SECTION PROBLEMS FOR GRAPH CONFIGURATION SPACES

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ABSTRACT. We consider, for a finite graph G, when the surjective map $\mathrm{Conf}_{n+1}(G) \to \mathrm{Conf}_n(G)$ of configuration spaces admits a section. We study when the answer depends only on the homotopy type of G, and give a complete answer. We also provide basic techniques for construction of sections.

1. INTRODUCTION

If $f : X \to Y$ is a surjective map of topological spaces, a natural question is to consider whether the map f admits a *section*, that is, a map $s : Y \to X$ such that $f \circ s = \mathrm{id}_Y$. Many questions in topology can be rephrased in terms of finding a section of a particular map.

Example 1.1. Here are two examples that show how topological problems can be restated in terms of the existence of a section to a particular map.

(a) Let M^n be a smooth manifold, and let TM be its tangent bundle. Let SM be a S^{n-1} bundle over M constructed as follows: Let Y be TM minus the image of the zero section, and let SM be the quotient of Y by the $\mathbb{R}_+^*$ action which scales each vector in its tangent space. We can see that $f : SM \to M$ is indeed an S^{n-1}-bundle. Note that a section of f is, by definition, a continuous choice of nonzero vector (up to scaling) in the tangent space of each vector, which is precisely a nonvanishing vector field on M (up to multiplication by a positive function).

(b) Now, let $E \to M$ be a rank k vector bundle over M. We can associate a $\mathrm{GL}_k(\mathbb{R})$-bundle $G(E)$ over M to E whose fiber over $x \in M$ is the set of linear isomorphisms $E_x \to \mathbb{R}^k$, where E_x denotes the fiber of x in E. A section of $G(E) \to M$ is precisely a continuous choice of isomorphism between each tangent space and $\mathbb{R}^k$, so these glue to a global isomorphism $E \cong M \times \mathbb{R}^k$, so $G(E) \to M$ admits a section precisely when E is a trivial vector bundle.

The main body of this thesis will be concerned with graph configuration spaces. By a *graph*, we mean a finite connected 1-dimensional cell complex. If n is a natural number, then define the *n-th ordered configuration space of G* to be the set

$$\mathrm{Conf}_n(G) = \{(x_1, \ldots, x_n) \in G^n : x_i \neq x_j \text{ for } i \neq j\}$$

equipped with the subspace topology in the product space G^n. There is a natural "forgetting map" $\phi_{n+1} : \mathrm{Conf}_{n+1}(G) \to \mathrm{Conf}_n(G)$ defined by:

$$(x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_n).$$

Later, we will study, in detail, when this map admits a section.

The general theory of section problems, (as well as some other related problems) are studied in the branch of topology known as *obstruction theory*. The general technique of obstruction theory is to attempt to construct maps of CW complexes one cell at a time

inductively on the skeleta. In Section 2, we will cover the basics of obstruction theory, and apply it to the case of Example 1.1(a).

The rest of the thesis proceeds as follows: In Section 3, we state our main results for graph configuration spaces. In Section 4, we introduce notation and terminology. In Section 5, we present elementary examples of sections. In Section 6, we introduce and develop the idea of *connectable components*, a technical tool useful for the proof of Theorem 3.1. Section 7 is devoted to the proof of Theorem 3.1. In Section 8, we introduce a combinatorial model for $\mathrm{Conf}_n(G)$ which is useful for proving more advanced existence results. In Section 9, we show that nothing can be concluded about the case where $\chi(G) < 0$ and $n < 2 - \chi(G)$ without appealing to more than the homotopy type of the graph.

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2. Obstruction Theory and the Euler Class

We will assume throughout that all spaces involved are CW complexes. The nicest case of the theory occurs when the map $f : X \to Y$ is an especially nice type of map called a *fibration*.

Definition 2.1. Let $f : X \to Y$ be a map. We say that f is a *(Hurewicz) fibration* if for every space S, and every commutative diagram of topological spaces of the following form:

$$
\begin{array}{ccc}
S \times \{0\} & \longrightarrow & X \\
\downarrow & & \downarrow{\scriptstyle f} \\
S \times I & \longrightarrow & Y,
\end{array}
$$

there is a map $S \times I \to X$ such that the diagram:

$$
\begin{array}{ccc}
S \times \{0\} & \longrightarrow & X \\
\downarrow & \nearrow & \downarrow{\scriptstyle f} \\
S \times I & \longrightarrow & Y
\end{array}
$$

commutes.

It is a general fact that every fiber bundle over a CW complex is a fibration, so the tools of obstruction theory apply to Example 1.1. For the remainder of this discussion, we assume that f is a fiber bundle with path-connected fiber space F. There are two groups which can act naturally on the homotopy groups $\pi_n(F)$. First, $\pi_1(F)$ acts via the change-of-basepoint isomorphisms. Suppose that, for some fixed n, this action is trivial. In this case, we can ignore the basepoint of F and identify $\pi_n(F)$ with the set of (unbased) homotopy classes of maps $S^n \to F$. Additionally, $\pi_1(X)$ acts on $\pi_n(F)$ in a slightly more complicated way. Given $\alpha \in \pi_1(X)$ and $\beta \in \pi_n(F)$, define a map $\rho : S^n \times I \to Y$ by $\alpha \circ \mathrm{proj}_2$, and a map $\eta : S^n \to X$ by sending S^n into F via β in the fiber over the basepoint of Y. The definition of a fibration implies that there is a map $\theta : S^n \times I \to X$ which restricts to η at time 0

and such that $f \circ \theta = \rho$. We then define $\alpha \cdot \beta = [\theta|_{S \times 1}]$. One can verify that this is also a well-defined group action. For simplicity, we also assume that this is a trivial group action.

We can now state the main theorem of obstruction theory.

Theorem 2.2. *Under the above assumptions, suppose that $\sigma : Y^{(n)} \to X$ is a partially defined section on the n-skeleton of Y. Then, there exists a cocycle $\theta(\sigma) \in C^{n+1}(Y, \pi_n(F))$ such that σ extends to a section defined on $Y^{(n+1)}$ if and only if $\theta(\sigma) = 0$. Further, the restriction $\sigma|_{Y^{(n-1)}}$ extends to a section on $Y^{(n+1)}$ if and only if $\theta(\sigma)$ represents the trivial cohomology class.*

For a detailed proof of this fact, see, for instance, Theorem 7.37 of [Kir]. We first give a brief explanation of how the cocycle is constructed. Suppose for simplicity that each $(n+1)$-cell of Y is embedded. For an $(n+1)$-cell e_i with attaching map $\phi_i : S^n \hookrightarrow Y^{(n)}$, then since e_i is contractible, the restriction of f to e_i is a trivial bundle. Therefore, $f^{-1}(e_i) \cong e_i \times F$. We define $\theta(\sigma)(e_i) = \sigma \circ \phi_i \in \pi_n(e_i \times F) \cong \pi_n(F)$. Our simplifying assumptions on F and Y ensure that this definition is independent of our choice of isomorphism $f^{-1}(e_i) \cong e_i \times F$. Note that $\theta(\sigma)(e_i) = 0$ if and only if the map $\sigma \circ \phi_i$ extends to e_i which is equivalent to saying that σ extends to a section defined on e_i. Therefore, $\theta(\sigma) = 0$ if and only if σ extends to a section on $Y^{(n)}$.

When F is $(n-2)$-connected, the above discussion implies that we can always define a section on the $(n-1)$-skeleton of Y. In this case, the cohomology class of the obstruction cocycle constructed above is called the *primary obstruction*. If Y is also an n dimensional CW complex, then it follows that the primary obstruction vanishes if and only if a global section exists. This situation is not as uncommon as it may seem. In fact, it applies to the situation in Example 1.1(a) (although we need the further assumption that M is oriented to ensure that $\pi_1(M)$ acts trivially on $\pi_{n-1}(S^{n-1})$). In this case, we call the primary obstruction the *Euler class* of M, and denote it by $e(M) \in H^n(M, \pi_{n-1}(S^{n-1})) \cong \mathbb{Z}$. In fact the Euler class is very closely related to the Euler characteristic. Indeed, under the identification $H^n(M, \pi_{n-1}(S^{n-1})) \cong \mathbb{Z}$, by Theorem VII.14.3 of [Bre93], $e(M) = \chi(M)$, so M admits a nonvanishing vector field if and only if $\chi(M) = 0$. This reduces determining the existence of a nonvanishing vector field to a simple computation of the Euler characteristic. More generally, the Euler class is defined for any S^{n-1} bundle over an n-manifold.

Example 2.3. We compute the Euler class of two different S^1-bundles over S^2. Throughout, we fix the standard cell structure on S^2 containing a pair of 2-cells D_+, D_-, corresponding to the northern and southern hemisphere respectively, with $D_+ \cap D_- = D_0$ is the unique 1-cell.

(a) (Hopf Fibration) There is a tautological map $\mathbb{C}^2 \setminus \{(0,0)\} \to \mathbb{C}P^1 \cong S^2$ with fiber $\mathbb{C}^*$. Restricting to the embedded copy of $S^3 \subset \mathbb{C}^2$ gives an S^1-bundle $f : S^3 \to S^2$ known as the *Hopf fibration*. We identify $S^2 \cong \mathbb{C}P^1$, so that $D_+ = \{[v : w], |v| \geq |w|\}$, $D_- = \{[v : w] : |w| \geq |v|\}$, and $D_0 = D_+ \cap D_- = \{[v : w] : |v| = |w|\}$. Then, we can define a partial section $\sigma : D_0 \to S^3$ by $\sigma([v : w]) = (1/\sqrt{2}, w/(\sqrt{2}z))$. Then, σ extends without issue across D_+ by the formula

$$\sigma([z : w]) = \frac{(1, w/z)}{\sqrt{1 + |w/z|^2}},$$

so $e(D_+) = 1$.

Over D_-, a homotopy equivalence $h^{-1}(D_-) \simeq S^1$ is given by $(z, w) \mapsto w/|w|$, so $\sigma|_{\partial D_-}([z : w]) = w/z$. We can choose $z = 1$, so that $\sigma_{\partial D_-}([1 : w]) = w$. This map is a generator of $\pi_1(S^1)$, so $e(D_-) = 1$, and $e([S^2]) = \theta(\sigma)(D_+) + \theta(\sigma)(D_-) = 0 + 1 = 1$, and the Euler class of h is 1. In particular, the Hopf fibration does not admit a section.

(b) (Unit Tangent Bundle) We now consider the S^1 bundle $f : SS^2 \to S^2$ constructed in 1.1(a). We will identify SS^2 with $\{(v, w) \in S^2 \times S^2 : v \cdot w = 0\}$, so that the map f corresponds to projection onto the first factor. On D_0, define $\sigma(v) = (v, (0, 0, 1))$. Since every element of D_0 has third coordinate zero, this is indeed a section on D_0. We can identify $f^{-1}(D_+)$ with $D_+ \times S^1$ as follows. Given $(v, w) \in f^{-1}(D_+)$, rotate S^2 so that v moves along the great circle arc connecting v to $(0, 0, 1)$, and so that the vectors orthogonal to this great circle are fixed. Let $x \in D_0$ be the endpoint of w under this transformation. Then, $(v, w) \mapsto (v, x)$ is the desired identification. Under this identification, we can see that $\sigma(v) \mapsto (v, -v)$, so that $\theta(\sigma)(D_+) = 1$. A similar computation shows that $\theta(\sigma)(D_-) = 1$, so $e([S^2]) = 1 + 1 = 2$, which agrees with the Euler characteristic of S^2. This proves that there does not exist a nonvanishing vector field on S^2, a fact that is also known as the hairy ball theorem.

3. STATEMENT OF RESULTS

The remainder of this thesis will be concerned with the graph configuration space $\mathrm{Conf}_n(G)$ introduced in Section 1. In particular, we will study whether the map ϕ_{n+1} admits a section, which we will denote by $s_n : \mathrm{Conf}_n(G) \to \mathrm{Conf}_{n+1}(G)$ when it exist. Our main result is the following, stated in terms of the Euler characteristic χ of graphs:

Theorem 3.1. *For any graph G with $\chi(G) < 0$ and natural number n with $n \geq 2 - \chi(G)$, no section s_n exists.*

Theorem 3.2. *For any graph G with $\chi(G) = 0$, a section s_n exists for all natural numbers n.*

Theorem 3.3. *For any graph G with $\chi(G) = 1$, a section s_n exists if and only if $n \geq 2$.*

We also show that when $\chi(G) < 0$ and $n < 2 - \chi(G)$, then a section may or may not exist by providing examples of both cases, which shows that it isn't possible to improve our results without appealing to more than the value of $\chi(G)$. We will fix a graph G and a natural number n throughout.

The tools of obstruction theory are not particularly applicable to this question since the map ϕ_{n+1} is not a fibration. This follows since the fibers of a fibration are all homotopy equivalent, whereas the fibers of ϕ_{n+1} can differ wildly depending on the point. As a result, our methods are primarily geometric, and make little use of algebraic topology. The related problem of finding a section to the forgetting map $\mathrm{Conf}_{n+1}(M) \to \mathrm{Conf}_n(M)$ where M is a manifold with dimension greater than 1 has been studied, for instance in [FN62], [GG03], [CGL20], and [Che19]. This problem is much more amenable to the tools of obstruction theory since the forgetting map is a fibration in this case.

Another related problem is to determine whether the map ϕ_{n+1} induces a split surjection on fundamental groups. Since $\mathrm{Conf}_n(G)$ is aspherical, as proven in Theorem 3.1 of [Ghr01], this is equivalent to finding a map $s_n : \mathrm{Conf}_n(G) \to \mathrm{Conf}_{n+1}(G)$ such that $\phi_{n+1} \circ s_n \simeq \mathrm{id}_{\mathrm{Conf}_n(G)}$. Proposition 5.6 of [LRM21] shows that such maps always exist.

4. Notation

We assume that each graph is equipped with its natural cellular structure such that when $G \not\cong S^1$, every vertex has degree $\neq 2$, and when $G \cong S^1$, then G only has one vertex. A *branched vertex* is a vertex with degree ≥ 3, and a *free vertex* is a vertex with degree 1. An edge e is a *loop* if it contains a single vertex. If we fix a homeomorphism between each edge and $I = [0, 1]$ (or S^1 in the case that the edge is a loop) we obtain a metric d on G by taking these homeomorphisms to be local isometries and using the shortest path metric.

A subset of a graph is called a *subgraph* if it is closed and connected (this is not the standard definition, but it will prove more useful and flexible for us). A graph is a *tree* if it is simply connected. A *maximal subtree* of a graph is a subgraph which is a tree, and which contains every vertex. It is a basic fact of graph theory that every graph contains a maximal subtree.

We call the elements of $\mathrm{Conf}_n(G)$ *configurations*, and the entries in a configuration *tokens*. The integers $1, \ldots, n$ are called *indices*. If p, q are distinct points in G which lie on a common edge, then (p, q) (resp. $[p, q]$) is the open (resp. closed) interval they bound. In the case, for example, where $G \cong S^1$ or p, q are vertices which are endpoints of multiple edges, there is ambiguity in this definition, which we ignore, but the particular choice of edge will be clear from context in all cases we consider. If $x = (x_1, \ldots, x_n)$ is a configuration, then we abuse notation, and denote by $G \setminus x$ the complement $G \setminus \{x_1, \ldots, x_n\}$.

5. Basic Examples of Sections

Here we present three elementary cases where sections always exist. First we discuss identifying functions:

Note that the data of a section is exactly the data of a function that continuously adds a token to each configuration of $\mathrm{Conf}_n(G)$. In other words, this is equivalent to asking if there exists a map $f : \mathrm{Conf}_n(G) \to G$ such that for any $x = (x_1, \ldots, x_n) \in \mathrm{Conf}_n(G)$ and any index j, we have that $f(x) \neq x_j$. We call any function of this form an *identifying function*. Note that identifying functions are in one-to-one correspondence with sections. For notational simplicity we will prefer to work with them henceforth.

Proposition 5.1. *Suppose $n = 1$. If $\chi(G) \leq 0$ then G admits an identifying function.*

Proof. If $\chi(G) \leq 0$, then there exists an embedded circle $S^1 \subset G$ and a retract $r : G \to S^1$. Define $f : G \to G$ by $f(x) = -r(x)$, the point on S^1 antipodal to $r(x)$. Then f is an identifying function. $\square$

Proposition 5.2. *If $n \geq 2$ and $\chi(G) = 1$, then G admits an identifying function.*

Proof. Note that $\chi(G) = 1$ implies that G is a tree. Therefore there exists a unique embedded line segment between x_1, x_2 for each configuration $x = (x_1, \ldots, x_n)$. Let

$$\delta(x) = \min(\{d(x_1, x_k) : k \neq 1\} \cup \{1\})/2.$$

Then let $f(x)$ be the point on this embedded line segment with distance $\delta(x)$ away from x_1. Then f is an identifying function. $\square$

Proposition 5.2 is one half of Theorem 3.3. We will prove the other direction in Proposition 6.12. We conclude this section by proving Theorem 3.2.

Proof of Theorem 3.2. The case $n = 1$ follows from Proposition 5.1, so assume $n \geq 2$. Our approach will be to orient each edge, and then add a point near x_1 with respect to this orientation. There is a unique embedded circle $S^1 \subset G$. Orient S^1, and orient each edge in the complement of S^1 so they are "pointing towards" S^1 (Note the condition $\chi(G) = 0$ implies that G is homeomorphic to a circle with trees attached, where each tree is attached at a single point). This gives us a unique "forwards direction" at each point on G. Let

$$\delta(x) = \min\{d(x_1, x_k) : k \neq 1\}/2,$$

and let $f(x)$ be the point a distance $\delta(x)$ away from x_1 in the "forward direction." Then f is an identifying function. $\square$

6. Connectable Components and Chasing

In this section we develop tools which can be used to prove the nonexistence of sections. Consider the configurations x, x' on the graph G, as depicted in Figure 1.
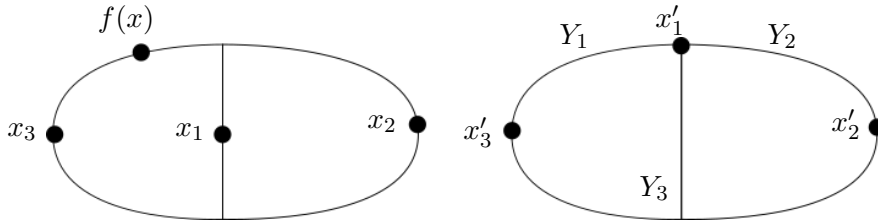

FIGURE 1. The configurations x, x'

As we can see, $G = \Theta$, the graph homeomorphic to a theta, $n = 3$, x and x' are the pictured configurations, and Y_1, Y_2, Y_3 are the connected components of $G \setminus x'$. Suppose that f is an identifying function on G, with $f(x)$ as shown on the left. We are interested in knowing which of Y_1, Y_2, Y_3 contains $f(x')$. Intuitively, we could imagine that if we move x_1 up along the central edge then we "force" $f(x)$ to choose between either Y_1 or Y_2, and that it is impossible for $f(x')$ to lie in Y_3.

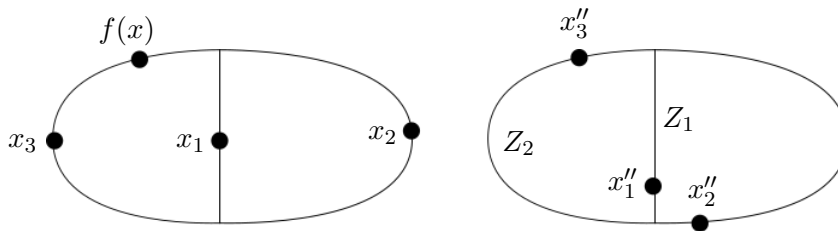Now consider the configuration x'' from Figure 2

FIGURE 2. The configuration x''

Note that x'' differs from x only by moving tokens within edges, without ever entering vertices. Therefore, we would intuitively expect that $f(x'') \in Z_1$. These facts are not difficult to prove individually using basic concepts from point-set topology, but we would like generalizations of them that hold for every graph and any n. The main idea is that tokens entering branched vertices "slice up" the complement into more components, and along paths where no token enters a branched vertex, the components of the complement do not increase in number.

6.1. **Connectable Components.** Suppose $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a path in $\mathrm{Conf}_n(G)$. We will define a relation between the components of $G \setminus \alpha(0)$ and the components of $G \setminus \alpha(1)$. First, we must define locally consistent systems of components along α.

Definition 6.1. Suppose $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a path in $\mathrm{Conf}_n(G)$, and suppose that for each $t \in I$ we have a connected component X_t of $G \setminus \alpha(t)$. The assignment $t \mapsto X_t$ is called a *system of components along α*. Such a system is *locally consistent* if for each $t \in I$ and each point $p \in X_t$, there exists an $\varepsilon > 0$ such that $p \in X_s$ for all $s \in (t - \varepsilon, t + \varepsilon)$.

We now define the relation on the components of $G \setminus \alpha(0)$ and the components of $G \setminus \alpha(1)$ where α is a path in $\mathrm{Conf}_n(G)$.

Definition 6.2. Let α be a path in $\mathrm{Conf}_n(G)$. If X is a component of $G \setminus \alpha(0)$ and Y is a component of $G \setminus \alpha(1)$, then we say that X and Y are *connectable by α*, or write $X \rightsquigarrow_\alpha Y$ if there exists a locally consistent system X_t of components along α such that $X_0 = X$ and $X_1 = Y$. We say that the system X_t *connects X to Y* (along α).

We also must consider globally consistent systems of components defined on subsets of $\mathrm{Conf}_n(G)$.

Definition 6.3. Suppose $A \subset \mathrm{Conf}_n(G)$. A *(globally) consistent system of components on A* consists of a connected component X_x of $G \setminus x$, for each $x \in A$, such that for any path $\alpha : I \to A$, the assignment $t \mapsto X_{\alpha(t)}$ is a locally consistent system of components along α.

The next proposition assures us of the essential fact that identifying functions give rise to consistent systems of components.

Proposition 6.4. *Suppose f is an identifying function. Let X_x be the component of $G \setminus x$ containing $f(x)$. Then $x \mapsto X_x$ is a consistent system of components on $\mathrm{Conf}_n(G)$.*

Proof. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be path in $\mathrm{Conf}_n(G)$, and let $X_t = X_{\alpha(t)}$. It suffices to show that the assignment $t \mapsto X_t$ is a locally consistent choice of components along α. Select $t \in I$ and $p \in X_t$. Let K be a line segment in X_t connecting p to $f(\alpha(t))$, and let $d(\alpha(t), K) = \delta > 0$. Let $\phi : I \to \mathbb{R}$ be the continuous function defined by

$$\phi(s) = d(\alpha(s), K \cup f(\alpha([s, t])))$$

(where we instead write $[t, s]$ when $t < s$). Since $\phi(t) = \delta > 0$, ϕ is positive in some neighborhood $(t - \varepsilon, t + \varepsilon)$ of t. For any $s \in (t - \varepsilon, t + \varepsilon)$, we can see that the connected set $K \cup f(\alpha([s, t]))$ contains p and $f(\alpha(s))$, and is disjoint from $\alpha(s)$. Therefore, it lies in a single component of $G \setminus \alpha(s)$, which is X_s, as desired. $\square$

Remark. Our main method of disproving the existence of sections is to show that consistent systems of components do not exist on $\mathrm{Conf}_n(G)$.

In this section, we will prove two important facts that will allow us to compute the relation $\rightsquigarrow_\alpha$ for two classes of paths. These facts formalize the intuitive ideas discussed in the opening portion of this section. Given an identifying function f, these facts will allow us to determine by Proposition 6.4 the component of $G \setminus x$ containing $f(x)$ by moving along paths in $\mathrm{Conf}_n(G)$.

Definition 6.5. We say that a path $\alpha : I \to \mathrm{Conf}_n(G)$ is a *Type I path* if no token ever *enters* a vertex. That is, if $\alpha_i(t)$ is not a vertex, then for all $s > t$, $\alpha_i(s)$ is also not a vertex. We say that a path is a *Type II path* if each coordinate of α except one is constant, and the

nonconstant coordinate moves directly at constant speed from the interior of an edge to a vertex. A path that is either Type I or Type II is called *basic*.

Remark. If α is any path in $\mathrm{Conf}_n(G)$ such that at least two tokens never enter a branched vertex simultaneously, then α is homotopic to a concatenation of basic paths via a homotopy h_t such that the topology of $G \setminus h_t(s)$ depends only on s, up to reparameterization. If two tokens simultaneously enter branched vertices, then we can slightly perturb α such that the tokens enter the branched vertices at different times, which does not change the relation $\leadsto_\alpha$. If α, β are paths such that the concatenation $\alpha * \beta$ is defined, and X is a component of $G \setminus \alpha(0)$ and Z is a component of $\beta(1)$, then it is clear that $X \leadsto_{\alpha * \beta} Z$ if and only if there is some component Y of $G \setminus \alpha(1)$ such that $X \leadsto_\alpha Y$ and $Y \leadsto_\beta Z$. Therefore, if we compute the relation $\leadsto_\alpha$ for basic paths α, we can determine it for all paths.

Our current goal is to determine exactly which components are connectable by basic paths.

6.2. **Type I Paths.**

Proposition 6.6. *If α is a Type I path, and X is a component of $G \setminus \alpha(0)$, then there is at most one component Y of $G \setminus \alpha(1)$ such that $X \leadsto_\alpha Y$.*

Proof. Suppose Y, Z are distinct components of $G \setminus \alpha(1)$, and that $X \leadsto_\alpha Y$ and $X \leadsto_\alpha Z$. Suppose that Y_t, Z_t are locally consistent systems of components along α connecting X to Y, Z respectively. Let

$$s = \inf\{t \in I : Y_t \neq Z_t\}.$$

If $Y_s = Z_s$, then for any $p \in Y_t$, we can take arbitrarily small $\varepsilon > 0$ such that $Y_{s+\varepsilon} \neq Z_{s+\varepsilon}$ and $p \in Y_{s+\varepsilon}$. However, this implies that $p \notin Z_{s+\varepsilon}$ for arbitrarily small ε, which contradicts the fact that Z_t is a locally consistent system of components. Therefore, $Y_s \neq Z_s$.

Now, select $p \in Y_s, q \in Z_s$. There exists an $\varepsilon > 0$ such that for all $r \in [s - \varepsilon, s)$, we have $p, q \in Y_r$. Let K be an embedded line segment in $Y_{s-\varepsilon}$ connecting p to q. Since $Y_s \neq Z_s$, there is some $r \in (s - \varepsilon, s]$ such that for some $i = 1, \ldots, n$ we have $\alpha_i(r) \in K$. We can in fact assume that $\alpha_i(r) \in \partial K$ (where ∂ refers to the topological boundary as a subspace of G) by possibly taking a smaller value of r. However, by our choice of ε, $\alpha_i(r) \neq p, q$, so $\alpha_i(r) = v$ for some branched vertex contained in K, since these are the only other boundary points of K. This contradicts the fact that α is a Type I path, and we are done. $\qquad\square$

We now describe explicitly, for any type I path α and any component X of $G \setminus \alpha(0)$, the unique component Y of $G \setminus \alpha(1)$ such that $X \leadsto_\alpha Y$.

Proposition 6.7. *Suppose $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a Type I path, and that X is a component of $G \setminus \alpha(0)$. Define a component Y of $G \setminus \alpha(1)$ as follows:*

   (i) *If X contains a vertex v, then Y is the component containing v.*
   (ii) *If X is an open interval $(\alpha_i(0), \alpha_j(0))$ in an edge e and $\alpha_i(t), \alpha_j(t) \in e$ for all t, then $Y = (\alpha_i(1), \alpha_j(1)) \subset e$.*
   (iii) *If X is an open interval $(\alpha_i(0), \alpha_j(0))$ in an edge e and $\alpha_i(0) = v$ is an endpoint of e, and $\alpha_i(t) \notin e$ for some t, then Y is the component containing v.*
   (iv) *If X is a loop e without its branched vertex $v = \alpha_i(0)$, then if α_i is constant, $Y = X$, and otherwise, Y is the component containing v.*
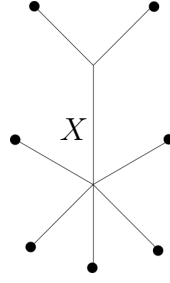
Then, $X \leadsto_\alpha Y$.
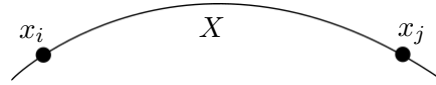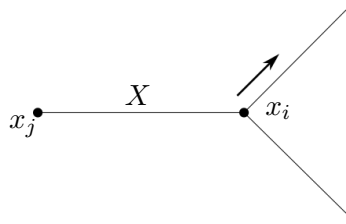
FIGURE 3. The case (i)

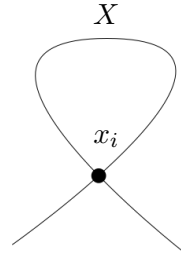FIGURE 4. The case (ii)

FIGURE 5. The case (iii)

FIGURE 6. The case (iv)

Proof. For any $t \in I$, consider the restricted path $\alpha|_{[0,t]}$. Since any restriction of a Type I path remains Type I, the statement of the proposition gives us a component Y_t of $G \setminus \alpha(t)$. We claim this is a locally consistent system of components along α. We proceed by proving each of the cases separately.

(i) Select $t \in I$ and $p \in Y_t$. Since α is Type I, $v \in Y_t$, so there exists a compact connected $K \subset Y_t$ containing p and v. Let $\zeta = d(K, \alpha(t))$, and select ε such that for all $s \in (t - \varepsilon, t + \varepsilon)$ and all $i = 1, 2, \ldots, n$, we have $d(\alpha_i(t), \alpha_i(s)) < \zeta$. Since $v \in K$, we have $K \subset Y_s$ for all $s \in (t - \varepsilon, t + \varepsilon)$, so $p \in Y_s$, as desired.

(ii) Select $t \in I$ and $p \in Y_t$. Let

$$\zeta = \min\{d(\alpha_i(t), p), d(\alpha_j(t), p)\},$$

and select ε such that $d(\alpha_\eta(t), \alpha_\eta(s)) < \zeta$ for $\eta = i, j$ and $s \in (t - \varepsilon, t + \varepsilon)$. Then $p \in (\alpha_i(s), \alpha_j(s)) = Y_s$ for all $s \in (t - \varepsilon, t + \varepsilon)$, as desired.

(iii) Select $t \in I$. If $\alpha_i(t + \varepsilon) = v$ for some $\varepsilon > 0$, then apply (ii) to the restricted path $\alpha|_{[0,t+\varepsilon]}$. If $\alpha_i(t - \varepsilon) \notin e$ for some $\varepsilon > 0$, then apply (i) to the restricted path $\alpha|_{[t-\varepsilon,1]}$. It suffices to consider the remaining case, where $\alpha_i(t) = v$ but $\alpha_i(t + \varepsilon) \notin e$ for all $\varepsilon > 0$. Select $p \in Y_t$. As before, we can set

$$\zeta = \min\{d(\alpha_i(t), p), d(\alpha_j(t), p)\},$$

and select ε such that for all $s \in (t - \varepsilon, t + \varepsilon)$ we have $d(\alpha_\eta(s), \alpha_\eta(t)) < \zeta$ for $\eta = i, j$. If $s \leq t$, then $Y_s = (\alpha_i(s), \alpha_j(s))$, which contains p. If $s > t$, then Y_s contains v. Since $\alpha_i(s) \notin e$, $[v, p]$ is disjoint from $\alpha(s)$, so $p \in Y_s$.

(iv) We can modify the cellular structure of G such that G has no loops, perhaps by adding a vertex to each loop. Since none of our techniques depend on the vertices of G being branched, we can reduce to part (i).

$\square$

6.3. **Type II Paths.** We now turn to Type II paths. We begin with a lemma.

Lemma 6.8. *Suppose α is a path in $\mathrm{Conf}_n(G)$, and X_t is a system of components along α. If $X_t \subset X_s$ whenever $t < s$, then X_t is the only locally consistent system of components along α starting at X_0.*

Proof. We can easily observe that X_t is locally consistent, so it suffices to show that this is the unique locally consistent system of components along α starting at X_0. Suppose Y_t is another such system, and let

$$s = \inf\{t \in I : X_t \neq Y_t\}.$$

By the argument from the proof of Proposition 6.6, we can see that in fact $X_s \neq Y_s$. Select $p \in Y_s$, and ε such that $p \in Y_r$ for all $r \in [s - \varepsilon, s]$. Since $Y_r = X_r$ by assumption, and $X_r \subset X_s$, we have $p \in X_s$, a contradiction. $\square$

We can now compute the relation $\rightsquigarrow_\alpha$ when α is a Type II path.

Proposition 6.9. *Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a Type II path where α_i is the unique nonconstant coordinate, and therefore has constant speed, and $\alpha_i(1) = v$ for some vertex v. Suppose that the components of $G \setminus \alpha(0), G \setminus \alpha(1)$ near v are labelled as in the picture below (note that X_1, X_2 and $Y_1, \ldots, Y_k$ are not necessarily distinct). If $X = X_1$, and X_1, X_2 are distinct, then $Y = Y_1$, and if $X = X_2$ then $Y = Y_j$ for some $j \geq 2$. If $X = Z$ for some other component Z, then $Y = Z$.*

Proof. First, suppose that $X = X_1$ and that X_1, X_2 are distinct. Then, we can define a system of components Z_t, such that Z_t is the component of $G \setminus \alpha(t)$ containing X_1. By Lemma 6.8, this is the unique locally consistent system of components starting at X_1, so $Y = Y_1$.

Now suppose that $X = X_2$, and that Z_t is a locally consistent system of components starting at X_2. It suffices to show that

$$Z_1 \subset Y_2 \cup \cdots \cup Y_k.$$

Select $p \in Z_1$, and ε such that $p \in Z_s$ whenever $s \in [1 - \varepsilon, 1]$. For any $s < 1$, we note that the restriction $\alpha|_{[0,s]}$ is a Type I path, which implies that Z_s is the component of $G \setminus \alpha(s)$ containing v by Proposition 6.7(i). When $Y_1 = Y_j$ for some $j \geq 2$, this implies that

$$Z_s = (Y_2 \cup \cdots \cup Y_k) \setminus \{\alpha_i(s)\},$$
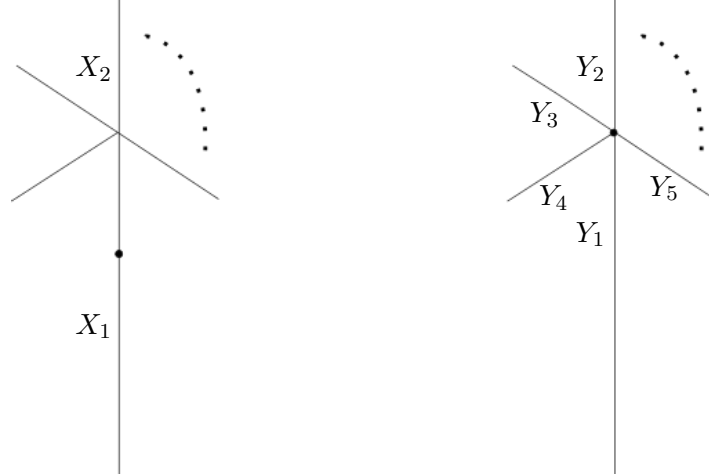
and since $p \in Z_{1-\varepsilon}$, this implies that p is in Y_j for some $j \geq 2$, as desired. When $Y_1 \neq Y_j$ for all $j \geq 2$, then

$$Z_s = Y_2 \cup \cdots \cup Y_k \cup [\alpha_i(s), v].$$

If we can show $p \notin [\alpha_i(1-\varepsilon), v]$, then we are done. However, this is impossible, since otherwise there would be some $s \in (1-\varepsilon, 1)$ such that $\alpha_i(s) = p$, which contradicts our choice of ε. This completes the proof. $\qquad\square$

Propositions 6.7 and 6.9 are generally used in conjunction with Proposition 6.4. If we have a basic path α, and an identifying function f with $f(\alpha(0)) \in X$ for some component X of $G \setminus \alpha(0)$, then we obtain by Proposition 6.4 a locally consistent choice of components along α. We can then use either Proposition 6.7 or Proposition 6.9 to determine the possible components of $G \setminus \alpha(1)$ which can contain $f(\alpha(1))$.

6.4. **Applications.** Our first important application of Propositions 6.7 and 6.9 is the "chasing" lemma:

Lemma 6.10 (Chasing). *Suppose f is an identifying function, and $x = (x_1, \ldots, x_n)$ is a configuration. Let X denote the connected component of $G \setminus x$ containing $f(x)$, and suppose that $X \cup \{x_1\}$ is simply connected. Then there exists some point $p \in X$ and an integer $2 \leq j \leq n$ such that if $x' = (p, x_2, \ldots, x_n)$, then $f(x') \in (x_j, p)$.*

Proof. We induct on the number of vertices k in X. If $k = 0$, then X is an interval, so we can merely take $p = x_1$.

Suppose we know the result for all $\ell < k$. Since $X \cup \{x_1\}$ remains connected when x_1 is removed, x_1 must have degree 1 in the closure of X, since any point which is not a free vertex disconnects a simply connected 1-complex when removed. Consider the Type II path α in $\mathrm{Conf}_n(G)$ such that $\alpha_1(0) = x_1$ and α_1 moves along the unique edge in the closure of X which is connected to x_1. Let $v = \alpha_1(1)$, and let X' be the connected component of $G \setminus \alpha(1)$ containing $f(\alpha(1))$. By Proposition 6.9, X' borders v and $x_1 \notin X'$. Therefore, $X' \cup \{v\}$ is simply connected, since it is a connected subset of a tree. Since $v \notin X'$ and $X' \subset X$, X' contains fewer than k vertices, so we are done by the inductive hypothesis. $\qquad\square$

Of course, the chasing lemma applies equally well when x_1 is replaced by any other token. The chasing lemma, while simple, is an important aspect of the proof of the main theorem. When we apply it (with x_i taking the role of x_1) we say that we are "chasing with x_i."

The following corollary is worth noting separately:

Corollary 6.11. *If f is an identifying function on G, and $T \subset G$ is a subtree of G which is connected to the rest of G by a single point p, then for any configuration of the form $x = (p, x_2, \ldots, x_n)$ with $x_2, \ldots, x_n \notin T$, we have $f(x) \notin T$.*

Proof. Suppose that $x = (p, x_2, \ldots, x_n)$ is such that $x_2, \ldots, x_n \notin T$, and $f(x) \in T$. Then, applying Lemma 6.10, there is some $i = 2, \ldots, n$ and $p' \in T$ such that if $x' = (p', x_2, \ldots, x_n)$, then $f(x') \in (p', x_i)$. If $p' \neq p$, then this is a contradiction since $x_i \notin T$, and if there is any interval where exactly one endpoint is in T, then that endpoint must be in ∂T, and therefore be equal to p. However, if $p' = p$, then $x' = x$, and we have $f(x) \notin T$, which is a contradiction, so we are done. $\square$

We can now prove the second half of Theorem 3.3.

Proposition 6.12. *Suppose $\chi(G) = 1$ and $n = 1$. Then G does not admit a section.*

Proof. Suppose $f : G \to G$ is an identifying function. Given any point $p \in G$, p separates G into subtrees $T_1, \ldots, T_k$. However, by Corollary 6.11, $f(p)$ can be contained in none of them, which is a contradiction. $\square$

Note that Proposition 6.12 is a generalization of the 1-dimensional Brouwer fixed-point theorem, as it says that for any tree T (not just the interval I), any continuous map $T \to T$ has a fixed point.

Remark. This discussion gives an algebraic necessity for the existence of a section. Let $PB_n(G) = \pi_1(\mathrm{Conf}_n(G))$ denote the *n'th pure braid group* of G. Suppose that X_x is a consistent system of components on $\mathrm{Conf}_n(G)$. Let $H < PB_{n+1}(G)$ be the set of homotopy classes of paths which can be represented by a path $\alpha = (\alpha_1, \ldots, \alpha_{n+1})$ such that $\alpha_{n+1}(t) \in X_{(\alpha_1(t), \ldots, \alpha_n(t))}$ for all $t \in I$. One can verify that H is a subgroup of $PB_{n+1}(G)$. Then, if $s : \mathrm{Conf}_n(G) \to \mathrm{Conf}_{n+1}(G)$ is a section whose corresponding system of components (in the sense of Proposition 6.4) is X_x, then the image of the induced map $s_* : PB_n(G) \to PB_{n+1}(G)$ must lie in H, since for any based loop β in $\mathrm{Conf}_n(G)$, the composition $s \circ \beta$ represents an element of H. Therefore, if a section exists then there exists some consistent choice of components X_x such that the restricted map:

$$H \hookrightarrow PB_{n+1}(G) \to PB_n(G)$$

admits a section.

7. Proof of Theorem 3.1

In this section, we prove Theorem 3.1 and some necessary lemmas.

7.1. **Distinguished Pairs.** First we define and discuss the basic properties of distinguished pairs, which occur when, for a configuration x and identifying function f, $f(x)$ lies between two tokens of x. We will show that the existence of such pairs allow us to draw broad conclusions about the existence of sections, culminating in the proof of Theorem 3.1. Recall the open interval (p, q) notation introduced at the start of Section 5.

Definition 7.1. Suppose $G \not\cong S^1$, and e is an oriented edge such that G remains connected when a single interior point of e is removed, and suppose that f is an identifying function on G. If i, j are a pair of distinct indices, we say that (i, j) forms a *distinguished pair (of f) on e* if there exists a configuration $x = (x_1, \ldots, x_n)$ such that:

(a) $x_i, x_j \in e$,
(b) x_i lies before x_j on e with respect to the orientation on e,
(c) $x_k \notin (x_i, x_j)$ for all k,
(d) $f(x) \in (x_i, x_j)$.

We then say that the configuration x is a *witness* to the distinguished pair (i, j).
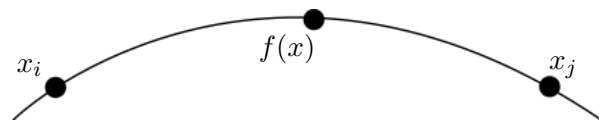

FIGURE 7. A witness to a distinguished pair (i, j)

Note that if f is an identifying function and $x = (x_1, \ldots, x_n)$ is a configuration which satisfies the hypotheses of Lemma 6.10 with respect to f, and additionally $G \setminus \{x_\ell\}$ is connected for each $\ell \geq 2$, then the conclusion of Lemma 6.10 in fact produces a distinguished pair containing 1.

Next we state and prove the most important fact about distinguished pairs.

Proposition 7.2. *If (i, j) is a distinguished pair on e, and $y = (y_1, \ldots, y_n)$ is a configuration which satisfies the properties (a)-(c) in Definition 7.1 with respect to (i, j) for f, then y also satisfies property (d), that is, it is a witness to (i, j).*

Proof. Let $x = (x_1, \ldots, x_n)$ to (i, j) be a witness to (i, j), and let $y = (y_1, \ldots, y_2)$ be as in the proposition statement. Our goal is to construct a path $\alpha = (\alpha_1, \ldots, \alpha_n)$ in $\mathrm{Conf}_n(G)$ connecting x, y such that α_i, α_j never leave e, which is a concatenation of basic paths. This will imply by Propositions 6.7 and 6.9 that $f(y) \in (y_i, y_j)$.

First consider the case where none of the x_ℓ with $\ell \neq i, j$ lie in e, and y_i, y_j lie in the interior of e. Then, we can take α to first move the i'th and j'th tokens to y_i, y_j respectively, and then move each of the remaining $n - 2$ tokens of x to the corresponding tokens of y. Since $\mathrm{Conf}_{n-2}(G \setminus [x_i, x_j])$ is connected by Theorem 2.7 of [Abr00], this is possible.

If (say) y_i is at an endpoint of e, then let y_i' be a point in e which is very close to y_i. We can modify the path α above by initially moving x_i to y_i', and then moving y_i' to y_i once the remaining $n - 2$ tokens agree with y.

If some of the x_ℓ lie in e for $\ell \neq i, j$, then we can first apply a path which moves each of these x_ℓ outside of e, and then reduce to the previous case.

All these paths can be taken to be concatenations of basic paths, so we are done. □

Proposition 7.2 gives rise to the very useful observation:

Corollary 7.3. *If (i, j) is a distinguished pair on e and (k, ℓ) is a distinguished pair on e', then $\{i, j\} \cap \{k, \ell\} \neq \varnothing$*

Proof. Suppose (i, j), (k, ℓ), are distinguished pairs such that $\{i, j\} \cap \{k, \ell\} = \varnothing$. Then, we can find a configuration $x = (x_1, \ldots, x_n)$ satisfying the hypotheses for Proposition (7.2) with respect to both distinguished pairs. This necessitates $f(x) \in (x_i, x_j)$ and $f(x) \in (x_k, x_\ell)$, which is a contradiction. □

7.2. **The Case** $n > 3$. We now present the proof of the main theorem in the case $n > 3$:
Suppose f is an identifying function, and consider the statements below:

(A) Every index belongs to some distinguished pair of f.
(B) No index belongs to every distinguished pair of f.

Lemma 7.4. *If $n > 3$ and $f : \mathrm{Conf}_n(G) \to G$ is an identifying function, then the statements (A), (B) are not simultaneously true of f.*

Proof. Suppose that statements (A), (B) are both true, and suppose without loss of generality that $(1, 2)$ is a distinguished pair. Then 3 is contained in some distinguished pair, $(i, 3)$. If $i \neq 1, 2$, then this contradicts Corollary 7.3, so suppose $i = 1$. Then, $(1, 4)$ must be a distinguished pair, since if 4 is in any other distinguished pair, we can produce a counterexample to Corollary 7.3. By (B), there is some distinguished pair (i, j) with $i, j \neq 1$. This contradicts Corollary 7.3 applied to either the pairs (i, j) and $(1, 2)$, the pairs (i, j) and $(1, 3)$, or the pairs (i, j) and $(1, 4)$, depending on i, j. $\square$

It therefore suffices to establish that (A) and (B) hold when $n \geq 2 - \chi(G)$ and $\chi(G) < 0$.

Proof of (A) when $n \geq 2 - \chi(G)$. Let T be a maximal subtree of G, let $k = 1 - \chi(G)$, and let $e_1, e_2, \ldots, e_k$ be the edges in the complement of T. Let A_1, A_2 be closed intervals contained in the interiors of e_1, e_2 respectively, and let

$$\gamma : I \to T \cup e_1 \cup e_2$$

be an embedded path connecting an endpoint of A_1 to an endpoint of A_2, such that the intersection of the image of γ with $A_1 \cup A_2$ is exactly these two endpoints. We can assume that T is not contained in the image of γ (if not, enlarge T slightly so that it is not homeomorphic to an interval). The situation is summarized in Figure 8.
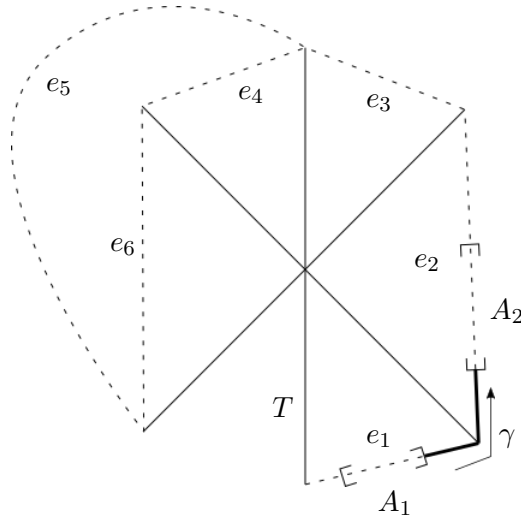

FIGURE 8. A graph G with labelled edges, with labelled A_1, A_2 and γ.

We will prove that 1 is contained in some distinguished pair. Let $x_1 \in G$ be a point in T not contained in the image of γ. Let $x_2, \ldots, x_{k-1}$ be points in the interiors of $e_3, \ldots, e_k$ respectively. Let $x_k, \ldots, x_{n-1}$ be points spaced out equally in A_1 in order such that x_{n-1} is on the endpoint contained in the image of γ, and let x_n be the endpoint of A_2 not contained in the image of γ. Let $x = (x_1, \ldots, x_n)$. The configuration x is pictured in Figure 9.
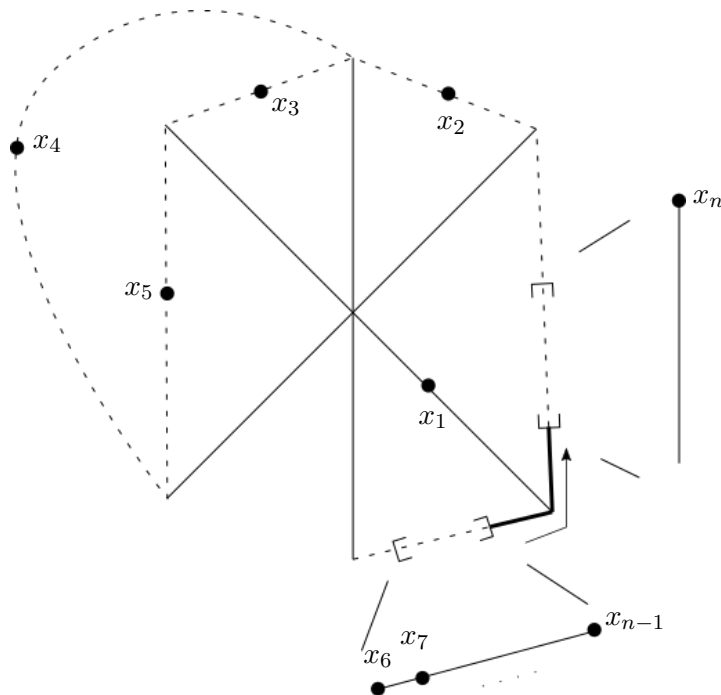
FIGURE 9. The configuration x

The connected components of $(G \setminus x) \cup \{x_1\}$ are the intervals $(x_k, x_{k+1}), \ldots, (x_{n-2}, x_{n-1})$, and a large simply connected component X which contains x_1. If $f(x) \in X$, then let $\tilde{X}$ be the component of $X \setminus \{x_1\}$ containing $f(x)$. Note that $\tilde{X} \cup \{x_1\}$ is simply connected, so we can apply Lemma 6.10 to produce a distinguished pair containing 1, as desired.

Otherwise, suppose that $f(x) \in (x_{n-2}, x_{n-1})$ (the cases when $f(x)$ lies in a different of these intervals is similar, by renumbering the indices and using Proposition 7.2). Let α be the path in $\mathrm{Conf}_n(G)$ which moves x_{n-1} along γ, and fixes each of the other tokens. Call the ending configuration

$$x' = (x_1, \ldots, x_{n-2}, \gamma(1), x_n),$$

and let X' be the big simply connected component in $(G \setminus x') \cup \{x_1\}$. For each t, let X_t be the component of $G \setminus \beta(t)$ which contains (x_{n-2}, x_{n-1}). Note that X_t is a system of components which satisfies the hypotheses of Lemma 6.8, which implies that $f(x') \in X_1 = X'$. As before, we can apply Lemma 6.10 to x' to produce a distinguished pair containing 1. $\qquad \square$

Proof of (B) when $n \geq 2 - \chi(G)$. Suppose for sake of contradiction that 1 is in every distinguished pair, and that $(1, 2)$ is a distinguished pair in e_1 (a distinguished pair exists by (A)). Recall that this equips e_1 with an specified orientation. Let T be a maximal subtree not containing e_1, and let e_2 be some other edge in $G \setminus T$. Suppose further that T "peeks in" to each edge, such that each point in ∂T is contained in the closure of exactly one edge in $G \setminus T$. Figure 10 shows an example of one of these constructions.

Let $e_3, \ldots, e_k$ be the other edges not contained in T, where $k = 1 - \chi(G)$, as before. Let H be the subgraph formed from $T \cup e_1 \cup e_2$ by removing free edges until every vertex has degree at least 3. Since H is a deformation retract of $T \cup e_1 \cup e_2$, we can see that $\chi(H) = -1$, and H has no free edges. Therefore H is homeomorphic to either the figure eight graph (∞), the theta graph (Θ), or the dumbbell graph D. The three possibilities are pictured in Figure 11.
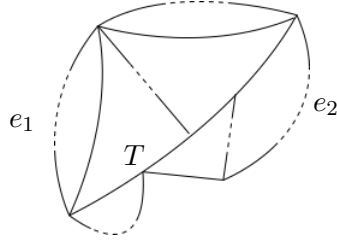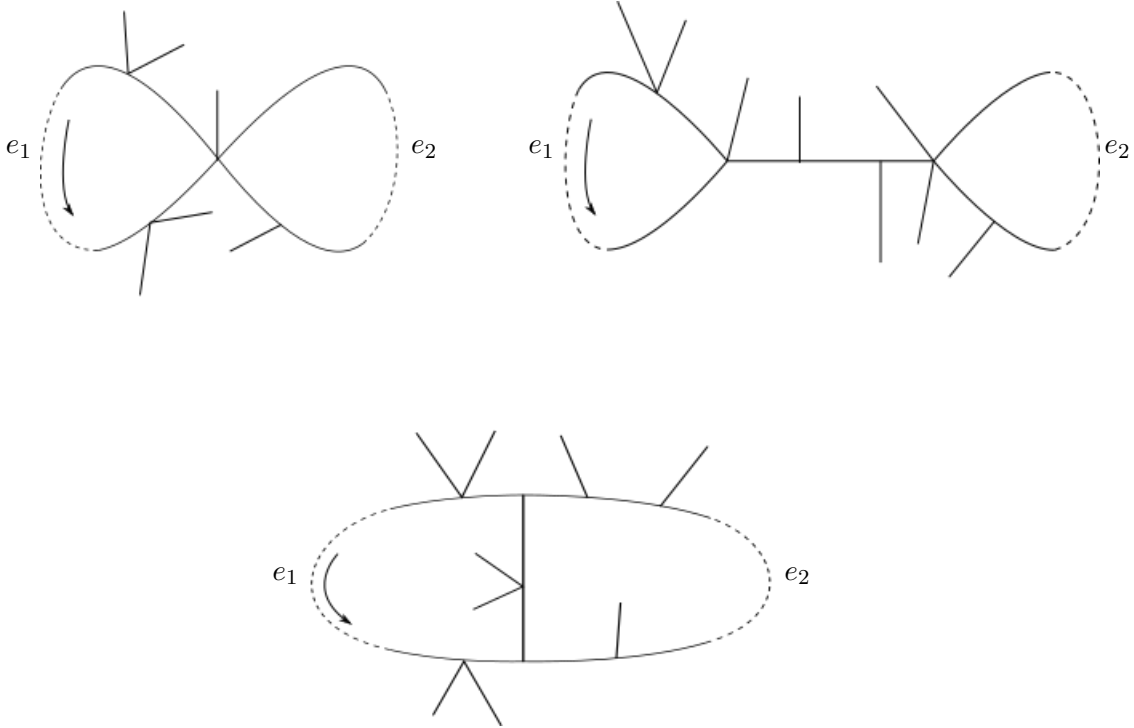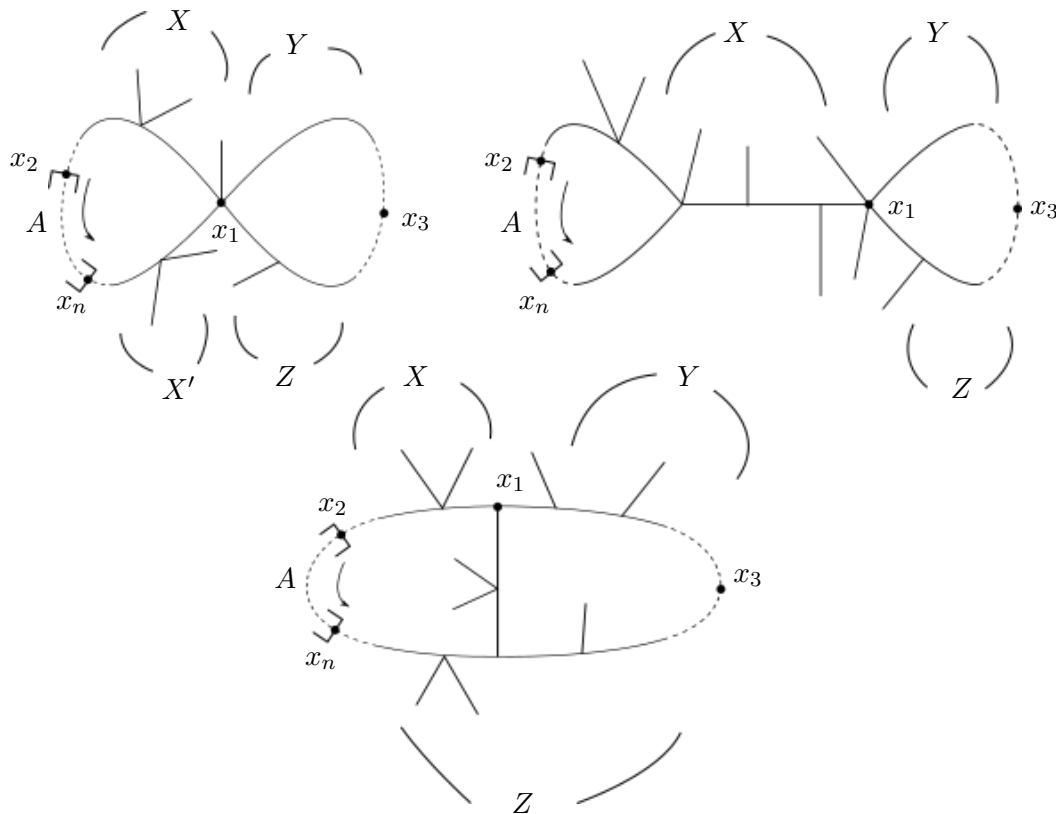
FIGURE 10

FIGURE 11. Neighborhoods of the possible subgraphs H

Define a configuration $x = (x_1, \ldots, x_n)$ as follows: Let $x_3, x_4, x_5, \ldots, x_{k+1}$ be points in the interior of $e_2, e_3, \ldots, e_k$, respectively. In the interior of e_1, place $x_2, x_{k+2}, \ldots, x_n$ equally spaced apart in e_1, such that x_2 is farthest back on e_1 with respect to the orientation on e_1, and let $A = [x_2, x_n]$. Finally, place x_1 on a branched vertex in $T \cap H$ as follows:

(i) When $H = \infty$, then x_1 is the unique branched vertex of H.
(ii) When $H = \Theta$, then x_1 is the branched vertex of H closer to the back end of e_1, such that when x_1 slides into e_1, we obtain a witness to $(1, 2)$.
(iii) When $H = D$, then x_1 is the branched vertex of H lying closer to e_2

Further, let $B = G \setminus (x \cup A)$, and label the components of $G \setminus B$ by X, Y, Z (and X' when $H = \infty$) as indicated in Figure 12.

When we slide x_1 into the back of e_1, we obtain a witness to $(1, 2)$ by Proposition 7.2, and the reverse path gives a locally system of components by Lemma 6.8, which implies exactly that $f(x) \in X$.

FIGURE 12. The components X, Y, Z, (and X') of $G \setminus B$.

Let $x' = (x_3, x_2, x_1, x_4, \ldots, x_n)$ be the configuration which is x except with x_1 and x_3 swapped. We will argue that $f(x')$ cannot lie in any component of B. If $f(x') \in X$ (or X'), then applying Lemma 6.10 with x_3 produces a distinguished pair which does not contain 1. Therefore we must have $f(x') \in Y$ or $f(x') \in Z$.

We will define two paths $\beta = (\beta_1, \ldots, \beta_n)$ and $\gamma = (\gamma_1, \ldots, \gamma_n)$ in $\mathrm{Conf}_n(G)$ from x to x' which will allow us to see that neither case is possible. Take each of these paths to be concatenations of basic paths which fix each token except for the first and third, and such that β moves x_1, x_3 around the right loop in H a half-turn *clockwise*, and γ moves x_1, x_3 around the right loop in H a half-turn *counter-clockwise*[1].

Note that for each t, there are exactly two components of $G \setminus \beta(t)$ which border both $\beta_1(t)$ and $\beta_3(t)$. Since β_1, β_3 both move clockwise, β_1, β_3 are each moving "towards" a specific well-defined component at each t, and they likewise are each moving "away" from a specific well-defined component at each time. Further, for t such that $\beta_1(t) \neq x_1$, the path β_1 is moving "away" from the component that β_3 is moving "towards." Our immediate goal is to show that $f(x') = f(\beta(1))$ must lie in the component of $G \setminus \beta(1)$ which β_1 is moving away from. We show this inductively on the basic pieces of β.

Consider a Type I piece of β. We can see that f must remain in the component β_1 is moving away from by Propositions 6.6 and 6.7.

Consider a Type II piece of β where only β_1 moves to a vertex v. Since the component that β_1 is moving away from is distinct from the components bordering v, we can deduce by

_____

[1]Here "clockwise" and "counter-clockwise" are with respect to the orientations implied by the figure

Proposition 6.9 that f must remain in the component which f is moving away from at the end of this basic piece.

Finally, consider a Type II piece of β where β_3 moves, and suppose that f takes values in a component of $G \setminus \beta$ which borders β_3. Since we inductively assume that f lies in the component of $G \setminus \beta$ which β_1 is moving away from, f must lie on the component of $G \setminus \beta$ that β_3 is moving towards. Therefore, once β_3 reaches the end of this Type II piece at some time t, f must lie in some component W of $G \setminus \beta(t)$ which borders $\beta_3(t)$ but is not behind β_3 at t by Proposition 6.9. If W does not border $\beta_1(t)$, then by our construction of the configuration x, $W \cup \{\beta_3(t)\}$ is simply connected, so we can apply Lemma 6.10 to produce a distinguished pair which does not contain 1. Therefore, W must border $\beta_1(t)$, so it must in fact be the component which β_3 is moving towards, and therefore the component which β_1 is moving away from.

We have shown that $f(x')$ is in the component which β_1 is moving away from. Since β moves clockwise, we must have $f(x') \in Y$. However, if we apply the exact same argument with γ, we get $f(x') \in Z$, which is a contradiction. $\square$

7.3. **The Case** $n = 3$. When $n = 3$ and $\chi(G) = -1$, the statements (A) and (B) do not contradict Corollary 7.3, so we must rely on a different technique. We first introduce a proposition which simplifies the proof.

Proposition 7.5. *Suppose $H \subset G$ is a deformation retract of G, and suppose there exists an identifying function on G. Then there exists an identifying function on H.*

Proof. Since H is a deformation retract of G, G can be obtained from H by attaching trees to H at single points. Therefore, there exists a retract $r : G \to H$ which collapses each of these trees to a single point. We define $g : \mathrm{Conf}_n(H) \to H$ by $g(x) = r(f(x))$. We can see that g is continuous, and we can show that it is also an identifying function. Suppose that $x = (x_1, \ldots, x_i)$ is a configuration, and suppose $x_i = r(f(x))$. Then $f(x) \in T$ for some tree T attached to G. Corollary 6.11 produces a contradiction. $\square$

Finally, we present another consequence of Proposition 7.2 which will be useful to us.

Proposition 7.6. *If (i, j) and (k, i) are distinguished pairs on e, then $j = k$.*

Proof. Suppose otherwise, then we can construct a configuration $x = (x_1, \ldots, x_n)$ which is a simultaneous witness to (i, j) and (k, i), by picking x_i, x_j, x_k all lying on e with $x_k < x_i < x_j$ with respect to the orientation on e, and no other tokens contained in the interval $[x_k, x_j]$. By Proposition 7.2, such a configuration is a witness to both distinguished pairs, necessitating $f(x) \in (x_i, x_j)$ and $f(x) \in (x_k, x_i)$, which is a contradiction. $\square$

We can now present the final portion of the proof of Theorem 3.1.

Proof when $n = 3$ and $\chi(G) = -1$. It suffices to consider this in the case where G has no free vertices by Prop 7.5. The only such graphs with $\chi(G) = -1$ are the graphs ∞, Θ, and D as before. Suppose for sake of contradiction that f is an identifying function on G.

We will define two functions
$$\mathrm{Hor} : \Sigma_3 \to \{\pm 1\},$$
$$\mathrm{Ver} : \Sigma_3 \to \{\pm 1\}.$$
For a permutation $\sigma \in \Sigma_3$, consider the configuration x_σ depicted in Figure 13.
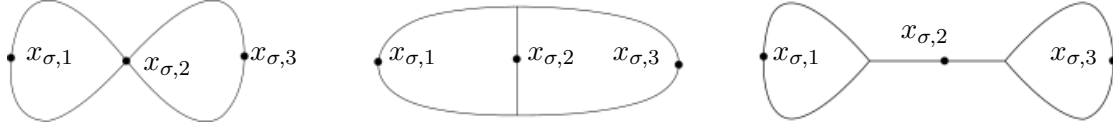
FIGURE 13. The configurations x_σ

Denote by $x_{\sigma,i}$ the i'th token of x_σ. We consider the components of $G \setminus x_\sigma$ to be either on the "top" or "bottom," (resp. "left," "right") corresponding to their depictions in the figure. When $G = \infty, \Theta$, then we say $\mathrm{Ver}(\sigma) = +1$ if $f(x_\sigma)$ lies in one of the top components of $G \setminus x_\sigma$, and $\mathrm{Ver}(\sigma) = -1$ otherwise. Similarly, when $G = \infty, D$ we say $\mathrm{Hor}(\sigma) = +1$ if $f(x_\sigma)$ lies in one of the right components of $G \setminus x_\sigma$ and $\mathrm{Hor}(\sigma) = -1$ otherwise. To compute Ver for $G = D$, define x'_σ to be the configuration obtained by moving along a Type II path which moves $x_{\sigma,2}$ towards $f(x_\sigma)$. By Proposition 6.9, $f(x'_\sigma)$ lies in either the top or bottom component of $G \setminus x'_\sigma$ contained in the loop containing $x_{\sigma,2}$. We say that $\mathrm{Ver}(\sigma) = +1$ if $f(x'_\sigma)$ is in the top component of $G \setminus x'_\sigma$ in this loop, and $\mathrm{Ver}(\sigma) = -1$ if $f(x'_\sigma)$ is in the bottom component of G. Similarly, for $G = \Theta$, define x'_σ to be the configuration obtained by moving along a Type II path which moves $x_{\sigma,2}$ towards $f(x_\sigma)$. As before, we can conclude by Proposition 6.9 that $f(x'_\sigma)$ lies on one of the components of $G \setminus x'_\sigma$ in the direction that $x_{\sigma,2}$ moved. We can therefore define $\mathrm{Hor}(\sigma) = +1$ if $f(x'_\sigma) \in (x'_{\sigma,2}, x'_{\sigma,3})$, and $\mathrm{Hor}(\sigma) = -1$ otherwise. Figure 14 shows a particular example of this construction.
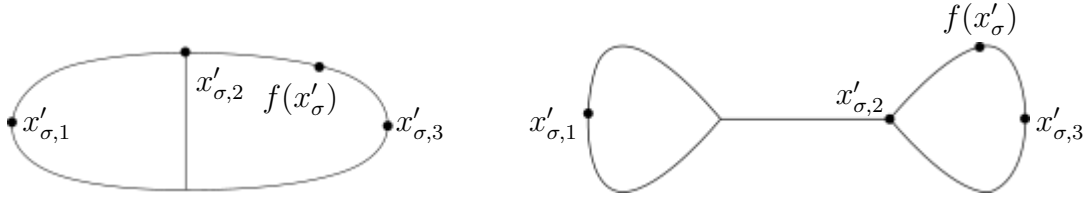

FIGURE 14

Label the edges containing $x_{\sigma,1}$ and $x_{\sigma,3}$ by e_1, e_3 respectively. Note that Ver and Hor associate a distinguished pair to each σ which contains $\sigma(2)$. Explicitly, this association is given by:

$$\sigma \mapsto (\sigma(2), \sigma(2 + \mathrm{Hor}(\sigma))) \text{ on } \mathrm{Ver}(\sigma) \cdot e_{2+\mathrm{Hor}(\sigma)},$$

where e_1, e_3 are equipped with appropriate orientation, and where $\{\pm 1\}$ acts on the oriented edges of G by changing orientation. Our next task is to determine how Ver and Hor interact with the multiplication in Σ_3. By possibly applying a symmetry to G, assume that $\mathrm{Hor}(\mathrm{id}) = \mathrm{Ver}(\mathrm{id}) = +1$. We make the following claims for all $\sigma \in \Sigma_3$:

$$\mathrm{Hor}(\sigma) = +1,$$
$$\mathrm{Ver}((12)\sigma) = \mathrm{Ver}(\sigma),$$
$$\mathrm{Ver}((23)\sigma) = -\mathrm{Ver}(\sigma).$$

If we can prove these claims, then the remainder of the proof is simple, since we can write:

$$\mathrm{Ver}(\mathrm{id}) = \mathrm{Ver}((123)(123)(123)) = \mathrm{Ver}((12)(23)(12)(23)(12)(23)) = -\mathrm{Ver}(\mathrm{id}),$$

a contradiction.

All that remains is to prove the claim. We will prove this only in the case $G = \infty$, but the other cases are not so different.

We will show that $\mathrm{Hor}((12)) = \mathrm{Hor}((23)) = \mathrm{Ver}((12)) = +1$, and $\mathrm{Ver}((23)) = -1$. This will imply the rest of the claim by replacing f with the identifying function f_σ, defined by

$$(x_1, x_2, x_3) \mapsto f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}).$$

Suppose the components of $G \setminus x_{\mathrm{id}}$ are labelled as in Figure 15
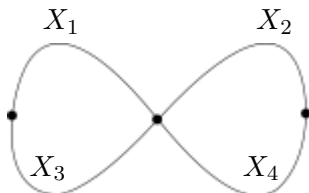

FIGURE 15. The components of $G \setminus x_{\mathrm{id}}$

By assumption, we have $f(x_{\mathrm{id}}) \in X_2$. Consider the path α between x_{id} and $x_{(23)}$ which rotates $x_{\mathrm{id},2}$ and $x_{\mathrm{id},3}$ a half turn clockwise around the loop on the right. By representing α as a concatenation of a Type I and Type II path, we can see using Propositions 6.7 and 6.9 that $f(x_{(23)}) \in X_4$, so $\mathrm{Ver}((23)) = -1$ and $\mathrm{Hor}((23)) = +1$.

Now, consider the path β_1 from x_{id} and $x_{(12)}$ which rotates $x_{\mathrm{id},1}$ and $x_{\mathrm{id},2}$ clockwise a half-turn around the right circle, and the path β_2 which rotates them counter-clockwise a half-turn. By representing these as concatenations of a Type I and a Type II path, we can use Propositions 6.7 and 6.9 to show that $X_2 \not\rightsquigarrow_{\beta_1} X_1$, and $X_2 \not\rightsquigarrow_{\beta_2} X_3$. Finally, Proposition 7.6 shows that $f(x_{(12)}) \notin X_3$, otherwise $(2,3), (3,1)$ would both be distinguished pairs in e_3. Therefore, $f(x_{(12)}) \in X_2$, so $\mathrm{Ver}((12)) = \mathrm{Hor}((12)) = +1$, thereby establishing the claim, and concluding the proof. $\square$

8. A Combinatorial Model

In this section, we discuss $K_n(G)$, a combinatorial approximation to the spaces $\mathrm{Conf}_n(G)$ which was introduced in [Lü14]. These spaces are very useful for constructing sections in the cases when they exist, as we will show that partially defined identifying functions on $K_n(G)$ often extend to $\mathrm{Conf}_n(G)$.

The outline of this section is as follows: we will first define a poset $P_n(G)$, which is defined slightly differently from the definition given in [Lü14], but the two are easily seen to be equivalent. We will define $K_n(G)$ as the cube complex whose face poset is $P_n(G)$, and state without proof the fact that $K_n(G)$ embeds as a deformation retract of $\mathrm{Conf}_n(G)$. We will conclude by discussing which identifying functions on $K_n(G)$ extend to $\mathrm{Conf}_n(G)$.

Let $\mathcal{E}$ be the set of oriented edges of G, and let $\bar{\mathcal{E}} \subset \mathcal{E}$ be the set of positively oriented edges, where we fix some positive orientation on each edge. Let $\mathcal{B}$ be the set of branched vertices of G. For each $e \in \mathcal{E}$, let v_e be the endpoint of e in the direction that e is pointing, and let $\bar{e}$ be the element of $\bar{\mathcal{E}}$ corresponding to e. Recall that an *index* denotes an integer $1, \ldots, n$. First we define the k-dimensional faces of $P_n(G)$.

Definition 8.1. A *k-face* F of $P_n(G)$ associates indices to the elements of $\bar{\mathcal{E}} \sqcup \mathcal{B} \sqcup \mathcal{E}$ as follows:

(i) To each positively oriented edge $\bar{e} \in \bar{\mathcal{E}}$, there is an ordered tuple of associated indices:
$$F(\bar{e}) = (i_{\bar{e},1}, \ldots, i_{\bar{e},\ell}).$$
(ii) To each branched vertex $v \in \mathcal{B}$, there is *at most* one associated index $F(v) = i_v$.
(iii) To *exactly* k oriented edges $e \in \mathcal{E}$ such that $v_e \in \mathcal{B}$, there is an associated index $F(e) = i_e$.
(iv) Each index occurs exactly once as an index associated to an element of $\bar{\mathcal{E}} \sqcup \mathcal{B} \sqcup \mathcal{E}$.
(v) For each vertex $v \in \mathcal{B}$, there is at most one index of the form $F(v)$ or $F(e)$, where $e \in \mathcal{E}$ is an oriented edge with $v_e = v$.

This gives a definition of $P_n(G)$ as a graded set. We now can define the partial ordering.

Definition 8.2. If F is a k-face, and $e \in \mathcal{E}$ is an oriented edge such that $F(e)$ is defined, then we will define two $(k-1)$-faces, F_e^+ and F_e^-, which we will use to define the order on $P_n(G)$. First, we define F_e^+ as follows:
(i) $F_e^+(\bar{e}') = F(\bar{e}')$ for all $\bar{e}' \in \bar{\mathcal{E}}$.
(ii) $F_e^+(v_e) = F(e)$. For $v \in \mathcal{B}$ with $v \neq v_e$, $F_e^+(v)$ exists if and only if $F(v)$ does, and if they are defined, the two are equal.
(iii) $F_e^+(e)$ is undefined, and for $e' \in \mathcal{E}$ with $e' \neq e$, $F_e^+(e')$ is defined if and only if $F(e)$ is, and if they are defined, the two are equal.

Now we define F_e^- as follows:
(i) $F_e^-(\bar{e})$ is equal to $F(\bar{e})$ with $F(e)$ attached to the back when $e = \bar{e}$ (that is, when e has positive orientation), and $F(\bar{e})$ with $F(e)$ attached to the front when $-e = \bar{e}$. For $\bar{e}' \in \bar{\mathcal{E}}$ with $\bar{e}' \neq \bar{e}$, define $F_e^-(\bar{e}') = F(\bar{e}')$.
(ii) For $v \in \mathcal{B}$, $F_e^-(v)$ exists if and only if $F(v)$ does, and when they both exist the two are equal.
(iii) $F_e^-(e)$ is undefined, and for $e' \in \mathcal{E}$ with $e' \neq e$, $F_e^-(e')$ exists if and only if $F(e')$ does, and when they both exist the two are equal.

One can verify that F_e^+ and F_e^- are indeed $(k-1)$-faces of $P_n(G)$. We define the ordering $\succ$ on $P_n(G)$ to be generated by the relations $F \succ F_e^+$ and $F \succ F_e^-$ for each face F and each oriented edge e such that $F(e)$ exists.

The important facts about $P_n(G)$ are summarized in the following theorem, whose statements are proven in [Lü14].

Theorem 8.3. *The graded poset $P_n(G)$ is the face poset of a unique (up to isomorphism) abstract cube complex $K_n(G)$. There is an embedding $i : K_n(G) \hookrightarrow \mathrm{Conf}_n(G)$, and a deformation retract $h_t : \mathrm{Conf}_n(G) \to \mathrm{Conf}_n(G)$ of $\mathrm{Conf}_n(G)$ onto the image of $K_n(G)$ in $\mathrm{Conf}_n(G)$. In addition, the following holds:*
(a) *The embedding i takes vertices of $K_n(G)$ exactly to the configurations where the tokens are spread out evenly on each edge.*
(b) *For any configuration $x \in \mathrm{Conf}_n(G)$, no tokens of x leave or enter any edges or vertices along the path $t \mapsto h_t(x)$. Alternatively, the paths $t \mapsto h_t(x)$ and $t \mapsto h_{1-t}(x)$ are both Type I for every x.*
(c) *Tokens of configurations in $K_n(G)$ are uniformly far from each other. That is, there exists some $\varepsilon > 0$ such that for all $x = (x_1, \ldots, x_n) \in K_n(G)$, we have $d(x_i, x_j) \geq \varepsilon$.*

Corollary 8.4. *If G has k branched vertices, then $\mathrm{Conf}_n(G)$ has the homotopy type of a $\min\{k, n\}$-dimensional cell complex.*

Proof. Indeed, one can check that $K_n(G)$ is $\min\{k, n\}$ dimensional. Details can be found in [Lü14]. $\qquad\square$

Intuitively, we can imagine the vertices of $K_n(G)$ to be the configurations of $\mathrm{Conf}_n(G)$ where the vertices in each edge are equally spaced, as suggested in Theorem 8.3(a). Given such a configuration x, we discuss how to define the associated vertex F of $P_n(G)$. For a positively oriented edge $\bar{e} \in \bar{\mathcal{E}}$, set $F(\bar{e})$ to be the tuple of indices whose corresponding tokens of x lie in $\bar{e}$, ordered consistently with the orientation on $\bar{e}$. For a branched vertex $v \in \mathcal{B}$, set $F(v)$ to be the index of the token of x, if it exists, occupying v.

An edge in $K_n(G)$ corresponds to a continuous move between vertices of $K_n(G)$ which moves a single token from a vertex into the interior of an edge. Similarly, a k-dimensional face corresponds to a set of k moves which can be performed consistently, that is, they involve distinct vertices and tokens.

In our construction of the poset $P_n(G)$, for a fixed face F, the indices associated with positively oriented edges $\bar{e}$ and vertices v correspond to tokens which are essentially fixed on the image of F, and the location of the token corresponding to such an index is determined by which positively oriented edge or vertex it is associated to. Conversely, the indices associated to oriented edges e correspond to tokens which can move from the interior of e to the vertex v_e throughout F. For such an e, the associated face F_e^+ corresponds to the boundary face of F where the moving token on e is instead taken to be stationary at v_e, and F_e^- corresponds to the boundary face where the moving token on e is taken to be stationary on the interior of e.

Example 8.5. Let G be the dumbbell graph pictured in Figure 16, with the given labels and choices of positive orientations, and let $n = 5$.
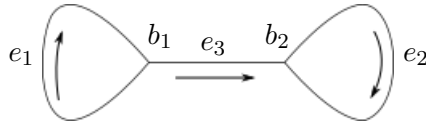

FIGURE 16

A vertex of $K_n(G)$ maps to a configuration like x, pictured in Figure 17, such that the vertices on each edge are evenly spaced apart.
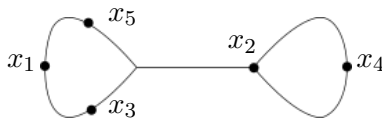

FIGURE 17

Let F_x be the 0-face which maps to x. Explicitly, we can compute:

$$F(\bar{e}_1) = (3, 1, 5),$$
$$F(\bar{e}_2) = (4),$$
$$F(b_2) = 2,$$

and every other value of F is trivial.

We now turn to the question of extending identifying functions defined on $K_n(G)$ to $\mathrm{Conf}_n(G)$.

The important fact is that under certain mild conditions, we can extend any identifying function defined on $K_n(G)$ to all of $\mathrm{Conf}_n(G)$.

Definition 8.6. Suppose $A \subset K_n(G)$ is a subset containing the 0-skeleton of $K_n(G)$, and $f : A \to G$ is a partially defined identifying function. We say that f is *extendable* if the following situation never occurs: Select a free edge e (edge connected to a free vertex) which is connected to a branched vertex v. Then there exists a vertex F of $K_n(G)$ where $F(v)$ exists, $F(\bar{e})$ is empty, and $f(F) \in e$.

Proposition 8.7. *A partially defined identifying function $f : K_n(G) \to G$ extends to an identifying function on $\mathrm{Conf}_n(G)$ if and only if f is extendable.*

Proof. Suppose f is extendable, and let $x = (x_1, \ldots, x_n)$ be a configuration, let $y = (y_1, \ldots, y_n) = h_1(x)$, and let $p = f(y)$. If p is a branched vertex of G, then define $f(x) = p$. Note that $p \neq x_1, \ldots, x_n$, since the path $t \mapsto h_{1-t}$ is Type I by Theorem 8.3(b). If p is not a branched vertex of G, then p lies in some component A of $G \setminus (h_1(x) \cup \mathcal{B})$. Note that A must be either: an open interval (a_1, a_2) or a half-open interval $(a_1, a_2]$, where a_2 is a free vertex of G. Let V be the set of (branched or free) vertices of G. Define a function

$$\psi : \{y_1, \ldots, y_n\} \sqcup V \to G$$

by:

$$\psi(q) = \begin{cases} x_j & q = y_j \\ q & q \in V \end{cases}.$$

We would like to define $f(x)$ to lie in the open interval $(\psi(a_1), \psi(a_2))$, where we take $a_1, a_2 \in \{y_1, \ldots, y_n\}$ if possible.

First we must check that $(\psi(a_1), \psi(a_2))$ is nonempty. If $\psi(a_1) = \psi(a_2)$, then we must have that a_2 is a free vertex of G and $a_1 = y_j$ for some j. Then, there is a path in $\mathrm{Conf}_n(G)$ from y to a vertex F of $K_n(G)$ such that y_j moves directly onto the nearest branched vertex v. By Lemma 6.8 and Proposition 6.9, if f extends, we must have $f(F) \in [a_2, v)$, which contradicts the fact that f is extendable. Therefore $(\psi(a_1), \psi(a_2))$ is a nonempty interval.

We define $f(x)$ to lie between $\psi(a_1), \psi(a_2)$ in the same proportion that $f(y)$ lies between a_1, a_2. This defines f on $\mathrm{Conf}_n(G)$, and it is easily verified to be a continuous identifying function.

Now suppose f is not extendable. Note that the situation in Definition 8.6 contradicts Corollary 6.11, so f does not extend to an identifying function on $\mathrm{Conf}_n(G)$. $\square$

Corollary 8.8. *If G has no free vertices, then every identifying function on $K_n(G)$ extends to $\mathrm{Conf}_n(G)$.*

9. SHARPNESS OF $n \geq 2 - \chi(G)$

In this section we use the methods of Section 8 to show that we cannot make any stronger claims about the existence or nonexistence of sections based only on the homotopy type of a graph, that is, whenever $n < 2 - \chi(G)$ and $\chi(G) < 0$, it is impossible to determine whether G admits an identifying function. First, we prove an existence result.

Proposition 9.1. *Suppose G contains a vertex w, such that $G \setminus w$ contains k components which are connected to w at least twice. Then, if $n \leq k$, G admits an identifying function.*

Proof. First, we consider the graph $H = \bigvee_1^k S_k^1$, a wedge of k circles, where $k \geq n$. Let v be the unique branched vertex, and let $p_1, \ldots, p_k$ be the points such that p_i is the point of S_i^1 antipodal to v. Let S be the set $\{1, 2, \ldots, k\}$, and let $\mathcal{F}$ be the class of functions $\psi : S \to \mathbb{N}$ such that

$$\sum_{s \in S} \psi(s) = n - 1.$$

Let $h : \mathcal{F} \to S$ be a function such that $\psi((h(\psi))) = 0$ for all $\psi \in \mathcal{F}$. Since $k \geq n$, such a function always exists.

Select some small $\varepsilon > 0$, and let $M \subset H^n$ be the subset:

$$M = \{(x_1, \ldots, x_n) : x_i \in H, \text{ there is at most one } i \text{ such that } d(x_i, v) < \varepsilon\}.$$

(note that we do not require the x_i to be distinct). We will define a Σ_n-invariant identifying function on M, that is, a function $f : M \to H$ such that $f(x_1, \ldots, x_n) \neq x_i$ for all $x = (x_1, \ldots, x_n) \in M$, and $i = 1, \ldots, n$ such that $f(\sigma x) = f(x)$, where Σ_n acts on M by permuting indices. Let $N \subset H$ be the ε-neighborhood of v, and let $A = (G \setminus N)^n$ and $B = N \times (G \setminus N)^{n-1}$. Note that A and B are disjoint, and every element of M is in the Σ_n-orbit of an element in A or B (but not both), so it suffices to define f on $A \cup B$. On A, we simply define f to be constantly equal to v. On B, we will define f separately on each component X of B. For each X, we can define a function $\psi_X : S \to \mathbb{N}$ by setting $\psi_X(i)$ to be the number of tokens of an element of X contained in $S_i^1 \setminus N$. Clearly, $\psi_X \in \mathcal{F}$. Let $\pi : B \to N$ be the projection of B onto N. We can define a map $\lambda : \bar{N} \to S^1_{h(\psi_X)}$ such that:

(a) $\lambda(v) = p_{h(\psi_X)}$
(b) For each free vertex u of $\bar{N}$, $\lambda(u) = v$
(c) λ has no fixed points, that is, $\lambda(q) \neq q$ for all $q \in N \cap S^1_{h(\psi_X)}$.

By the definition of h, $\psi_X(h(\psi_X)) = 0$, so $S_{h(\psi_X)}$ contains no tokens of any element of X by the definition of ψ_X. Therefore, such a map exists. We define $f = \lambda \circ \pi$ on X. This is an identifying function on X, and by applying this process to all components of B, we get an identifying function on B. Further, by taking f to be Σ_n invariant, we get an identifying function on M, as desired, by noting that the local definitions of f glue continuously.

Now, we turn to the task of defining an identifying function on G. By our assumptions on w, there exists an embedding $i : H \hookrightarrow G$ such that $i(v) = w$, and a retract $r : G \to H$ such that $r^{-1}(w) = \{v\}$. Note that r induces a map $R : K_n(G) \to H^n$, by applying r to each token of a configuration. By Theorem 8.3(c), there is some ε such that at most one token of any configuration of $K_n(G)$ is within ε of w. Taking sufficiently small ε, we can see that this implies that $R(K_n(G)) \subset M$. Then, let $g = i \circ f \circ R$, where f is as constructed above. For any $x \in K_n(G)$, note that none of the tokens of x lie in the image of the circle which contains $g(x)$, so g is a partially defined identifying function. Further, the image of H in G intersects no free edges, so g is extendable, and by Proposition 8.7, we conclude. $\qquad\square$

Now we present a class of examples with $n < 2 - \chi(G)$ where sections do not exist.

Example 9.2 (Wedge of balloons)**.** Let B be the balloon graph, as pictured in Figure 18

Suppose $G = \bigvee_1^k B$ is a wedge of balloons, glued at the free vertices of each balloon, and suppose that $n \leq k$. If $n \geq 2$, then statement (A) holds, and if $n \geq 3$, statement (B) holds, and the proofs of these are quite similar to the proof of Theorem 3.1. The key observation is that whenever we have a configuration x with $f(x)$ in a copy of B and a token (say x_1)
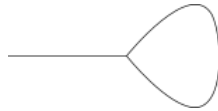
FIGURE 18

as depicted in Figure 19, there always exists a distinguished pair $(1, j)$ in the circle in that copy of B (in fact for any $j \neq 1$, possibly after changing orientation).
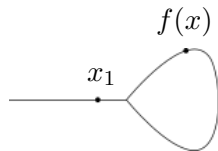

FIGURE 19

To see why, note that we can slide any token x_j into B, and continue through the path depicted by Figure 20.
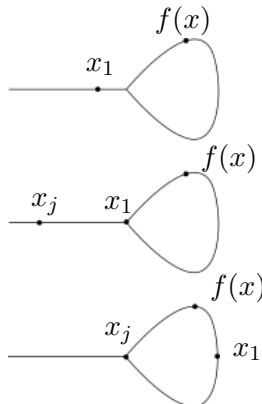

FIGURE 20

To prove (A) and (B), note that we can avoid placing tokens in any balloons we would like, and copy the proof of Theorem 3.1, and the fact above when necessary.

Therefore we have shown that when $n < 2 - \chi(G)$ (and $n \geq 4$), we cannot say for sure whether or not G admits a section.

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