Testability of the exclusion restriction in continuous instrumental variable models

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Abstract
In this note we prove Pearl's conjecture (Pearl 1995b), showing that the exclusion restriction of an instrument cannot be tested without structural assumptions in general instrumental variable models with a continuously distributed endogenous variable. This stands in contrast to models with a discretely distributed endogenous variable, where tests have been devised in the literature, and shows that there is a fundamental difference between the continuous and the discrete case.

1 Introduction

Estimating the causal effect of an endogenous variable $X$ on an outcome $Y$ using a potential instrument $Z$ for $X$ is a standard problem in economics. One fundamental problem in this setting is to check whether $Z$ satisfies the exclusion restriction of the model. Formally, this can be achieved by deriving testable conditions under which $Z$ satisfies the exclusion restriction.

In the case where $X$ is discrete, this problem is close to being solved. In particular, Pearl (1995b) was the first to derive an “instrumental inequality” in this setting, a necessary condition for $Z$ to satisfy the exclusion restriction when $X$ is discrete, which makes the exclusion restriction principle testable in this setting. Manski (2003) arrived at the same result as Pearl in the missing data context, using a different approach. Extending these results, Kitagawa (2015) derived a test when $Y$ is continuous and $X$ and $Z$ are binary, also testing monotonicity of the instrument. Kitagawa (2009), again for binary $X$ and $Z$, shows that Pearl’s instrumental inequality gives a sharp testable implication allowing $Y$ to have arbitrary support. Kédagni & Mourifié (2015) reached the same conclusion for the case in which $Y$ is binary. A test for Pearl’s instrumental inequality in the missing data context was developed in Kitagawa (2010).

The case of continuous $X$ is more delicate and less well understood. In fact, Pearl (1995b) points out that there are fundamental differences between models with discrete $X$ and models with continuous $X$. One important difference is the fact that the conditional probability mass function corresponding to a discretely distributed $X$ is bounded above by 1, which is not the case for the probability density function (if it exists), so that an analogous approach to derive an instrumental inequality in this setting fails. This,

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in conjunction with some other idiosyncracies, lead to Pearl’s conjecture that the exclusion restriction of \( Z \) cannot be tested in the most general nonparametric setting when the endogenous variable \( X \) is continuous.

In this note we provide the first proof of this conjecture, showing that without any structural assumptions the exclusion restriction for \( Z \) can in fact not be tested in this case; this even holds if \( Z \) is not continuous and has finitely many atoms. The idea for proving nontestability of the instrument is to show that every possible observable measure \( P_{Y,X|Z=z} \) can be generated by a general instrumental variable model since this would imply that there cannot exist testable restrictions on the model. In fact, if a model manages to replicate every possible data generating process, then it can never be tested, as there are no settings in which it can fail to explain the observables. In the process for proving this, we introduce a connection to the voter paradox in majority voting systems with an uncountable number of individuals. In particular, we show that proving a slight extension of Pearl’s conjecture can be achieved by constructing a Condorcet cycle in infinite dimensional state spaces. In this respect, our proof of Pearl’s conjecture shows that one needs to allow for very general production functions in order to arrive at this impossibility theorem.

Our result is purely theoretical, but has interesting implications for the theory of nonseparable triangular models \cite{Chesher2003,ImbensNewey2009,Gunsilius2017} and references therein). In particular, it suggests that, theoretically, there is a discrete “jump” in the testability of the exclusion restriction between the discrete and the continuous case. This strict distinction is akin to the distinction between the discrete and continuous case in many nonparametric point-identification results: point-identification by definition is a binary function, so the lack of continuity of the identification function means that one cannot hope, in general, to prove theoretical point-identification results by approximating the continuous case via a sequence of discrete cases (also see the discussion in Schennach 2018). Testability, like point-identification, is also inherently a binary function. One should therefore be careful when deducing or proving theoretical results for the general continuous case by methods which rely on \textit{a priori} discretizations.

2 Statement, proof, and implications of Pearl’s conjecture

Throughout, we will need a structural representation of an instrumental variable model:

\[
\begin{align*}
Y &= h(X,V) \\
X &= g(Z,U).
\end{align*}
\]

Here, \( Y \) is the outcome variable, \( X \) is endogenous in the sense that it depends on the unobservable variable \( V \), and \( Z \) is a potential instrument satisfying the relevance condition \( Z \perp \!\!\!\!\perp X \), where “\( \perp \!\!\!\!\perp \)" denotes independence. Both \( V \) and \( U \) are unobserved random variables and can be of arbitrary and even of infinite dimension, just like \( Y \), \( X \), and \( Z \). The production functions \( h \) and \( g \) are unknown.

Model (1) represents the most general structural form of an instrumental variable model \cite{Pearl1995,Heckman2001,HeckmanVytlacil2003} and references therein) and is general enough to encompass all important counterfactual relations, while being precise enough to enable all necessary mathematical derivations. The main assumption made in these models is the exclusion restriction, which is defined by \( Z \perp \!\!\!\!\perp V \). This assumption implies that the only influence the instrument \( Z \) has on the outcome \( Y \) is through \( X \), i.e. \( Z \perp \!\!\!\!\perp Y|X,V \). This follows from the fact that \( Z \) is not present in the second stage of model (1) in conjunction with the fact that both \( g \) and \( U \) are unobserved, so that we can always pick a \( g \) such that

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\cite{Chesher2003,ImbensNewey2009,Gunsilius2017} Other representations of instrumental variable models are graphs \cite{Pearl1995} and counterfactual notation \cite{Rubin1974}. The appendix in \cite{Pearl1995} gives a brief overview of the connections between the different representations.

\normalsize
Z ⊥ U; then, even though U can influence V, it must hold that there are no unobserved variables which jointly affect Z and Y.

The whole point of this note is to analyze the testability of the exclusion restriction when X is continuous. Our definition of a continuous random variable is the most general. In fact, we consider continuous random variables to be nonatomic. A nonatomic probability measure \( P_X \) is one where for every measurable set \( A \) in the Borel \( \sigma \)-algebra \( \mathcal{A}_X \) with \( P_X(A) > 0 \), there exists \( B \in \mathcal{A}_X \) with \( B \subset A \) and \( P_X(A) > P_X(B) > 0 \). For example, probability measures which are absolutely continuous with respect to Lebesgue measure are nonatomic, but there exist many nonatomic measures which do not possess a density with respect to Lebesgue measure. If the respective \( \sigma \)-algebras are nonatomic if and only if every singleton set has measure zero.

We now state and prove Pearl’s conjecture\(^2\), showing that the exclusion restriction \( Z \perp V \) is not testable without structural assumptions on the production function. We prove a slightly stronger result than required by the conjecture by showing that the exclusion restriction on \( Z \) is not testable even if \( g(z, U) \) is assumed to be invertible in \( U \). Moreover, we show that the conjecture holds even if the instrument \( Z \) is allowed to have (finitely many) atoms, as long as the conditional measure \( P_{X|Z=z} \) is nonatomic for almost every \( z \in Z \), which covers basically every probability measure encountered in practice. Our method of proof does not cover the case where \( P_Z \) has a countably infinite number of atoms, but those distributions are rather pathological and are unlikely to appear in practice. Considering the first extension of Pearl’s conjecture allows us to make a connection to the voter paradox in majority voting. In fact, the key for proving the stronger version of the conjecture is to construct a Condorcet cycle in uncountable state space.

In our set-up using nonatomic measures, we can state Pearl’s conjecture as well as its slight extension as follows.

**Conjecture**\(^3\) (Pearl 1995b). Let \( Y, X, \) and \( Z \) be random variables inducing measures \( P_Y, P_X, \) and \( P_Z \) on the respective spaces, where \( P_Z \) is a general probability measure with at most finitely many atoms on the space \( X, Y, \) and \( Z \), where \( Z \) is an instrument for \( X \) and \( P_{Y,X|Z=z} \) is the observable conditional measure. If the marginal measure \( P_{X|Z=z} \) of \( P_{Y,X|Z=z} \) is nonatomic for almost every \( z \in Z \), then \( P_{Y,X|Z} \) can be generated through model (1):

\[
y = h(x, v)
\]
\[
x = g(z, u).
\]

This even holds if the function \( g(z, u) \) is assumed to be invertible in \( U \).

Let us give two important remarks on this conjecture. First, note that the above statement is slightly more technical than the wording of Pearl’s original conjecture; in particular, Pearl simply stated that “if the variable \( X \) is continuous, then every joint density \( f_{Y,X|Z=z} \) can be generated by model (1)”. Since we work in an instrumental variable model, the important probability measures are \( P_{Y,X|Z=z} \) and \( P_{X|Z=z} \), so that the continuity assumption needs to be upheld with respect to the conditional measure \( P_{X|Z=z} \) and not the unconditional measure \( P_X \). In fact, this is what Pearl meant when he stated the conjecture, as the proof of the conjecture relies on a Lemma in Pearl 1995b, which explicitly relies on the fact that the measure \( P_{X|Z=z} \) is nonatomic almost everywhere—Pearl himself exclusively worked with density functions and hence implicitly assumed that all relevant distributions, especially \( P_{X|Z=z} \) and \( P_{Y,X|Z=z} \), are absolutely

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\(^2\)I thank Susanne Schennach for this remark.

\(^3\)Calligraphic letters denote general sets. Note that the spaces can be of arbitrary dimension in principle as long as they are Polish, i.e. well-behaved spaces in the sense that they are separable, complete, and metrizable. Throughout, nothing is lost by assuming that \( Y, X, Z \) are subsets of Euclidean spaces.
continuous with respect to Lebesgue measure, a stronger condition than we uphold.

Second, the above statement of the conjecture is more general in that it allows for the assumption that $g$ is invertible in $U$, which is a structural assumption in this model. Also, we only require $P_{X|Z=z}$ to be nonatomic for almost every $z \in Z$ and only require that $Z$ has at most finitely many atoms, but the other conditional distributions including $Y$ such as $P_{Y,X|Z=z}$ is left completely unspecified, in contrast to Pearl’s original statement, who used density functions $f_X|Z=z$ and $f_{Y,X|Z=z}$. The key for this is, again, Lemma 1 below, as it lets us reduce the proof of the conjecture to find appropriate production functions for the probability measure $P_{X|Z=z}$.

For the proof of the conjecture we need to introduce three concepts, generators, measure preserving isomorphisms, and disintegrations, together with some of their properties. The concept of disintegrations gives meaning to the restriction of a joint probability measure $P_{Y,X}$ to a subset of Lebesgue measure zero, for instance the conditional measure $P_{Y|X=x}$ when $X$ is a continuous random variable inducing a nonatomic probability measure $P_X$. A disintegration $P_{Y|X=x}(A)$ for some Borel set $A$ is a version of the standard conditional expectation $E(1\{Y \in A\}|\mathcal{F}_X)$ for some filtration $\mathcal{F}_X \subset \mathcal{B}_X$ when it exists, where $1\{E\}$ denotes the standard indicator function which is 1 if the event $E$ happens and 0 otherwise.

The existence of a disintegration can be shown under very general circumstances and is guaranteed in our setting (see Theorem 1 in Chang & Pollard 1997). Working with disintegrations instead of general conditional expectations makes the proof more accessible.

The second formal concept we require for the proof of the conjecture is the notion of measure preserving isomorphisms. A map $T : \mathcal{X} \to \mathcal{Y}$ transporting a probability measure $P_X$ onto another probability measure $P_Y$ is measure preserving if it is measurable and

$$P_Y(E) = P_X(T^{-1}E) \quad (2)$$

for every set $E$ in the Borel $\sigma$-algebra $\mathcal{B}_Y$ corresponding to $Y$. If $T$ is invertible and its inverse is also measure preserving, it is called a measure-preserving isomorphism. For all our work, we only need measure preserving isomorphisms up to sets of measure zero, as measure preserving isomorphisms can only be identified up to sets of measure zero anyways, so that from now on we mean “measure preserving isomorphism modulus sets of measure zero” when we write “measure preserving isomorphism”.

Third, we introduce generators, a term coined in Pearl (1995a).

**Definition (Generator).** Given a probability measure $P_{X|Z}$, a function $x = g(z,u)$ is a generator of $P_{X|Z}$ if and only if there exists some probability measure on the domain of $U$ such that $g(z,U)$ is distributed as $P_{X|Z=z}$. A generator is one-to-one if and only if for almost every $z_i, z_j \in Z$ and $u \in U \ g(z_i, u) = g(z_j, u)$ implies $z_i = z_j$.

We need generators because of the following lemma, which is required for proving the conjecture. The special case of this lemma for probability density functions was derived and proved in Pearl (1995a).

**Lemma 1.** Any probability measure $P_{Y,X|Z}$ whose marginal $P_{X|Z}$ has a one-to-one generator can be generated by $(1)$, even if we require $U$ and $V$ to be univariate.

**Proof.** Let $g(Z,U)$ be a one-to-one generator of the measure $P_{X|Z}$ and factor $P_{Y,X|Z} = P_{Y|X,Z}P_{X|Z}$. Use $x = g(z,u)$ to generate $P_{X|Z=z}$ via $P_U$ and some other function $y = h'(x,z,v')$ to generate $P_{Y|X=x,Z=z}$.

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4 Measurability of $T$ means that $\mathcal{B}_X = T^{-1}\mathcal{B}_Y$, where $\mathcal{B}_Y$ and $\mathcal{B}_X$ are the Borel $\sigma$-algebras corresponding to $Y$ and $X$, respectively.

5 $T^{-1}E$ denotes the set of points $x \in \mathcal{X}$ such that $Tx \in E$. 

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Conjecture 2, as we can construct some measure preserving isomorphism is without loss of generality as for every probability measure Proof of the conjecture. We will assume that all random variables take values in the unit interval, which choosing \( g \) maps. 

\[ g \in \text{9.2.2 in Bogachev (2007). Note also that to be a measure preserving isomorphism, i.e. to be invertible with measure preserving inverse by Theorem Lemma 2. \] 

Fix some probability measure nonatomic one can pick \( \nu \) to be Lebesgue measure (Bogachev 2007, Theorem 9.2.2), so that there always exists a probability measure for \( V \) which is the pushforward of a probability measure for \((U, V')\). 

This lemma reduces the problem of proving the conjecture to simply proving that there exists a one-to-one generator for each possible \( P_{X|Z} \). To show in a simple example that restrictions on the dimension of \( U \) and \( V \) cannot help, assume that all random variables take values in the unit interval \([0, 1]\). It then holds that \( (u, v') \in [0,1]^2 \) while \( v \in [0,1] \). In this case one can construct a standard Hilbert- or Peano curve \( \mathcal{H} : [0,1] \to [0,1]^2 \) which is a measure preserving isomorphism from the unit interval to the unit square. Theorem 9.2.2 in Bogachev (2007) shows that this is a special case of a more general property, which is the property we need for the impossibility result. Also note that by using Lemma I we do not make any assumptions on the distribution of \( Y \), so that we can allow for general distributions here too, which does not change the result. In fact, the whole proof works with the properties of \( P_{X|Z=z} \), which we assume to be nonatomic for almost all \( z \in Z \).

For the construction of the Condorcet cycle in our proof, we need the following technical lemma about measure preserving isomorphisms, which is proved in Halmos (1956, p. 74).

**Lemma 2.** Fix some probability measure \( m \) on a measurable space \((\mathcal{X}, \mathcal{B}_\mathcal{X})\), where \( \mathcal{X} \subset \mathbb{R} \) is an interval. If \( E \) and \( F \) are Borel sets of the same measure in the interval \( \mathcal{X} \), i.e. \( m(F) = m(E) \), then there exists a measure preserving isomorphism \( T : \mathcal{X} \to \mathcal{X} \) such that \( m(TE + F) = 0 \) on \( \mathcal{X} \).

We are now ready to prove the conjecture. Note that we prove a slightly stronger statement that Conjecture 2 as we can construct some \( g \) which is a measure preserving isomorphism in both \( U \) and \( Z \).

**Proof of the conjecture.** We will assume that all random variables take values in the unit interval, which is without loss of generality as for every probability measure \( \mu \) on a Polish space there exists a measure preserving isomorphism onto the unit interval equipped with some probability measure \( \nu \); in case \( \mu \) is nonatomic one can pick \( \nu \) to be Lebesgue measure (Bogachev 2007, Theorem 9.2.2).

To begin notice that \( g(z, \cdot) \) in model II is by definition a measure-preserving map \( g(z, \cdot) : [0, 1] \to [0, 1] \) for almost every \( z \). In fact, since \( P_{X|Z=z} \) for almost all \( z \) and \( P_Y \) are probability measures, we can require \( g \) to be a measure preserving isomorphism, i.e. to be invertible with measure preserving inverse by Theorem 9.2.2 in Bogachev (2007). Note also that \( g \) is only specified up to sets of measure zero, so that in our proof we only have to specify it modulus sets of measure zero, too. Of course, if we can prove the conjecture when choosing \( g \) to be a measure preserving isomorphism, we have also proved it for general measure preserving maps.

We proceed by first proving the conjecture for nonatomic \( P_Z \) with support \( Z \subset [0, 1] \). After that we

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\(^6\)For two sets \( A \) and \( B \), \( A + B \) denotes their disjoint sum \( (A \setminus B) \cup (B \setminus A) \).

\(^7\)The support \( Z \) of a measure \( P_Z \) is defined by two criteria. First, it is the closed set \( Z \) on which the measure \( P_Z \) concentrates, i.e. \( P_Z([0,1] \setminus Z) = 0 \). Second, it is such that for every open set \( G \) such that \( G \cap Z \neq \emptyset \), it holds that \( P(G \cap Z) > 0 \), i.e. every open set intersecting the support has positive measure. This is a straightforward generalization of the support of a density function and the reader can always think about the latter. It is also important to note that every probability measure on \( \mathbb{R}^d \) has a support.
extend the result to allow for $P_Z$ with atoms. In light of Lemma 1, we only have to show that $P_{X|Z=z}$ admits a one-to-one generator modulus sets of measure zero. We hence need to show that for each $z \in [0, 1]$ there exists a measure preserving isomorphism $g(z,u)$ between $P_U$ and $P_{X|Z=z}$ such that $g(z_i,u) \neq g(z_j,u)$ for almost all $z_i, z_j \in [0, 1], i \neq j$ and $u \in [0, 1]$.

We show that such $g$ exists by factoring it as $g(z,u) = T_z f_z(u)$. Here, $f_z : [0, 1] \to [0, 1]$ is some measure preserving isomorphism between $P_U$ and the respective $P_{X|Z=z}$. We require $T_z$ to be a measure preserving isomorphism on the measure space $([0,1], \mathcal{F}_{[0,1]}, P_{X|Z=z})$ garbling the map $f_z$. To be more precise, since $f_z$ is allowed to be any measure preserving isomorphism, it may well happen that $f_{z_i}(u) = f_{z_j}(u)$ for some $z_i, z_j, u \in [0, 1]$. We therefore need to show that there always exists a collection $T_z$ of measure preserving isomorphisms such that $T_z f_{z_i}(u) \neq T_z f_{z_j}(u)$ for almost all $z_i, z_j, u \in [0, 1]$. The idea to achieve this is to prove a simple generalization of the Condorcet Paradox to uncountable state space.

For illustrative purposes, let $X$, $Z$, and $U$ be discrete with three values each: $x_1, x_2, x_3$ each with probability $\frac{1}{3}$, $z_1, z_2, z_3$ each with probability $\frac{1}{3}$, and the same for $U$. Then we can build a matrix where the $m^{th}$ row represents $f_{z_m}(U)$ and the $n^{th}$ column represents $x_n$. Each cell $x(m,n)$ of the matrix contains the index $i \in \{1, 2, 3\}$ of $u_i$ assigned to $x(m,n)$ by the measure preserving map $f_{z_m}$. For example the matrix could look like this:

$$
\begin{bmatrix}
3 & 2 & 1 \\
3 & 1 & 2 \\
1 & 2 & 3 \\
\end{bmatrix}
$$

In this case, the map $f_z$ is a generator, but clearly not a one-to-one generator as needed, since it maps $u_3$ onto $x_1$ for $z_1$ and $z_2$.

In order for $T_z f_z(u)$ to be a one-to-one generator, one simply needs to guarantee that no column contains two or more equal numbers. However, since we have already assumed that $g(z,u)$ is a measure preserving isomorphism in both variables, we must also guarantee that $T_z f_z(u)$ is a one-to-one generator in both variables. This means that neither columns nor rows must contain two equal numbers. This requirement is hence analogous to constructing a Condorcet cycle from voter theory, which is possible for $n = m$:

$$
\begin{bmatrix}
1 & 2 & 3 \\
3 & 1 & 2 \\
2 & 3 & 1 \\
\end{bmatrix}
$$

This idea extends naturally to the uncountable case modulus sets of measure zero. Constructing a Condorcet cycle for an uncountable number of individuals is hence equivalent to constructing a one-to-one generator in both variables. The proof is therefore complete if we can show that a Condorcet cycle can always be constructed in the case where $X = [0, 1] = Z = U$ when $T_z$ is a measure preserving isomorphism.

To show this, let $P_U$ be Lebesgue measure on $[0, 1]$ and let $f_z(u)$ be any measure preserving isomorphism from $P_U$ to $P_{X|Z=z}$ for all $z$. If $f_z(u)$ is a one-to-one generator, we just let $T_z$ be the identity for all $z$ and nothing is to prove. So assume that there are Borel sets $E_u \subset [0, 1]$ and $E_Z$ of measure $P_U(E_u) = \varepsilon_u$ and

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8Note that this covers all types of probability measures with finitely many atoms, even continuous singular ones on $\mathbb{R}$ as a general finite measure can be uniquely decomposed into a nonatomic measure and a purely atomic measure under the assumption that they are singular with respect to one another, see Theorem 2.1 in [Johnson (1970)]. In fact, continuous singular measures like Cantor measures are nonatomic measures with a set of Hausdorff dimension smaller than one as support.

9Note that $T_z$ is a measure preserving isomorphism which maps to the same measure space, whereas $f_z$ maps between different measure spaces. We need this set-up since we want to use Lemma 2 which works for isomorphisms acting on the same measure space.

10We assume $U$ to follow the uniform distribution on $[0, 1]$ for convenience, we could specify any other distribution; also note that this is without any loss of generality, as we are free to choose the distribution of $U$ for this proof.
We construct a measure preserving permutation $T_z : [0, 1] \to [0, 1]$ which is garbling of $f_z$ in the sense that $T_z f_z(u) \neq T_z f_z(u)$ almost all $u \in E_u$ and $z_i, z_j \in E_Z$.

To do this, partition $E_Z = E^1_Z \cup E^2_Z$ into two disjoint parts $E^1_Z$ and $E^2_Z$ of equal measure

$$P_Z(E^1_Z) = P_Z(E^2_Z) = \frac{1}{2} \varepsilon_Z$$

and do the same with $E_u$, i.e. $E_u = E^1_u \cup E^2_u$ with

$$P_U(E^1_u) = P_U(E^2_u) = \frac{1}{2} \varepsilon_u.$$

Since $f_z$ is a measure preserving isomorphism for every $z \in [0, 1]$, it must be the case that $f_z(E^1_u)$ and $f_z(E^2_u)$ are disjoint modulus sets of measure zero and that

$$P_X|_{Z = z}(f_z(E^1_u)) = P_X|_{Z = z}(f_z(E^2_u)) = \frac{1}{2} \varepsilon_u \quad \text{for all } z \in E_Z.$$

Define $T_z$ to be the identity for $z \in E^1_Z$; for $z \in E^2_Z$ let it be such that $T_z f_z(E^1_u) = f_z(E^2_u)$ and $T_z f_z(E^2_u) = f_z(E^1_u)$, i.e. switching $f_z(E^1_u)$ and $f_z(E^2_u)$). Lemma guarantees that this is always possible. Now partition $E^2_Z$ into two parts of equal measure $E^{21}_Z$ and $E^{22}_Z$, so that

$$P(Z(E^{21}_Z)) = P(Z(E^{22}_Z)) = \frac{1}{4} \varepsilon_Z$$

and do the same with $E^2_u$. Then on $E^{21}_Z$, let $T_z$ be the same as on $E^2_Z$ and on $E^{22}_Z$ let it be such that $T_z f_z(E^{21}_u) = f_z(E^{22}_u)$ and $T_z f_z(E^{22}_u) = f_z(E^{21}_u)$.

At stage $n \in \mathbb{N}$ with sequences $E^i_u$ and $E^i_Z$, the inductive step is to split $E^1_u$ and $E^i_Z$ into two disjoint Borel subsets of equal measure $E^{i\land 1}_u$ and $E^{i\lor 2}_u$ as well as $E^{i\land 1}_Z$ and $E^{i\lor 2}_Z$. Then on $E^{i\land 1}_Z$ let $T_z$ be identical to $T_z$ on $E^2_Z$ and on $E^{i\lor 2}_Z$ let it be such that

$$T_z f_z(E^{i\land 1}_u) = T_z f_z(E^{i\lor 2}_u) \quad \text{and} \quad T_z f_z(E^{i\lor 2}_u) = T_z f_z(E^{i\land 1}_u),$$

which is possible by Lemma. Now let us order the set $\{1, 2\}^N$ as follows: Start with the sequence of all ones, which is the minimal element. Then change the first digit from a 1 to a 2, keeping all other digits. Then change the first digit back to 1 and the second to a 2, keeping all others as 1. Let the digit 2 “run through all positions” up to infinity. After this change the first two digits to a 2, keeping all other digits at 1, and let the second 2 run through all positions, keeping the first position at 2. Change the first position back to 1 and keep the second position a 2 while running the second 2 through all positions. Do the same with three 2’s, four 2’s and so on. This is a well-ordering since every subset of $\{1, 2\}^N$ has a smallest element (Aliprantis & Border, 2006, p. 18). Therefore, we can proceed by transfinite induction for all $i \in \{1, 2\}^N$ over this well-ordered set, which yields an uncountable Condorcet cycle for $T_z f_z(u)$ modulus sets of measure zero in the sense that $T_z f_z(u) \neq T_z f_z(u)$ almost all $u \in E_u$ and $z_i, z_j \in E_Z$.

This construction only uses values within $E_Z$ and $E_u$, respectively, and can hence be applied separately to every combination of Borel sets $E_Z$ and $E_u$ for which $f_z(u) = f_z(u)$ for $u \in E_u$, yielding a Condorcet cycle up to sets of measure zero for all of $[0, 1]$ if we let $T_z$ be the identity for all other Borel sets. This

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11The notation $i \land 1$ means appending the number 1 to the sequence $i$. 

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proves the conjecture in the case where \( E_Z \) is uncountable since \( g \) can only be specified modulus sets of measure zero.

For the case where \( Z \) has finitely many atoms, the above construction can be adapted as follows. If there is no atomic \( z \in [0,1] \) such that \( f_z(u) = f_{z_i}(u) \) for some other \( z_i \in [0,1] \) and some \( u \in [0,1] \), then the same construction as above works. If there is some number of atomic \( z_j, j = \{1, \ldots, k\} \), for which there is a Borel set \( E_u \) and some Borel set \( E_Z \) such that \( f_{z_j}(u) = f_{z_i}(u) \) for all \( z_i \in E_Z \) and all \( u \in E_u \), then in the construction above can be adjusted as follows.

Consider the Borel set \( F := \bigcup_{j=1}^k z_j \cup E_Z \) and partition \( E_Z \) into two disjoint Borel subsets \( E^1_Z \) and \( E^2_Z \) of equal measure
\[
P_Z(E^1_Z) = P_Z(E^2_Z) = \frac{1}{2} \varepsilon_Z
\]
and the corresponding \( E_u \) into \( k + 2 \) disjoint Borel sets \( E^1_u, \ldots, E^{k+2}_u \) of equal measure, i.e.
\[
P_U(E^1_u) = \ldots = P_U(E^{k+2}_u) = \frac{1}{k+2} \varepsilon_u.
\]

Then for \( z_1 \) let \( T_{z_1} \) be the identity. For the other values \( z_2, \ldots, z_k \) as well as any \( z \in E^1_Z \) and \( z' \in E^2_Z \) let \( T_{z_j} \) be a cyclic map (which can be done by Lemma \([2]\)), i.e. for \( z_2 \)
\[
T_{z_2} f_{z_2}(E^k_u) = f_{z_2}(E^1_u), \quad T_{z_2} f_{z_2}(E^1_u) = f_{z_2}(E^2_u), \quad \ldots, \quad T_{z_2} f_{z_2}(E^{k+1}_u) = f_{z_2}(E^{k+2}_u),
\]
for \( z_k \)
\[
T_{z_k} f_{z_k}(E^1_u) = f_{z_k}(E^k_u), \quad T_{z_k} f_{z_k}(E^2_u) = f_{z_k}(E^{k+1}_u), \quad \ldots,
\]
for \( z \in E^1_Z \)
\[
T_z f_z(E^1_u) = f_z(E^k_u), \quad T_z f_z(E^2_u) = f_z(E^{k+1}_u), \quad \ldots,
\]
and for \( z' \in E^2_Z \)
\[
T_{z'} f_{z'}(E^1_u) = f_{z'}(E^k_u), \quad T_{z'} f_{z'}(E^2_u) = f_{z'}(E^{k+1}_u), \quad \ldots.
\]

Then at each iteration \( n \) of the construction split \( E^1_Z, i_z \in \{1,2\}^n \), into two disjoint Borel subsets \( E^{i^1}_Z \) and \( E^{i^2}_Z \) of equal measure
\[
P_Z(E^{i^1}_Z) = P_Z(E^{i^2}_Z) = \frac{1}{2n} \varepsilon_Z
\]
and \( E^{i_u} \), \( i_u \in \{1, \ldots, k+2\}^n \), into \( k + 2 \) disjoint Borel subsets \( E^{i_u^1} \), \ldots, \( E^{i_u^{k+2}} \) of equal measure
\[
P_U(E^{i_u^1}) = \ldots = P_U(E^{i_u^{k+2}}) = \frac{1}{(k+2)^n} \varepsilon_u,
\]
leave \( T_{z_1} \) unchanged from period \( n - 1 \) on \( E^{i^1}_Z \), and construct it to be cyclic (which again can be done by Lemma \([2]\)) for the other \( z \), i.e. for \( z_2 \)
\[
T_{z_2} f_{z_2}(E^{i_u^{k+2}}) = f_{z_2}(E^{i_u^1}), \quad \ldots, \quad T_{z_2} f_{z_2}(E^{i_u^{k+1}}) = f_{z_2}(E^{i_u^{k+2}}),
\]

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for \( z_3 \)
\[
T_{z_3} f_{z_3} (E_u^{i_u \wedge k+1}) = f_{z_2} (E_u^{i_u \wedge 1}), \ldots, T_{z_2} f_{z_2} (E_u^{i_u \wedge k}) = f_{z_2} (E_u^{i_u \wedge k+2}),
\]
for \( z_k \)
\[
T_{z_k} f_{z_k} (E_u^{i_u \wedge 1}) = f_{z_k} (E_u^{i_u \wedge k}), \quad T_{z_k} f_{z_k} (E_u^{i_u \wedge 2}) = f_{z_k} (E_u^{i_u \wedge k+1}), \ldots,
\]
for \( z \in E_Z^{i_u \wedge 1} \)
\[
T_z f_z (E_u^{i_u \wedge 1}) = f_z (E_u^{i_u \wedge k+1}), \quad T_z f_z (E_u^{i_u \wedge 2}) = f_z (E_u^{i_u \wedge k+2}), \ldots,
\]
and for \( z' \in E_Z^{i_u \wedge 2} \)
\[
T_{z'} f_{z'} (E_u^{i_u \wedge 1}) = f_{z'} (E_u^{i_u \wedge k+2}), \quad T_{z'} f_{z'} (E_u^{i_u \wedge 2}) = f_{z'} (E_u^{i_u \wedge 1}), \ldots
\]
Then again by transfinite induction as in the previous case one obtains a Condorcet cycle \( T_z (f_z (u)) \) modulus sets of measure zero in the sense that \( T_z, f_z, (u) \neq T_z, f_z, (u) \) almost all \( u \in E_u \) and \( z, z' \in E_Z \) as well as \( z_1, \ldots, z_k \), as required.

Finally, the case where there are only finitely many atoms \( z_1, z_2, \ldots, z_k \) such that \( f_{z_i} (u) = f_{z_i} (u) \) for some Borel set \( E_u \subset [0, 1] \) with measure \( P_U (E_u) = \varepsilon_u > 0 \) is a special case of the above construction. The above construction covers all cases for uncountable or discrete subsets \( E_Z \) where \( f_z (u) \) is not a one-to-one generator. In all cases we were able to construct a Condorcet cycle for all of \([0, 1]\) by letting \( T_z \) be the identity map on all other sets except those sets \( E_Z \); by definition, a Condorcet cycle is equivalent to a one-to-one generator. Therefore, we can apply Lemma 3 to finish the proof.

Five remarks about this result are in order. First, in order for the construction of the permutation \( T_z \) in the proof to work, a nonatomic \( P_{X|Z=x} \) for almost all \( z \) is crucial. In fact, every Borel set \( E \subset [0, 1] \) of positive measure contains an uncountable number of elements, so that even if there is an uncountable number of \( z \), one can always find a permutation of \( E \) such that almost every \( x \in E \) corresponds to exactly one \( z \). In this respect, note that nothing in the above proof depends on particular topological properties of the unit interval, so that the conjecture is actually proven for general Polish spaces.

Second, recall that we proved an even stronger version of the conjecture by constructing a Condorcet cycle whilst assuming that \( g(z, u) \) which is one-to-one in \( z \) as well as \( u \), which one could call a “super-generator”. The definition of a one-to-one generator requires \( g \) to only be invertible in \( z \). A Condorcet cycle is equivalent to the stronger case where \( g(z, u) \) is one-to-one in both variables. This implies that even the assumption that \( g \) be invertible in \( u \) is too weak to make the exclusion restriction for \( Z \) testable.

Third, note that our proof shows that Pearl’s Conjecture holds even if we allow \( Z \) to have finitely many atoms, so that even for instruments with general distribution is the exclusion restriction untestable if the endogenous variable is nonatomic. Also note that we did not make any assumptions on the distribution of \( Y \), so that we can allow for general distributions here too, which does not change Pearl’s conjecture. In fact, the whole proof works with the properties of \( P_{X|Z=x} \), which we assume to be nonatomic for almost every \( z \) (see the first point). The reason is Lemma 4 which reduces the proof to constructing a Condorcet cycle for the measure preserving isomorphism between \( P_U \) and \( P_{X|Z=x} \) for almost every \( z \in Z \). It is also likely that the pathological case of measures with countably infinitely many atoms can be handled by slightly different arguments.

Fourth, the proof of Pearl’s conjecture also gives a simple explanation for why the exclusion restriction of \( Z \) can be testable when \( X \) is discrete. As an example, let \( X \) and \( Z \) each take three values, \( x_1, x_2, \) and \( x_3 \) as well as \( z_1, z_2, z_3 \), and let \( U \) be uniformly distributed on the unit interval. Moreover, assume that
$P_{X|Z=z_1}(x_1) + P_{X|Z=z_2}(x_1) > 1$. In this case, there cannot exist a Condorcet cycle. In fact, no matter how we partition the unit interval for $U$, the fact that $P_{X|Z=z_1}(x_1) + P_{X|Z=z_2}(x_1) > 1$ always implies that there is some Borel set $E_u \subset [0, 1]$ of measure

$$P_U(E_u) = P_{X|Z=z_1}(x_1) + P_{X|Z=z_2}(x_1) - 1 = \varepsilon_u > 0,$$

which gets mapped to $x_1$ for $z_1$ and $z_2$, no matter the measure preserving map $g(z, u)$, implying that there cannot be a Condorcet cycle or even a one-to-one generator as depicted in the illustration of the proof.

Fifth, note that we can intuitively understand the construction of the Condorcet cycle in our proof as a very special construction of the function $g(z, u)$, which has a non-standard form. In the literature on nonseparable triangular models, one usually assumes monotonicity and continuity of $g$, which are very strong assumptions. In contrast, for our non-testability result we need to allow for very general classes of functions $g$ in order to always being able to replicate the observable joint distribution $F_{Y,X|Z=z}$. In this sense, it might be possible to reestablish testability of the exclusion restriction even in the continuous setting by restricting the set of allowable functions $g$.

\section{Conclusion}

In this note we have provided a proof of a generalization of Pearl’s conjecture \cite{Pearl1995b}, showing that the exclusion restriction of an instrument cannot be tested in general instrumental variable models with a continuous endogenous variable. The idea is to construct a general measure preserving isomorphism for the first stage, which is akin to the construction of an (almost everywhere) Condorcet cycle in uncountable state space. This result has several interesting implications for the general research on instrumental variable models. In particular, it implies that the continuous case is fundamentally different from the discrete case and that one should be cautious when arguing about testability of the continuous case by using discretization. Moreover, the construction of a Condorcet cycle implies that we need to allow for very general production functions in order to arrive at the impossibility theorem. This suggests that testability can be reestablished in this setting under some weak structural form assumptions.

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\footnote{\textsuperscript{12}I thank Toru Kitagawa for this remark.}


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