate to an alternative indifferent to \( a \) in \( \theta \), hence we should have that \( aS_i^\theta b \).

**Proof of Lemma 5.** Towards a contradiction assume that \( f \) is RT-implementable by means of a participative mechanism \( g \) but \( f \) does not satisfy WNWA. Then there exists \( \theta \in \Theta \) and \( a \in f(\theta) \), \( i, j \in N \) with \( i \neq j \) such that \( a \) is the worst alternative in state \( \theta \) for both \( i \) and \( j \). Furthermore since \( g \) is participative, \( |M_i| \geq 2 \) and \( |M_j| \geq 2 \). Partition \( M_i \) and \( M_j \) into two non-empty disjoint sets, say \( M_i^1, M_i^2 \) and \( M_j^1, M_j^2 \). Let the tie breaking rule be such that if \( j \) sends a message in \( M_i^k \) for \( k = 1, 2 \) then \( i \) wants to send a message in \( M_i^k \), where \( t \neq k \). If \( i \) sends a message in \( M_i^k \) for \( k = 1, 2 \) then \( j \) wants to send a message in \( M_j^k \). Since \( g \) RT-implements \( f \), there exists an equilibrium \( m^* \) of \( g \) in state \( \theta \) such that \( \pi(m^*) = a \). Since \( a \) is the worst alternative for both \( i \) and \( j \) both of these agents should be materially indifferent between any message they can send when others are announcing their part of \( m^* \). But then under \( m^* \), \( i \)'s message is a best response to \( j \)’s (meaning \( i \) sends a message in \( M_i^k \) and \( j \) sends a message on \( M_j^k \) with \( k \neq t \)) and \( j \)'s message is a best response to \( i \)'s (meaning \( i \) sends a message in \( M_i^k \) and \( j \) sends a message on \( M_j^k \) with \( k = t \)). A contradiction. \( \square \)

**Proof of Proposition 5.** See Lemma 6 and Proposition 7. \( \square \)

**Proof of Proposition 6.** First claim directly follows from the definitions. Here, I will prove the second claim. Consider the following example. \( N = \{1, 2, 3, 4, 5\}, \Theta = \{\theta_1, \theta_2, \theta_3\}, Z = \{a, b, z\} \) and preferences and the SCC is given in Table 2. Any mechanism that strict Nash implements \( f \) should assign more than one message to agent 1, this is because agent 1 is the only agent whose preference changes as we move from state \( \theta_2 \) to \( \theta_3 \). But \( a \) cannot be supported as a strict Nash equilibrium outcome in state \( \theta_1 \) with such a mechanism, as \( a \) is the worst outcome for agent 1 in state \( \theta_1 \). Hence \( f \) is not strict Nash implementable, by Proposition 6, \( f \) is not SRT-implementable.

Finally I will show that \( f \) is RT-implementable. Let \( M_i = \Theta \times Z \times N \) for every \( i \in N \). The outcome function is defined by the following rules:

**Rule 1:** If for all \( i \in N \), \( m_i = (\theta, x, 1) \) where \( x \in f(\theta) \) then \( \pi(m) = x \).

**Rule 2:** If \( m_i = (\theta_1, a, 1) \) for \( i \neq 1 \), but \( m_1 = (\theta, x, n) \neq (\theta_1, a, 1) \) then \( \pi(m) = a \).

**Rule 3:** If for all \( j \neq i \) (where \( i \neq 1 \)) and \( j \neq 1 \), \( m_j = (\theta_1, a, 1) \) and \( m_i = (\theta', y, n) \neq (\theta_1, a, 1) \) then \( \pi(m) = y \) if \( aS_i^{\theta_1} y \) and \( yS_i^{\theta'} x \), otherwise \( \pi(m) = z \).

**Rule 4:** Otherwise, if for all \( j \neq i \), \( m_j = (\theta, x, 1) \), where \( x \in f(\theta) \) and \( m_i = (\theta', y, n) \neq (\theta, x, 1) \) then \( \pi(m) = y \) if \( xS_i^{\theta'} y \) and \( yS_i^{\theta'} x \), otherwise \( \pi(m) = z \).

**Rule 5:** In all other cases \( \pi(m) = m_i^2 \) where \( t = (\sum_{i \in N} m_i^3) (mod \ n) \).

\[
\begin{array}{c|ccc}
\theta & 1 & 2,3 & 4,5 \\
\hline
\theta_1 & aibiz & aSbsz & bSaSz \\
\theta_2 & bSaSz & bSaSz & aSbsz \\
\theta_3 & aSbsz & bSaSz & aSbsz \\
\end{array}
\]

Table 2: \( f(\theta_1) = f(\theta_3) = a \) and \( f(\theta_2) = b \)

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**Step 1:** For all \( \theta \in \Theta \), we have \( f(\theta) \subseteq N(g, \succeq_{T,\theta}) \) for all \( T \in \mathcal{T} \).

Take any \( T \in \mathcal{T} \). Let \( m^* \in M_1 \) be agent 1’s optimal message (or one of the optimal messages) with his extended preferences when \( m_i = (\theta_1, a, 1) \) for all \( i \neq 1 \). Note that since \( M_1 \) is finite by Corollary 5, \( m^* \) exists. In state \( \theta_1 \), \( m_1 = m^* \) and \( m_i = (\theta_1, a, 1) \) is an equilibrium since 1 cannot change the outcome by deviating and he is already sending the optimal response to other’s messages and any agent other than 1 can only induce an outcome which is strictly less preferred to \( a \).

Take any \( \theta \in \Theta \), \( \theta \neq \theta_1 \) and \( x \in f(\theta) \). Then \( m_i = (\theta, x, 1) \) for all \( i \in N \) is a Nash equilibrium with the extended preferences, because the agents can only deviate and induce outcomes that are either strictly worse than \( x \).

**Step 2:** For all \( \theta \in \Theta \), we have \( N(g, \succeq_{T,\theta}) \subseteq f(\theta) \) for all \( T \in \mathcal{T} \).

Take any \( T \in \mathcal{T} \). Since \( N(g, \succeq_{T,\theta}) \subseteq N(g, R^\theta) \) it is enough to show that for all \( \theta \in \Theta \), \( N(g, R^\theta) \subseteq f(\theta) \). Suppose there is an equilibrium in Rule 1 or Rule 2 in state \( \theta' \), but the announced state \( \theta \) is not the true state. Let \( x \) be the alternative implemented. Then there exists an individual \( i \) and an alternative \( y \) such that \( yS_i^\theta x \) and \( xS_i^\theta y \). This individual strictly prefers \( y \) over \( x \) and can induce \( y \), hence there is a profitable deviation.

Suppose there is an equilibrium in Rule 3, 4 or 5. Either agents 2 and 3 or agents 4 and 5 are not getting their most preferred alternative, but then there is at least one agent who can deviate and get his most preferred alternative. Contradiction. There is no equilibrium in rules 3, 4 or 5.

**Proof of Proposition 6.**

**Step 1:** \( f \) is SRT-implementable \( \Rightarrow \) \( f \) is doubly implementable

Suppose that \( f \) is SRT-implementable. Let \( g \) be a mechanism that implements \( f \). I will show that \( g \) doubly implements \( f \) in Nash and strict Nash equilibrium.

**Step 1-a:** Show that for all \( \theta \in \Theta \) we have \( f(\theta) \subseteq SN(g, R^\theta) \) (hence, \( f(\theta) \subseteq N(g, R^\theta) \)).

Take any \( \theta \in \Theta \) and \( a \in Z \) such that \( a \in f(\theta) \). Since \( g \) SRT-implements \( f \), in state \( \theta \) there exists a message profile \( m^* \) such that for all \( T \in \mathcal{T} \) we have \( m^* \in \hat{N}(g, \succeq_{T,\theta}) \) and \( \pi(m^*) = a \). I will show that \( m^* \) is a strict Nash equilibrium. Suppose not, there exists \( i \) and \( m' \in M \) such that \( m' = (m', m_{-i}^*) \neq m^* \) for some \( m \in M_i \) and \( \pi(m')R_i^\theta \pi(m^*) \). Let \( T \) be such that \( T_i(\theta, g) = \{ (m', m^*) \} \), but then \( m' \succeq_{T,\theta}^i m^* \) and hence \( m^* \) is not a Nash equilibrium of \( g \) with the extended preferences corresponding to \( T \). A contradiction.

**Step 1-b:** Show that for all \( \theta \in \Theta \) we have \( N(g, R^\theta) \subseteq f(\theta) \) (hence, \( SN(g, R^\theta) \subseteq f(\theta) \)).

Let \( T(\theta, g) = \emptyset \) for all \( g \) and \( \theta \). Note that \( N(g, R^\theta) = N(g, \succeq_{T,\theta}^\emptyset) \subseteq f(\theta) \) for all \( \theta \in \Theta \).

**Step 2:** \( f \) is doubly implementable \( \Rightarrow \) \( f \) is SRT-implementable.

Suppose that \( f \) is doubly implementable in strict Nash and Nash equilibrium. Let \( g \) be a mechanism that implements \( f \). I will show that \( g \) SRT-implements \( f \).

**Step 2-a:** Show that for all \( \theta \in \Theta \) and \( a \in f(\theta) \) there exists a message profile \( m^* \) such that for all \( T \in \mathcal{T} \) we have \( m^* \in \hat{N}(g, \succeq_{T,\theta}) \) and \( \pi(m^*) = a \).

Take any \( \theta \in \Theta \) and \( a \in Z \) such that \( a \in f(\theta) \). Since \( g \) doubly implements \( f \), in state \( \theta \) there exists a strict Nash equilibrium \( m^* \) of \( g \) such that \( \pi(m^*) = a \). I will show that \( m^* \in \hat{N}(g, \succeq_{T,\theta}) \) for all \( T \in \mathcal{T} \). As the extended preferences agree with the underlying preferences on the asymmetric part of the underlying preference and since \( m^* \) is a strict Nash equilibrium, we have that there is no deviation from \( m^* \) for any possible rule.

**Step 2-b:** Show that for all \( \theta \in \Theta \) and \( T \in \mathcal{T} \) we have \( N(g, \succeq_{T,\theta}) \subseteq f(\theta) \).

For all \( \theta \in \Theta \) and \( T \in \mathcal{T} \) we have \( N(g, \succeq_{T,\theta}) \subseteq N(g, R^\theta) \subseteq f(\theta) \). This is because an equilibrium under a rule \( T \) is always a refinement of Nash equilibrium and the second inequality is because \( g \) Nash implements \( f \).
Proof of Proposition 7. Suppose \( f \) satisfies SQM, NWA and NVP. By NWA, for each \( \theta \in \Theta \), \( i \in N \) and \( a \in f(\theta) \), there exists \( z_i^{a,\theta} \in Z \) such that \( a S_i^\theta b \) and \( b S_i^\theta a \). I will use a slight variant of the canonical mechanism to prove the result. Let \( M_i = \Theta \times Z \times N \) for every \( i \in N \). The outcome function is defined by the following rules:

**Rule 1**: If for all \( i \in N \), \( m_i = (\theta, a, 0) \) where \( a \in f(\theta) \) then \( \pi(m) = a \).

**Rule 2**: If for all \( j \neq i \), \( m_j = (\theta, a, 0) \) and \( m_i = (\theta', b, n) \neq (\theta, a, 0) \) then \( \pi(m) = b \) if \( a S_i^\theta b \) and \( b S_i^\theta a \), otherwise \( \pi(m) = z_i^{a,\theta} \).

**Rule 3**: In all other cases the outcome is the one announced by the agent announcing the highest integer. (Ties are broken in favor of the agent with the lowest index).

I will show that the mechanism doubly implements the SCC in Nash and strict Nash, which implies that the mechanism SRT-implements the SCC, which implies that the mechanism RT-implements the SCC.

**Step 1**: For all \( \theta \in \Theta \), we have \( f(\theta) \subseteq SN(g, R^\theta) \) (hence, \( f(\theta) \subseteq N(g, R^\theta) \)).

Take any \( \theta \in \Theta \) and \( a \in f(\theta) \). Then \( m_i = (\theta, a, 0) \) for all \( i \) is a strict Nash equilibrium in \( \theta \), since the agents can only deviate to an alternative strictly worse than \( a \).

**Step 2**: For all \( \theta \in \Theta \), we have \( N(g, R^\theta) \subseteq f(\theta) \) (hence, \( SN(g, R^\theta) \subseteq f(\theta) \)).

Suppose there is a Nash equilibrium in Rule 1 in which everybody announce \( (\theta, a, 0) \). If \( \theta \) is the true state of the world then \( a \in f(\theta) \). So, suppose the true state of the world is \( \theta' \neq \theta \). By SQM there exists \( i \in N \) and \( b \in Z \) such that \( a S_i^\theta b \) and \( b S_i^\theta a \), but \( i \) can induce \( b \), hence there is a profitable deviation. A contradiction.

By NVP, if \( a \) is an equilibrium outcome in state \( \theta \) under Rule 2 or Rule 3 then \( a \in f(\theta) \).

\[ \square \]

**References**


