1 Introduction

The goal of this project is increase accessibility of certain topics within the field of algebraic topology to a wider audience of undergraduate mathematicians.

The first sections of this work define the de Rham complex and de Rham cohomology on $\mathbb{R}^n$ and smooth manifolds. The bulk work is giving and using techniques for calculating de Rham cohomology of a variety of smooth manifolds, including homotopies, Mayer-Vietoris, and Poincaré Duality. We summarize our calculations in the tables below. We denote a torus with genus g as $\mathbb{T}_g$, and an n-dimensional torus as T^n:

Manifold	H^0	H^i	H^n	H_c^0	H_c^i	H_c^n
$\mathbb{R}^n$	$\mathbb{R}$	0	0	0	0	$\mathbb{R}$
S^n	$\mathbb{R}$	0	$\mathbb{R}$	$\mathbb{R}$	0	$\mathbb{R}$
$\mathbb{RP}^n$ n even	$\mathbb{R}$	0	0	$\mathbb{R}$	0	0
$\mathbb{RP}^n$ n odd	$\mathbb{R}$	0	$\mathbb{R}$	$\mathbb{R}$	0	$\mathbb{R}$
$\mathbb{T}^n$	$\mathbb{R}$	$\mathbb{R}^{\binom{n}{i}}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}^{\binom{n}{i}}$	$\mathbb{R}$

Manifold	H^0	H^1	H^2	H_c^0	H_c^1	H_c^2
$\mathbb{T} - \{*\}$	$\mathbb{R}$	$\mathbb{R}^2$	0	0	$\mathbb{R}^2$	$\mathbb{R}$
$\mathbb{T}_g$	$\mathbb{R}$	$\mathbb{R}^{2g}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}^{2g}$	$\mathbb{R}$
Möbius Strip	$\mathbb{R}$	$\mathbb{R}$	0	0	0	0
Klein Bottle	$\mathbb{R}$	$\mathbb{R}^2$	0	$\mathbb{R}$	$\mathbb{R}^2$	0

The remainder of the written work is proof of the Lefschetz fixed-point theorem for de Rham cohomology, as well as a couple examples of its application. It is my hope that any math undergraduate is able to read and digest the content here, so to be inspired to continue learning algebraic topology.

2 The de Rham Complex and de Rham Cohomology

Our goal is to study the topological properties of any C^∞ manifold by constructing algebraic structures and examining their algebraic structures. In general, a good way to do this is to look at the set of functions coming out of object. Additionally, if you pick the right target, you can give the set of all functions more structure. Here, we can use the structure of our manifold M to look at the set of all C^∞ functions $M \to \mathbb{R}$. First, we restrict our view to subsets of $\mathbb{R}^n$.

2.1 The de Rham Complex on $\mathbb{R}^n$

Fix U an open subset of $\mathbb{R}^n$. We know what it means for functions $U \to \mathbb{R}$ to be C^∞. Since their target is a ring, the set

$$\{C^\infty \text{ functions } U \to \mathbb{R}\}$$

is an $\mathbb{R}$-algebra. We now define another $\mathbb{R}$-algebra, Ω^*, which we use to construct the de Rham complex.

Definition 2.1.1. We define Ω_n^* to be the generated as an $\mathbb{R}$ module by the symbols dx_i for $i \in \{1, \dots, n\}$ subject only to the condition that $dx_i dx_j = -dx_j dx_i$ for all i, j. Note that this implies $(dx_i)^2 = dx_i dx_i = -dx_i dx_i = 0$ for all i. Since it is an $\mathbb{R}$ module, we have formal sums and scaling by real numbers. This means we have a basis of Ω^* as a vector space over $\mathbb{R}$:

$$\big\{1, dx_1, \dots, dx_n, (dx_1 dx_2), \dots, (dx_{n-1} dx_n), \dots, (dx_{i_1} \cdots dx_{i_j} \cdots dx_{i_k}), \dots, (dx_1 \cdots dx_n)\big\}$$

such that $i_j < i_k$ for all $j < k$.

Definition 2.1.2 (The de Rham complex over U)**.** For U open subset of $\mathbb{R}^n$:

$$\Omega^*(U) := \Omega_n^* \otimes_{\mathbb{R}} \{C^\infty \text{ functions } U \to \mathbb{R}\}$$

We say $\Omega^*(U)$ is the de Rham complex over U.

We often use ω to represent an element of $\Omega^*(U)$. We can uniquely represent $\omega = \sum_I f_I\, dx_I$, where the multi-index $dx_I = dx_{i_1} \ldots dx_{i_k}$. In this case we say the degree of ω is k. We see there is a natural grading on $\Omega^*(U) = \bigoplus_i \Omega^i(U)$, where $\Omega^k(U)$ is the subset of differential forms with degree k. While is possible to have a differential form that is the sum of elements of different degrees, we will not consider them, as we have no use for them. The differential is given by the map

$$d : \Omega^n(U) \to \Omega^{n+1}(U)$$

defined such that for $f \in \Omega^0(U)$ and $\omega \in \Omega^n(U)$ such that $\omega = \sum_I f_I\, dx_I$,

$$df = \sum_i \frac{\partial f}{\partial x_i}\, dx_i \qquad\qquad d\omega = \sum_I df_I\, dx_I$$

Example 2.1.3. In this first, example, we let $\omega = xyz$ and calculate $d\omega$:

$$d\omega = yz\, dx\, dy + xz\, dy\, dy + xy\, dz\, dx = yz\, dx\, dy - xy\, dx\, dz$$

We repeat, now with $\omega = e^{xy}\, dz$. We see:

$$d\omega = ye^{xy}\, dx\, dz + xe^{xy}\, dy\, dz$$

Proposition 2.1.4. We claim there is a simple equation for $d(\omega\eta)$. We claim

$$d(\omega\eta) = (d\omega)\eta + (-1)^{\deg \omega}\omega(d\eta)$$

Proof. We suppose that $\omega = f\, dx_I$ and $\eta = g\, dx_J$ for some multi-indices I, J. We see that

$$\begin{aligned}
d(\omega\eta) &= d\left(f g\, dx_I\, dx_J\right) \\
&= \left(\sum_i \frac{\partial f g}{\partial x_i}\, dx_i\right) dx_I\, dx_J \\
&= \left(\sum_i \left(g\frac{\partial f}{\partial x_i} + f\frac{\partial g}{\partial x_i}\right)\, dx_i\right) dx_I\, dx_J \\
&= \sum_i g\frac{\partial f}{\partial x_i}\, dx_i\, dx_I\, dx_J + \sum_i f\frac{\partial g}{\partial x_i}\, dx_i\, dx_I\, dx_J \\
&= \sum_i \frac{\partial f}{\partial x_i}\, dx_i\, dx_I\, g\, dx_J + (-1)^{\deg \omega} f\, dx_I \left(\sum_i \frac{\partial g}{\partial x_i}\, dx_i\, dx_J\right) \\
&= (d\omega)\eta + (-1)^{\deg \omega}\omega(d\eta)
\end{aligned}$$

□

Definition 2.1.5 (Exact and Closed Forms)**.** A form $\omega \in \Omega^*(U)$ is *closed* if $d\omega = 0$. A form ω is *exact* if there exists another form η such that $d\eta = \omega$.

Proposition 2.1.6 ($d^2 = 0$)**.** This is equivalent to saying all exact forms are closed. First we check for

0-forms. Let f be a smooth function. Then we see that:

$$\begin{aligned}
d^2 f &= d\left(\sum_i \frac{\partial f}{\partial x_i}\, dx_i\right)\\
&= \sum_i \left(\sum_j \frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_i}\, dx_j\right) dx_i\\
&= \sum_{i,j} \frac{\partial^2 f}{\partial x_j \partial x_i}\, dx_j\, dx_i\\
&= \sum_{i>j} \frac{\partial^2 f}{\partial x_j \partial x_i}\, dx_j\, dx_i + \sum_{i<j} \frac{\partial^2 f}{\partial x_j \partial x_i}\, dx_j\, dx_i + \sum_i \frac{\partial^2 f}{\partial x_i \partial x_i}\, dx_i\, dx_i\\
&= \sum_{i>j} \frac{\partial^2 f}{\partial x_j \partial x_i}\, dx_j\, dx_i - \sum_{i<j} \frac{\partial^2 f}{\partial x_j \partial x_i}\, dx_i\, dx_j + 0\\
&= \sum_{i>j} \frac{\partial^2 f}{\partial x_j \partial x_i}\, dx_j\, dx_i - \sum_{i>j} \frac{\partial^2 f}{\partial x_i \partial x_j}\, dx_j\, dx_i\\
&= \sum_{i>j} \left(\frac{\partial^2 f}{\partial x_j \partial x_i} - \frac{\partial^2 f}{\partial x_i \partial x_j}\right) dx_j\, dx_i\\
&= 0
\end{aligned}$$

With the last equality coming from the fact that mixed partial derivatives are equal when f is smooth. Take an arbitrary form $\eta = f\, dx_I$ for some multi-index I. We then calculate:

$$d^2\eta = d\left(\sum_i df\, dx_I\right) = \sum_i \left(d^2 f\, dx_I + (-1) df\, d^2 x_I\right) = 0$$

Definition 2.1.7 (de Rham Cohomology)**.** We know that every exact form is closed, but is every closed form exact? This is a important question, and by studying it, we will learn important topological properties. To measure which closed forms are not exact, we define de Rham cohomology to be

$$H^q_{dR}(U) = \{\text{closed } q\text{-forms}\} \,/\, \{\text{exact } q\text{-forms}\}$$

We often drop the dR notation when the meaning is clear. We know the exact forms are all closed, and it is clear that scaling any form by a real number does not change exactness. Moreover, by linearity of d, the sum of exact forms is exact, and the sum of closed forms is closed. Therefore, $H^q(U)$ has an abelian group structure, and can be scaled by elements in $\mathbb{R}$, thus it is a vector space over $\mathbb{R}$.

Proposition 2.1.8. Let ω be a closed form, and $d\eta$ be an exact form. We claim that $\omega d\eta$ is an exact form.

Proof. Then we construct:

$$(-1)^{\deg\omega} d(\omega\,\eta) = (-1)^{\deg\omega}\left(d\omega\,\eta + (-1)^{\deg\omega}\omega\, d\eta\right) = \omega\, d\eta$$

Therefore, $\omega\, d\eta$ is exact. □

Example 2.1.9 ($U = \mathbb{R}$)**.** First, we recall $\Omega^i(\mathbb{R}) \cong 0$ for $i > 1$ since there are no forms with degree higher than the dimension of $\mathbb{R}$. We see that the kernel of d in $\Omega^0(\mathbb{R})$ is exactly the constant functions. Thus,

$$H^0(\mathbb{R}) = \mathbb{R}$$

Every form in $\Omega^1(\mathbb{R})$ is closed, and we can find a preimage of any form $\omega = f dx$ by setting

$$\eta = \int_0^x f(t)\, dt$$

Then $d\eta = \omega$, so every closed form is exact. Therefore:

$$H^1(\mathbb{R}) = 0$$

Example 2.1.10 (U is n disjoint open intervals of $\mathbb{R}$)**.** We can apply the same logic as before to get

$$H^1(U) = 0$$

However, the closed 0-forms are the *locally* constant functions on U, which is equivalent to choosing n constant functions, one for each interval. Therefore,

$$H^0(U) = \mathbb{R}^n$$

Indeed, you should consider H^0 as a way to tell us the number of connected components of our space.

2.2 de Rham Complex with Compact Support

Definition 2.2.1 (Compact Support)**.** The *support* of a function $f : X \to \mathbb{R}$ is the closure of the set where f is not zero:

$$\operatorname{Supp}(f) = \overline{\{x \in X : f(x) \neq 0\}}$$

We say a form $\omega = \sum_I f_I d_I$ has *compact support* if $\bigcup_I \operatorname{Supp}(f_I)$ is compact.

We can similarly define the compactly supported de Rham complex and compactly supported de Rham cohomology:

$$\Omega^*_c(U) = \Omega^* \otimes_{\mathbb{R}} \{C^\infty \text{ functions with compact support } U \to \mathbb{R}\}$$

We denote the compactly supported cohomology by $H^q_c(U)$.

Example 2.2.2 ($U = \mathbb{R}$)**.** Unlike before, the constant functions are not closed 0-forms, since they are not compactly supported. Thus,

$$H^0_c(\mathbb{R}) = 0$$

Now, we try to find the exact compactly generated 1 forms. Note that integrating over all of $\mathbb{R}$ gives a map

$$\int_{\mathbb{R}} : \Omega^1_c(\mathbb{R}) \to \mathbb{R} \qquad\qquad \int_{\mathbb{R}} : f(x)\, dx \mapsto \int_{\mathbb{R}} f(x)\, dx$$

Note that if $f\, dx$ is exact, or equivalently $f(x)\, dx = dg(x) = \frac{dg(x)}{dx}\, dx$, then we know that

$$\int_{\mathbb{R}} \frac{dg(x)}{dx}\, dx = \lim_{a\to-\infty} \lim_{b\to\infty} g(b) - g(a) = 0$$

Thus, we have that $\operatorname{im} d \subset \ker \int_{\mathbb{R}}$. Now we suppose that for some compactly supported 1-form $f\, dx$ we know that $\int_{\mathbb{R}} f dx = 0$. Then we can define a 0-form:

$$g(x) = \int_{-\infty}^{x} f(t)\, dt$$

which we know must be compactly supported. Note that if $\int_{\mathbb{R}} f\, dx \neq 0$, such a form would not be compactly supported. Then, clearly $dg = f\, dx$. Therefore, $\ker \int_{\mathbb{R}} \subset \operatorname{im} d$ and they are equal. Then,

$$H^1_c(\mathbb{R}) = \frac{\Omega^1_c(\mathbb{R})}{\ker \int_{\mathbb{R}}} \cong \mathbb{R}$$

We know the last isomorphism since the any group homomorphism defines an isomorphism from the domain modulo the kernel to the image.

This is a surprising result, the importance of which should not be overlooked.

2.3 Properties of $\Omega^*(-)$

For those familiar with category theory, we point out that $\Omega^*(-)$ is a contravariant functor on the category of smooth open subspaces of Euclidean space. For those unfamiliar with category theory, all we are saying is that if $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are both open, with a smooth function $f : U \to V$, there exists an $\mathbb{R}$-linear homomorphism $f^* : \Omega^*(V) \to \Omega^*(U)$.

For an arbitrary zero form $g \in \Omega^0(V)$, we have $f^*(g) = g \circ f \in \Omega^0(U)$. We wish to extend this property further, but we wish to do so in a manner to ensure that $d \circ f^* = f^* \circ d$. Let x_i be the usual coordinates of R^n:

$$f^* \left(\sum_I g_I \, dx_I \right) = \sum_I (g_I \circ f) \, df_I$$

where f_i is the i-th component of f.

Proposition 2.3.1. We claim that $d \circ f^* = f^* \circ d$.

Proof. Since f^*, d are both linear, it suffices to check on $g_I \, dx_I$:

$$\begin{aligned} d(g_I \, dx_I) &= \sum_i \frac{\partial g_I}{\partial x_i} \, dx_i \, dx_I \\ f^*(d(g_I \, dx_I)) &= \sum_i \frac{\partial g_I}{\partial x_i} \circ f \, df_i \, df_I \\ f^*(g_I \, dx_I) &= g_I \circ f \, df_I \\ d(f^*(g_I \, dx_I)) &= \sum_i \frac{\partial}{\partial x_i} (g_I \circ f) \, dx_i \, df_I \end{aligned}$$

By chain rule we can see that $\frac{\partial}{\partial x_i} (g_I \circ f) \, dx_i = \frac{\partial g_I}{\partial x_i} \circ f \, df_i$, so we have shown that $d \circ f^* = f^* \circ d$. $\square$

2.4 de Rham Complex on Manifolds

We now have enough to define differential forms on a manifold. First, we recall the definition of a manifold.

Definition 2.4.1 (Manifold)**.** Intuitively, we wish for a manifold to look like $\mathbb{R}^n$, classical Euclidean space. Indeed, this would allow us to do calculus locally, and by piece it together in the appropriate way, do calculus on a topological space. A manifold M of dimension n is a topological space that comes equipped with an atlas: a (countable) cover of open sets $(U_\alpha)_A$ with homeomorphisms $\phi_\alpha : U_\alpha \to \mathbb{R}^n$. Moreover, we will restrict our view to smooth manifolds, which requires that the transition functions:

$$g_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha \left(U_\alpha \cap U_\beta \right) \to \phi_\beta (U_\alpha \cap U_\beta)$$

be smooth for every α, β. (We also require M to be second countable as a topological space, so that there exist a countable basis of open sets. This technical requirement will not be explicitly used in any proofs in this paper.)

Definition 2.4.2 (A good cover)**.** A good cover of a manifold M is a collection of open sets $\{U_i\}$ such that every finite intersection of U_i is diffeomorphic to $\mathbb{R}^n$. It has been shown that every manifold has good cover [BT95, Theorem 5.1].

Definition 2.4.3 (A differential form on M)**.** A differential form ω on M is a collection of differential forms $\omega_\alpha \in \Omega^* \left(U_\alpha \right)$ with the requirement that for the inclusions

$$\begin{tikzcd} & U_\alpha \cap U_b \arrow[dl, "i_\alpha"'] \arrow[dr, "i_\beta"] & \\ U_\alpha & & U_\beta \end{tikzcd}$$

we have that $i_\alpha^* \omega_a = i_\beta^* \omega_b \in \Omega^* \left(U_\alpha \cap U_\beta \right)$.

Example 2.4.4 ($H^0(M)$)**.** Since any 0-form is defined locally, we see that any class in $H^0(M)$ is made from local classes. Since no 0-form is exact, we just have to find the closed forms. If a collection of forms is closed, each must individually be closed. Thus, each local form must be locally constant. Since local forms must agree on intersections, a locally constant 0-form is determined exactly by the connected components. Therefore, we see that for a manifold with n connected components:

$$H^0(M) = \mathbb{R}^n$$

Example 2.4.5 (The punctured plane)**.** We let $X = \mathbb{R}^2 - \{0\}$, which we call the punctured plane. Since X is an open subset of a manifold, it is a smooth manifold. Therefore, by the argument above:

$$H^0(X) = \mathbb{R}$$

We now claim that $H^1(X) \cong \mathbb{R}$. Let us first construct two 1-forms on X. We know we have polar coordinates on X using $r(x, y) = \sqrt{x^2 + y^2}$, $\theta(x, y) = \arctan(y/x)$, although $\theta(x, y)$ cannot be defined on all of X at once. Indeed, we see that we could choose to give the point $(1, 0)$ any of the angles $0, 2\pi, 4\pi, \cdots \in 2\pi\mathbb{Z}$. Notice how the jump in angle is always a multiple of 2π, which is always a constant. This is motivation for the argument that $d\theta$ should be well defined, since $d(c) = 0$ for any constant c. Indeed, if we define:

$$\begin{aligned}
d\theta &= \frac{\partial}{\partial x} \arctan\left(\frac{y}{x}\right) dx + \frac{\partial}{\partial x} \arctan\left(\frac{y}{x}\right) dy \\
&= \frac{1}{1 + \left(\frac{y}{x}\right)^2} \frac{-y}{x^2} dx + \frac{1}{1 + \left(\frac{y}{x}\right)^2} \frac{1}{x} dy \\
&= \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy \\
&= \frac{x\, dy - y\, dx}{x^2 + y^2}
\end{aligned}$$

We can similarly calculate for the radius $r = \sqrt{x^2 + y^2}$:

$$dr = \frac{x\, dx + y\, dy}{\sqrt{x^2 + y^2}}$$

We can check that dr and $d\theta$ are linearly independent by calculating the determinant of the linear map:

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

Therefore, the determinant of this matrix is $r^2(\sin^2\theta + \cos^2\theta) = r^2$. Since $r^2 > 0$ for all points in X, we see that this is always invertible, thus dr and $d\theta$ are linearly independent. Therefore, any 1-form ω on X can be written:

$$\omega = f\, dr + g\, d\theta$$

We know dx and dr are both exact, and therefore closed. We observe $dx = \cos\theta\, dr - r\sin\theta\, d\theta$, so $d\theta$ must also be closed. We also see that $f(r, \theta)\, dr$ is an exact form since:

$$d\int_1^\infty f(t, \theta)\, dt = f(r, \theta)\, dr$$

Therefore, we know that the cohomology class $[\omega] = [g\, d\theta]$. If ω is closed, then $\frac{\partial g}{\partial r} = 0$. Suppose that $\frac{\partial g}{\partial \theta} \neq 0$. We define a one form:

$$\eta = \int_0^\theta g(r, t)\, dt$$

Then, we see that $d\eta = \omega$ and ω is exact, unless g is constant. We see that $d\theta$ is not exact, since

$$\int_0^{2\pi} d\theta = 2\pi \neq 0$$

Thus, the closed 1-forms that are not exact can be written $c\,d\theta$ for $c \in \mathbb{R}$, so $H^1(X) \cong \mathbb{R}$.

We can see that any closed two form can be written $\omega = f(r,\theta)\,drd\theta$. Then we can define

$$\eta = \left(\int_1^r f(t,\theta)\,dt\right) d\theta$$

Then we can see that $d\eta = \omega$, so all closed two forms are exact. Therefore,

$$H^0(X) \cong \mathbb{R} \qquad H^1(X) \cong \mathbb{R} \qquad H^2(X) \cong 0$$

Then we see that $d\theta \in H^1(X)$, so $H^1(X) \neq 0$. Showing that $d\theta$ generates all of $H^1(X)$ is more difficult, and we will show that the dimension of $H^1(X)$ is exactly 1 in a later example.

3 Integration

3.1 Orientation

Recall Riemannian integration:

$$\int_{\mathbb{R}^n} f\,|dx_1 \cdots dx_n| = \lim_{\Delta x_i \to 0} \sum_{x \in \mathbb{Z}^n} f\left(\frac{x_1}{\Delta x_1}, \ldots, \frac{x_n}{\Delta x_n}\right) \Delta x_1 \cdots \Delta x_n$$

We adopt the notation of surrounding the dx_i with absolute value to emphasize that the order of them does not matter. We define the integral of an n-form $\omega = f dx_1 \cdots dx_n$ on $\mathbb{R}^n$ to be

$$\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} f\,|dx_1 \cdots dx_n|$$

Note that now the integration does depend on the order of the dx_i. Indeed, there is a possible sign change. This sign is determined by the orientation of the manifold. Note that integration depends on the coordinates. Suppose we change coordinates, by some diffeomorphism $A : \mathbb{R}^n \to \mathbb{R}^n$ with coordinates x_i, y_i respectively. Then, without too much work, it is possible to show that

$$dy_1 \cdots dy_n = J(A) dx_1 \cdots dx_n$$

where $J(A) = \det\left(\frac{\partial y_i}{\partial x_j}\right)$. This is the determinant of the Jacobi matrix. Similarly, we have that

$$|dy_1 \cdots dy_n| = |J(A)||dx_1 \cdots dx_n|$$

by the change of variables formula. Thus, we have

$$\int_{\mathbb{R}^n} A^*\omega = \text{sign}\,\big(J(A)\big) \int_{\mathbb{R}^n} \omega$$

We say a diffeomorphism A is orientation preserving if sign $J(A) = 1$. For a general manifold (M, ϕ_α), the atlas is *oriented* if all the transition functions $g_{\alpha,\beta}$ are orientation preserving, and the whole manifold is *orientable* if there exists an oriented atlas.

Proposition 3.1.1. A manifold is orientable if and only if there is a nowhere vanishing n-form.

Proof. Any two nowhere vanishing n-forms ω, ω' satisfy the equation $f\omega = \omega'$ for some nowhere vanishing C^∞ function f. Either f is entirely positive or entirely negative. We can make two equivalence classes, where two nowhere vanishing n-forms are equivalent if f is a positive function. These equivalence classes are called orientations on M. The usual orientation on $\mathbb{R}^n$ is $dx_1 \cdots dx_n$. □

3.2 Partitions of Unity

Definition 3.2.1 (Partition of unity)**.** A partition of unity on a manifold is a collection of smooth non-negative functions $\{\rho_\alpha\}_{\alpha \in A}$ such that any $x \in M$ has an open neighborhood where the following sum has only finitely many nonzero terms and is

$$\sum_i \rho_\alpha(x) = 1$$

Proposition 3.2.2. For any open cover $\{U_\alpha\}_{\alpha \in A}$, there exists a partition of unity *subordinate* to the cover. This means that the partition of unity is indexed by the same set, and $\mathrm{Supp}(\rho_\alpha) \in U_\alpha$ for all $\alpha \in A$ [War71, p. 10].

3.3 Stokes' Theorem

Given an orientation $[M]$ and a compactly supported form $\omega \in \Omega_c^n(M)$ we define integration by

$$\int_{[M]} \omega = \sum_\alpha \int_{U_\alpha} \rho_\alpha \omega$$

where, for the trivialization ϕ_α,

$$\int_{U_\alpha} \rho_\alpha \omega = \int_{\mathbb{R}^n} \left(\phi_\alpha^{-1}\right)^* (\rho_\alpha \omega)$$

Proposition 3.3.1. The integral does not depend on choice of atlas or partition of unity [BT95, Proposition 3.3].

Definition 3.3.2 (Manifold with Boundary)**.** A manifold with boundary (with dimension n) is a topological space with an atlas $\{U_\alpha, \phi_\alpha\}$ where U_α is homeomorphic to either $\mathbb{R}^n$ or the upper half plane $\mathbb{H}^n$ by a homeomorphism ϕ_α.

The boundary of M, ∂M, is a submanifold with dimension $n-1$.

Proposition 3.3.3. An orientation on M determines an orientation on ∂M [BT95, Lemma 3.4].

Theorem 3.3.4 (Stokes' Theorem)**.** If ω is a compactly supported $(n-1)$-form with compact support on an oriented manifold M with dimension n, and ∂M is given by the induced orientation, then

$$\int_M d\omega = \int_{\partial M} \omega$$

Importantly, we see that for any manifold without boundary, the integral of any exact form is 0. Therefore, since integration is linear, integration is well defined on cohomology classes.

3.4 Homotopic Manifolds

Theorem 3.4.1. We wish to show that

$$H^*(M \times \mathbb{R}) \cong H^*(M)$$

Proof. We let $s : M \to M \times \mathbb{R}$ be the function $x \mapsto (x, 0)$. We call s the zero-section. We also define the projection $\pi : M \times \mathbb{R} \to M$ which acts as $(x, a) \mapsto x$. We can see that $\pi \circ s = \mathrm{id}_M$ since $x \mapsto (x, 0) \mapsto x$. However $s \circ \pi$ is not the identity on $M \times \mathbb{R}$. We wish to show that $\pi^* \circ s^*$ is the identity on cohomology. All we need to check is for any closed form $\mathrm{id} - \pi^* \circ s^*$ is an exact form. One way to show this is to construct a linear map called a *homotopy operator* $K : \Omega^*(M \times \mathbb{R}) \to \Omega^{*-1}(M \times \mathbb{R})$ such that

$$\mathrm{id} - \pi^* \circ s^* = \pm (dK \pm Kd)$$

We can see that the right hand side of the equation sends closed forms to exact forms. Given a form ω on M, we can classify forms on $\Omega^*(M \times \mathbb{R})$ in two types:

$$(\pi^* \omega)\, h(x, y) \qquad\qquad (\pi^* \omega)\, h(x, y) dy$$

Next we define K in the following way:

$$K\left(\left(\pi^{*}\omega\right)h(x,y)\right)=0 \qquad\qquad K\left(\left(\pi^{*}\omega\right)h(x,y)dy\right)=\int_0^y h(x,t)dt$$

Then we check the identity on a degree m form

$$\left(\mathrm{id}\,-\pi^{*}\circ s^{*}\right)\left(\pi^{*}\,\omega\right)h(x,y)=\left(\pi^{*}\,\omega\right)h(x,y)-\left(\pi^{*}\,\omega\right)h(x,0)$$

And the other side:

$$\begin{aligned}
(Kd-dK)\left(\pi^{*}\,\omega\right)h(x,y)&=Kd\left(\pi^{*}\,\omega\right)h(x,y)\\
&=K\left(d\pi^{*}\omega+(-1)^{m}\pi^{*}\omega\sum\frac{\partial h}{\partial x_i}\,dx+(-1)^{m}\pi^{*}\omega\,\frac{\partial h}{\partial y}\,dy\right)\\
&=(-1)^{m}K\left(\pi^{*}\omega\,\frac{\partial h}{\partial y}\,dy\right)\\
&=(-1)^{m}\pi^{*}\omega\,\int_0^y\frac{\partial h}{\partial y}\,dy\\
&=(-1)^{m}\pi^{*}\omega\,\left(h(x,y)-h(x,0)\right)\,dy
\end{aligned}$$

Therefore, we have $\mathrm{id}\,-\pi^{*}\circ s^{*}=(-1)^{m}\left(Kd-dK\right)$ of the first type of form. Then we check the other type of form. Recall that $f^{*}d=df^{*}$ by construction. Thus, $s^{*}dy=ds^{*}y=0$. Therefore:

$$\left(1-\pi^{*}\circ s^{*}\right)\left(\left(\pi^{*}\omega\right)h(x,y)\,dy\right)=\left(\pi^{*}\omega\right)h(x,y)\,dy$$

Next we check

$$\begin{aligned}
Kd\big(\left(\pi^{*}\omega\right)h(x,y)\,dy\big)&=K\left(\left(d\pi^{*}\omega\right)h(x,y)\,dy+(-1)^{m-1}\left(\pi^{*}\omega\right)\sum\frac{\partial h}{\partial x_i}\,dx_i\,dy\right)\\
&=\left(d\pi^{*}\omega\right)\int_0^y h(x,t)\,dt+(-1)^{m-1}\left(\pi^{*}\omega\right)\sum\int_0^y\frac{\partial h(x,t)}{\partial x_i}\,dt\,dx_i\\
dK\big(\left(\pi^{*}\omega\right)h(x,y)\,dy\big)&=d\left(\left(\pi^{*}\omega\right)\,\int_0^y h(x,t)\,dt\right)\\
&=\left(d\pi^{*}\omega\right)\,\int_0^y h(x,t)\,dt+(-1)^{m-1}\left(\pi^{*}\omega\right)\left(\sum\int_0^y\frac{\partial h(x,t)}{\partial dx_i}\,dt\,dx_i+h(x,y)\,dy\right)\\
(Kd-dK)\big(\left(\pi^{*}\omega\right)h(x,y)\,dy\big)&=(-1)^{m}\left(\pi^{*}\omega\right)h(x,y)dy
\end{aligned}$$

Thus we have for both forms that

$$\mathrm{id}\,-\pi^{*}\circ s^{*}=(-1)^{m}\left(Kd-dK\right)$$

Therefore, s^{*} and π^{*} are isomorphisms on homology. Note that we could have chosen any section s_n which sends $x\mapsto(x,n)$. We have now proven that

$$H^{*}(M)\cong H^{*}(M\times\mathbb{R})$$

□

Example 3.4.2 ($H^{*}(\mathbb{R}^{n})$)**.** We calculated already that

$$H^{1}(\mathbb{R})\cong\mathbb{R} \qquad\qquad H^{i\neq 0}(\mathbb{R})\cong 0$$

By using the proposition above, we can see inductively that $H^{*}(\mathbb{R}^{n})\cong H^{*}(\mathbb{R}^{n-1})\cong H^{*}(\mathbb{R})$. Thus,

$$H^{1}(\mathbb{R}^{n})\cong\mathbb{R} \qquad\qquad H^{i\neq 0}(\mathbb{R}^{n})\cong 0$$

Example 3.4.3 ($H^*(S^1)$)**.** We see that the punctured plane is diffeomorphic to a cylinder $S^1 \times \mathbb{R}$ via polar coordinates. Therefore, $H^*(S^1) \cong H^*(\mathbb{R}^2 - \{(0,0)\})$. Therefore:

$$H^0(S^1) \cong \mathbb{R} \qquad\qquad H^1(S^1) \cong \mathbb{R}$$

Since S^1 is compact, we have $H^*(S^1) \cong H^*_c(S^1)$.

Example 3.4.4 (Solid Torus)**.** We see that the solid torus is exactly $S^1 \times \mathbb{R}^2$. Therefore, the solid torus $\mathbb{T}_s$ has cohomology:

$$H^0\left(\mathbb{T}_s\right) \cong \mathbb{R} \qquad H^1\left(\mathbb{T}_s\right) \cong \mathbb{R} \qquad H^2\left(\mathbb{T}_s\right) \cong 0 \qquad H^3(\mathbb{T}_s) \cong 0$$

Definition 3.4.5 (Homotopies)**.** A *homotopy* between two smooth maps between manifolds $f : M \to N$ and $g : M \to N$ is a map $F : M \times \mathbb{R} \to N$ such that

$$F(x,t) = f(x) \text{ for } t \geq 1$$
$$F(x,t) = g(x) \text{ for } t \leq 0$$

Note this is equivalent to defining taking the sections $s_0, s_1 : M \to M \times \mathbb{R}$ (defined by $s_0(x) = (x,0)$ and $s_1(x) = (x,1)$) and defining F such that $f = F \circ s_1$ and $g = F \circ s_0$. Then we can see that

$$f^* = s_1^* \circ F^* \qquad\qquad g^* = s_0^* \circ F^*$$

Since both s_0^* and s_1^* are inverses of π^* as shown before, we have that they are the same isomorphism. Thus, we see that $f^* = g^*$.

Definition 3.4.6 (Homotopic manifolds)**.** We say two smooth manifolds M, N are *homotopic* if there exists smooth maps $f : M \to N$ and $g : N \to M$ and a homotopy from $g \circ f$ to the identity on M and a homotopy from $f \circ g$ to the identity on N. If two manifolds M, N are homotopic, then we use the notation $M \simeq N$. This induces an isomorphism on cohomologies.

Example 3.4.7 (Deformation retractions)**.** A deformation retraction from a topological space X onto a subspace Y is a continuous map $F : X \times [0,1] \to X$ such that for all $x \in X$ and $y \in Y$, $F(x,0) = x$, $F(x,1) \in A$, and $F(y,1) = y$. The existence of a deformation retraction proves that $X \simeq Y$, since F is the homotopy from $F(-,0)$ which is the identity on X, to $F(-,1)$ which is the retraction from X to Y.

Example 3.4.8 (Möbius Strip)**.** Although the Möbius strip is non-orientable, there does exist a deformation retract from the Möbius strip onto S^1. Therefore, if M denotes the Möbius strip, then

$$H^0(M) \cong \mathbb{R} \qquad\qquad H^1(M) \cong \mathbb{R} \qquad\qquad H^2(M) \cong 0$$

3.5 Calculating $H^*_c(\mathbb{R}^n)$

Theorem 3.5.1. Similar to before, we wish to show that

$$H^k_c(M) \cong H^{k+1}_c(M \times \mathbb{R})$$

Proof. Note how the degree changes! As before, there are two types of forms on $M \times \mathbb{R}$. Let ϕ be a form on M and π the projection $M \times \mathbb{R} \to M$. While it may not be the case that $\pi^*\phi$ has compact support, all the forms on $M \times \mathbb{R}$ are either $\pi^*\phi f(x,y)$ or $\pi^*\phi f(x,y)dy$ where f has compact support.

Unlike before, we cannot pull back, but there is a map $H^{k+1}_c(M \times \mathbb{R}) \to H^k_c(M)$ for all k which we call integration on the fiber. Often denoted π_*, it is defined as such:

$$\pi_* : \pi^*\phi\, f(x,y) \mapsto 0 \qquad\qquad \pi_* : \pi^*\phi\, f(x,y)\, dy \mapsto \phi \int_{\mathbb{R}} f(x,t)\, dt$$

We claim that $d\pi_* = \pi_* d$, to show that $\pi_* : H_c^{*+1}(M \times \mathbb{R}) \to H_c^*(M)$ is a chain map. We see on the first type of form of degree q:

$$\begin{aligned}
d(\pi^*\phi\, f(x,y)) &= \pi^*\, d\phi f(x,y) + (-1)^q \pi^*\phi \sum_i \frac{\partial f}{\partial x_i}\, dx_i + (-1)^q \pi^*\phi\, \frac{\partial f}{\partial y}\, dy \\
\pi_*\, d(\pi^*\phi\, f(x,y)) &= 0 + 0 + (-1)^q \phi \int_{\mathbb{R}} \frac{\partial f}{\partial y} dt \\
&= (-1)^q \phi\, (f(x,\infty) - f(x,-\infty)) \\
&= 0
\end{aligned}$$

This implies $\pi_*\, d = d\,\pi_*$ on forms that look like $\pi^*\phi\, f(x,y)$. On forms with degree q that look like $\pi^*\phi\, f(x,y)\, dy$, we see that

$$\begin{aligned}
d\,\pi_*(\pi^*\phi\, f(x,y)\, dy) &= d\left(\phi \int_{\mathbb{R}} f(x,t)\, dt\right) \\
&= d\phi \int_{\mathbb{R}} f(x,t)\, dt + (-1)^{q-1} \phi \sum_i \frac{\partial}{\partial x_i} \int_{\mathbb{R}} f(x,t)\, dt\, dx_i
\end{aligned}$$

Now we compute

$$\begin{aligned}
d\,(\pi^*\phi\, f(x,y)\, dy) &= \pi^* d\phi\, f(x,y)\, dy + (-1)^{q-1} \pi^*\phi \left(\sum_i \frac{\partial f}{\partial x_i}\, dx_i + \frac{\partial f}{\partial y}\, dy\right) dy \\
&= \pi^* d\phi\, f(x,y)\, dy + (-1)^{q-1} \pi^*\phi \left(\sum_i \frac{\partial f}{\partial x_i}\, dx_i\right) dy \\
\pi_*\, d\,(\pi^*\phi\, f(x,y)\, dy) &= d\phi \int_{\mathbb{R}} f(x,t)\, dt + (-1)^{q-1} \phi \sum_i \int_{\mathbb{R}} \frac{\partial f}{\partial x_i}\, dt\, dx_i
\end{aligned}$$

Since the last integral is with respect to y, and the partials are with respect to x_i, we can change their order. Indeed this exactly shows that $d\,\pi_* = \pi_*\, d$. Next we construct a map the other direction. In the following construction, we let ρ be any compactly supported smooth function $\mathbb{R} \to \mathbb{R}$ with total integral 1. Then we define a map $\rho_* : \phi \mapsto \pi^*\phi \wedge \rho(y)\, dy$. We can see that $d(\rho(y)\, dy) = 0$. Therefore,

$$(d \circ \rho_*)\phi = d\,(\pi^*\phi \wedge \rho(y)\, dy) = (d\pi^*\phi) \wedge \rho(y)\, dy = \pi^*(d\phi) \wedge \rho(y)\, dy = (\rho_* \circ d)\phi$$

so we see that ρ_* is a map in cohomology. We can see that $\pi_* \circ \rho_*$ is the identity on $\Omega_c^*(M)$. Then we want to find a homotopy operator K such that $(\mathrm{id} - \rho_*\pi_*) = \pm(Kd - dK)$. We define $K : H_c^{*+1}(M \times \mathbb{R}) \to H_c^*(M \times \mathbb{R})$

$$K : \pi^*\phi f(x,y) \mapsto 0 \qquad K : \pi^*\phi f(x,y)\, dy \mapsto \pi^*\phi \left(\int_{-\infty}^{y} f(x,t)\, dt - \int_{-\infty}^{y} \rho(t)\, dt \int_{-\infty}^{\infty} f(x,t)\, dt\right)$$

We check first on the first type of form. We know that $(\mathrm{id} - \rho_*\pi_*)\pi^*\phi f(x,y) = \pi^*\phi f(x,y)$ since $\pi_*\pi^*\phi f(x,y) = 0$. Then we let $\deg\phi = q$ and check

$$\begin{aligned}
(Kd - dK)\pi^*\phi f(x,y) &= Kd\,(\pi^*\phi f(x,y)) \\
&= K\left(\pi^* d\phi + (-1)^q \pi^*\phi \sum_i \frac{\partial f}{\partial x_i}\, dx_i + (-1)^q \pi^*\phi \frac{\partial f}{\partial y}\, dy\right) \\
&= K\left((-1)^q \pi^*\phi \frac{\partial f}{\partial y}\, dy\right) \\
&= (-1)^q \pi^*\phi \left(\int_{-\infty}^{y} \frac{\partial f}{\partial t}\, dt - \int_{-\infty}^{y} \rho(t)\, dt \int_{-\infty}^{\infty} \frac{\partial f}{\partial t} dt\right) \\
&= (-1)^q \pi^*\phi \left(f(x,y) - \int_{-\infty}^{y} \rho(t)\, dt\, (f(x,\infty) - f(x,-\infty))\right) \\
&= (-1)^q \pi^*\phi f(x,y)
\end{aligned}$$

Now we check on forms $\pi^*\phi f(x,y)\,dy$, this time where $\deg\phi = q-1$. We see that

$$\begin{aligned}(\text{id} - \rho_*\pi_*)\,(\pi^*\phi\, f(x,y)\,dy) &= \pi^*\phi\, f(x,y)\,dy - \rho_*\left(\phi\int_{\mathbb{R}} f(x,t)dt\right)\\ &= \pi^*\phi\, f(x,y)\,dy - \pi^*\phi\,\rho(y)\int_{\mathbb{R}} f(x,t)dt\,dy\end{aligned}$$

Now we check:

$$\begin{aligned}Kd\,(\pi^*\phi\, f(x,y)\,dy) &= K\left(\pi^* d\phi f(x,y)\,dy + (-1)^{q-1}\pi^*\phi\left(\sum_i \frac{\partial f}{\partial x_i}\,dx_i\right)\,dy\right)\\ &= \pi^* d\phi\left(\int_{-\infty}^{y} f(x,t)\,dt - \int_{-\infty}^{y}\rho(t)dt\int_{-\infty}^{\infty} f(x,t)dt\right)\\ &+ (-1)^{q-1}\pi^*\phi\sum_i\left(\int_{-\infty}^{y}\frac{\partial f}{\partial x_i}\,dt - \int_{-\infty}^{y}\rho(t)\,dt\int_{-\infty}^{\infty}\frac{\partial f}{\partial x_i}\,dt\right)\,dx_i\\ dK\,(\pi^*\phi\, f(x,y)\,dy) &= d\left(\pi^*\phi\left(\int_{-\infty}^{y} f(x,t)\,dt - \int_{-\infty}^{y}\rho(t)\,dt\int_{-\infty}^{\infty} f(x,t)\,dt\right)\right)\\ &= \pi^* d\phi\left(\int_{-\infty}^{y} f(x,t)\,dt - \int_{-\infty}^{y}\rho(t)\,dt\int_{-\infty}^{\infty} f(x,t)\,dt\right)\\ &+ (-1)^{q-1}\pi^*\phi\, d\left(\int_{-\infty}^{y} f(x,t)\,dt - \int_{-\infty}^{y}\rho(t)\,dt\int_{-\infty}^{\infty} f(x,t)\,dt\right)\\ &= \pi^* d\phi\left(\int_{-\infty}^{y} f(x,t)\,dt - \int_{-\infty}^{y}\rho(t)\,dt\int_{-\infty}^{\infty} f(x,t)\,dt\right)\\ &+ (-1)^{q-1}\pi^*\phi\sum_i\left(\int_{-\infty}^{y}\frac{\partial f}{\partial x_i}\,dt - \int_{-\infty}^{y}\rho(t)\,dt\int_{-\infty}^{\infty}\frac{\partial f}{\partial x_i}\,dt\right)\,dx_i\\ &+ (-1)^{q-1}\pi^*\phi\left(f(x,y)\,dy - \rho(y)\int_{-\infty}^{\infty} f(x,t)\,dt\right)\\ (-1)^q(Kd - dK)\,(\pi^*\phi\, f(x,y)\,dy) &= \pi^*\phi\left(f(x,y)\,dy - \rho(y)\int_{-\infty}^{\infty} f(x,t)\,dt\right)\\ &= (\text{id} - \rho_*\pi_*)\,(\pi^*\phi\, f(x,y)\,dy)\end{aligned}$$

We see regardless of type of form, we see that $(\text{id} - \rho_*\pi_*) = (-1)^q(Kd - dK)$. Therefore, we see that π_* and ρ_* are isomorphisms on de Rham cohomology with compact support, so

$$H_c^*(M) \cong H_c^{*+1}(M\times\mathbb{R})$$

□

Example 3.5.2 ($H_c^*(\mathbb{R}^n)$)**.** Since we have calculated $H_c^0(\mathbb{R}) = 0$ and $H_c^1(\mathbb{R}) \cong \mathbb{R}$, we see that by induction:

$$\begin{aligned}H^0\left(\mathbb{R}^n\right) &\cong \mathbb{R} & H^{i>0}\left(\mathbb{R}^n\right) &\cong 0\\ H_c^{j<n}\left(\mathbb{R}^n\right) &\cong 0 & H_c^n\left(\mathbb{R}^n\right) &\cong \mathbb{R}\end{aligned}$$

Example 3.5.3 ($H_c^*(\mathbb{R}^2 - \{0\})$)**.** We recall the the punctured plane is diffeomorphic to $S^1\times\mathbb{R}$. Thus,

$$H_c^0(\mathbb{R}^2 - \{0\}) \cong 0 \qquad H^1(\mathbb{R}^2 - \{0\}) \cong \mathbb{R} \qquad H^2(\mathbb{R}^2 - \{0\}) \cong \mathbb{R}$$

4 The Mayer-Vietoris Sequence

4.1 Generalized Cohomology

Definition 4.1.1 (An exact sequence)**.** An exact sequence is a sequence of objects and maps

$$\cdots A^n \xrightarrow{f^n} A^{n+1} \xrightarrow{f^{n+1}} \cdots$$

such that $\ker\left(f^{n+1}\right) = \mathrm{im}\,(f^n)$ for all n. A short exact sequence (SES) is when the sequence is exact and of the form:

$$0 \to A \to B \to C \to 0$$

Definition 4.1.2 (Cochain complex)**.** A cochain complex is usually denoted A^* and a collection of abelian groups indexed by the integers such that there is a morphism $d^n : A^n \to A^{n+1}$ and $d^{n+1} \circ d^n = 0$ for all n.

Definition 4.1.3 (Cohomology)**.** We can define the cohomology of any cochain complex. Since $d^2 = 0$, we see that $\mathrm{img}\,(d^{n-1}) \subset \ker(d^n)$. Since all the groups are abelian, this means $\mathrm{img}\,(d)$ is a normal subgroup of $\ker(d)$, and we define the nth-cohomology of A^* to be:

$$H^i(A^*) := \frac{\ker(d^n)}{\mathrm{img}\,(d^{n-1})}$$

Example 4.1.4. We see that $\Omega^*(M)$ is a cochain complex, and it has the additional structure of a vector space.

A morphism of cochain complexes, often called a cochain map, is a function $f^* : A^* \to B^*$ that consists of a group homomorphism (in our case a linear map) $f^n : A^n \to B^n$ such that $f^{n+1} \circ d_A^n = d_B^{n+1} \circ f^n$. In other words, the diagram below commutes (any path you follow gives the same map):

$$\begin{tikzcd}
\cdots \arrow[r] & A^{n-1} \arrow[r, "d_A^{n-1}"] \arrow[d, "f^{n-1}"] & A^n \arrow[r, "d_A^n"] \arrow[d, "f^n"] & A^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] & \cdots \\
\cdots \arrow[r] & B^{n-1} \arrow[r, "d_B^{n-1}"] & B^n \arrow[r, "d_B^n"] & B^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

Proposition 4.1.5. Any short exact chain sequence of cochain complexes induces a long exact sequence of their cohomology groups. Suppose the following is a SES:

$$0 \to A^* \xrightarrow{f^*} B \xrightarrow{g^*} C^* \to 0$$

Writing this out in more detail, we see that every vertical composition of arrows is a SES of abelian groups.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A^{n-1} \arrow[r, "d_A^{n-1}"] \arrow[d, "f^{n-1}"] & A^n \arrow[r, "d_A^n"] \arrow[d, "f^n"] & A^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] & \cdots \\
\cdots \arrow[r] & B^{n-1} \arrow[r, "d_B^{n-1}"] \arrow[d, "g^{n-1}"] & B^n \arrow[r, "d_B^n"] \arrow[d, "g^n"] & B^{n+1} \arrow[r] \arrow[d, "g^{n+1}"] & \cdots \\
\cdots \arrow[r] & C^{n-1} \arrow[r, "d_C^{n-1}"] \arrow[d] & C^n \arrow[r, "d_C^n"] \arrow[d] & C^{n+1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

There exists a map $\delta^* : H^*(C) \to H^{*+1}(A)$ which makes the following sequence exact [BT95, Eq 2.4]:

$$\cdots \longrightarrow H^{n-1}(C) \xrightarrow{\delta^{n-1}} H^n(A) \xrightarrow{f^n} H^n(B) \xrightarrow{g^n} H^n(C) \xrightarrow{\delta^n} H^{n+1}(A) \longrightarrow \cdots$$

4.2 Mayer-Vietoris for de Rham Cohomology

Suppose that M is a manifold and $M = U \cup V$ for open sets U, V. We see there are two maps from $U \cap V \rightrightarrows U \coprod V$, one of which we call ∂_U that embeds the intersection in the copy of U, and the other ∂_V which embeds the intersections in the copy of V. Moreover, there is a map $U \coprod V \to M$ which just embeds U, V in M. This gives a sequence:

$$M \longleftarrow U \coprod V \underset{\partial_V}{\overset{\partial_U}{\leftleftarrows}} U \cap V$$

We can then apply the Ω^* functor and get:

$$\Omega^*(M) \longrightarrow \Omega^*\left(U \coprod V\right) \underset{\partial_V^*}{\overset{\partial_U^*}{\rightrightarrows}} \Omega^*\left(U \cap V\right)$$

We observe that $\Omega^*(U \coprod V) = \Omega^*(U) \oplus \Omega^*(V)$. Moreover, we can combine the right two maps into one by subtracting them from each other. We can extend this to get a short exact sequence.

$$0 \longrightarrow \Omega^*(M) \longrightarrow \Omega^*\left(U \coprod V\right) \xrightarrow{\partial_U^* - \partial_V^*} \Omega^*\left(U \cap V\right) \longrightarrow 0$$

The first nonzero morphism is simply restriction of forms. This is injective. We argue that the image of the restriction is the kernel of the difference. Indeed, if a pair of forms on U, V came from a form on $U \cup V$, then the forms must agree on $U \cap V$. Similarly, if a pair of forms (ω, η) is in the kernel of the difference map, then they agree on $U \cap V$. This means we can glue them together to form a form on M that restricts to the ω on U and η on V.

Lastly, we need to see that the difference map is surjective. Let $\omega \in \Omega^*(U \cap V)$. Then we wish to extend ω to be a form on U and V. Observe that for a partition of unity $\{\rho_U, \rho_V\}$ subordinate to $\{U, V\}$ we have $\rho_U\omega$ is a well deformed form on V and $\rho_V\omega$ is a well defined form on U. Then, we see $(\rho_V\omega, -\rho_U\omega) \mapsto \rho_V\omega + \rho_U\omega = \omega$. Thus, the sequence is exact. This leads to the long exact sequence:

$$\longrightarrow H^{n-1}(U \cap V) \xrightarrow{\delta^{n-1}} H^n(M) \longrightarrow H^n(U) \oplus H^n(V) \xrightarrow{\partial_U^* - \partial_V^*} H^n(U \cap V) \xrightarrow{\delta^n} H^{n+1}(M)$$

Example 4.2.1 (Punctured Plane)**.** We let $X = \mathbb{R} - \{*\}$ be the punctured plane. We have shown that it has nonzero H^1, but now we calculate it explicitly. Since it is connected, we know that $H^0(X) = \mathbb{R}$. To find the rest, we cover the punctured plane with two open sets U, V such that

$$U = \{(x, y) : x < 1 \text{ when } y \neq 0,\ x < 0 \text{ when } y = 0\}$$

and $V = -U$. Then we see that U, V are open and diffeomorphic to $\mathbb{R}$, $U \cup V = \mathbb{R} \setminus \{(0, 0)\}$, and

$$U \cap V = \{(x, y) | x \in (-1, 1) y > 0\} \cup \{(x, y) | x \in (-1, 1) y < 0\}$$

Therefore, $U \cap V$ is diffeomorphic to $\mathbb{R} \sqcup \mathbb{R}$. Then we construct the long exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & H^0(X) \arrow[r] & H^0(U) \bigoplus H^0(V) \arrow[r] & H^0(U \cap V) \arrow[dll] & \\
& H^1(X) \arrow[r] & H^1(U) \bigoplus H^1(V) \arrow[r] & H^1(U \cap V) \arrow[dll] & \\
& H^2(X) \arrow[r] & H^2(U) \bigoplus H^2(V) \arrow[r] & H^2(U \cap V) \arrow[r] & 0
\end{tikzcd}$$

Plugging in $H^q(\mathbb{R})^2$ for $H^q(U) \bigoplus H^q(V)$ and $H^q(U \cap V)$, and $\mathbb{R}$ we see that

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r] & \mathbb{R}^2 \arrow[r] & \mathbb{R}^2 \arrow[dll] & \\
& H^1(X) \arrow[r] & 0 \arrow[r] & 0 \arrow[dll] & \\
& H^2(X) \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

By exactness, this immediately implies that $H^2(X) = 0$. We see that the image of $\mathbb{R} \to \mathbb{R}^2$ is $\mathbb{R}$ since it has zero kernel. Therefore, the kernel and image of the $\mathbb{R}^2 \to \mathbb{R}^2$ map is also $\mathbb{R}$. Similarly, the kernel and image of the $\mathbb{R}^2 \to H^1(X)$ map is $\mathbb{R}$. Since the kernel of the map $H^1(X) \to 0$ is $\mathbb{R}$, and the map sends all of $H^1(X) \to 0$, we get that $H^1(X) \cong \mathbb{R}$. Therefore,

$$H^0(X) \cong \mathbb{R} \qquad H^1(X) \cong \mathbb{R} \qquad H^{i>1}(X) \cong 0$$

Example 4.2.2 ($H^*(S^n)$)**.** We have calculated $H^*(S^1)$. Now we determine $H^*(S^n)$ for $n \neq 1$. We can cover S^n with two open sets, in the following way. For some $\eta \in (0,1)$, we define:

$$U = \{(x_0, \ldots, x_n) \in S^n : x_0 > -\eta\} \qquad V = \{(x_0, \ldots, x_n) \in S^n : x_0 < -\eta\}$$

We see that $U \cong V \cong \mathbb{R}^n$ and $U \cap V \cong \mathbb{R}^n - \{0\}$. However, we see that $\mathbb{R}^n - \{0\} \cong S^{n-1} \times \mathbb{R}$. Then we can check inductively. We recall that $H^0(S^n) \cong \mathbb{R}$ since S^n is connected. Next we check $H^1(S^n)$:

$$0 \longrightarrow H^0(S^n) \longrightarrow H^0(U) \bigoplus H^0(V) \longrightarrow H^1(S^{n-1}) \longrightarrow H^1(S^n) \longrightarrow H^1(U) \bigoplus H^1(V)$$

We evaluate:

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R} \longrightarrow H^1(S^n) \longrightarrow 0$$

And we see that $H^1(S^n) \cong 0$ for $n \neq 1$. Next, we check $i > 1$:

$$H^{i-1}(U) \bigoplus H^{i-1}(V) \longrightarrow H^{i-1}(S^{n-1}) \longrightarrow H^i(S^n) \longrightarrow H^i(U) \bigoplus H^i(V)$$

Again we evaluate for what we can:

$$0 \longrightarrow H^{i-1}(S^{n-1}) \longrightarrow H^i(S^n) \longrightarrow 0$$

This shows for $i > 1$, $H^{i-1}(S^{n-1}) \cong H^i(S^n)$. Therefore, we have, for $j \neq 0, n$:

$$H^0(S^n) \cong \mathbb{R} \qquad H^n(S^n) \cong \mathbb{R} \qquad H^j(S^n) \cong 0$$

Example 4.2.3 ($H^*(\mathbb{R}^n - (0))$)**.** We see that $\mathbb{R}^n - \{0\}$ is diffeomorphic to $S^{n-1} \times \mathbb{R}$. Thus, $H^*(\mathbb{R}^n - \{0\}) \cong H^*(S^{n-1})$.

Example 4.2.4 ($H^*(\mathbb{T} - \{*\})$)**.** We let $\mathbb{T} - \{*\}$ denote the torus without a single point. We recall one way of defining the torus is by gluing opposite sides of a square together. Suppose the point missing is the point made by identifying the four corners. We now choose two open cylinders, call them U, V, each missing a line in $\mathbb{T} - \{*\}$ made by one pair of edges that have been identified on the square. Clearly, $U \cup V = \mathbb{T} - \{*\}$ and $U \cap V \cong \mathbb{R}^2$. Then we can calculate

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathbb{T} - \{*\}) \arrow[r] & H^0(U) \oplus H^0(V) \arrow[r] & H^0(\mathbb{R}^2) \arrow[dll] & \\
& H^1(\mathbb{T} - \{*\}) \arrow[r] & H^1(U) \oplus H^1(V) \arrow[r] & H^1(\mathbb{R}^2) \arrow[dll] & \\
& H^2(\mathbb{T} - \{*\}) \arrow[r] & H^2(U) \oplus H^2(V) \arrow[r] & H^2(\mathbb{R}^2) \arrow[r] & 0
\end{tikzcd}$$

Evaluating:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r] & \mathbb{R}^2 \arrow[r] & \mathbb{R} \arrow[dll] \\
& H^1(\mathbb{T} - \{*\}) \arrow[r] & \mathbb{R}^2 \arrow[r] & 0 \arrow[dll] \\
& H^2(\mathbb{T} - \{*\}) \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

We then see that $H^1(\mathbb{T} - \{*\}) \cong \mathbb{R}^2$ and $H^2(\mathbb{T} - \{*\}) \cong 0$.

Example 4.2.5 (Many Punctured Plane)**.** We let $X^n := \mathbb{R}^3 - \{a_1, \dots, a_n\}$ be the plane with n distinct points removed. We can move these points to all lie on the real line by some diffeomorphism, in order $a_i < a_j$ if $i < j$. We let $a = \frac{1}{2}(a_n - a_{n-1})$. We can then cover X^n with two open sets $U := \{(x, y) : x < a + \eta\}$, $V := \{(x, y) : x > a - \eta\}$ for any $\eta < \frac{1}{2}(a_n - a_{n-1})$. Then it is clear that $U \cong X^{n-1}$ and $V \cong X^1$ and $U \cap V \cong \mathbb{R}^2$. Then we can solve it inductively using Mayer-Vietoris:

$$\begin{tikzcd}
0 \arrow[r] & H^0(X^n) \arrow[r] & H^0(X^{n-1}) \oplus H^0(X^1) \arrow[r] & H^0(\mathbb{R}^2) \arrow[dll] & \\
& H^1(X^n) \arrow[r] & H^1(X^{n-1}) \oplus H^1(X^1) \arrow[r] & H^1(\mathbb{R}^2) \arrow[dll] & \\
& H^2(X^n) \arrow[r] & H^2(X^{n-1}) \oplus H^2(X^1) \arrow[r] & H^2(\mathbb{R}^2) \arrow[r] & 0
\end{tikzcd}$$

We evaluate and find

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r] & \mathbb{R}^2 \arrow[r] & \mathbb{R} \arrow[dll] & \\
& H^1(X^n) \arrow[r] & H^1(X^{n-1}) \oplus \mathbb{R} \arrow[r] & 0 \arrow[dll] & \\
& H^2(X^n) \arrow[r] & H^2(X^{n-1}) \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

So we see that $H^1(X^n) \cong H^1(X^{n-1}) \oplus \mathbb{R}$ and $H^2(X^n) \cong H^2(X^{n-1})$. Since we know the values for when $n = 1$, we see that

$$H^0(X^n) \cong \mathbb{R} \qquad H^1(X^n) \cong \mathbb{R}^n \qquad H^2(X^n) \cong 0$$

4.3 Mayer-Vietoris for Compactly Supported de Rham Cohomology

There is a similar short exact sequence and induced long exact sequence for compactly supported forms. Notably, Ω_c^* is not a contravariant functor since the pullback of a compactly supported function need not be compactly supported. If we restrict our view to open inclusions, then Ω_c^* is a covariant functor, since we can extend a compactly supported form to a larger open set by setting it to be zero outside of its original domain. Thus, the diagram

$$M \longleftarrow U \coprod V \underset{\partial_V}{\overset{\partial_U}{\leftleftarrows}} U \cap V$$

is sent to

$$\Omega_c^*(M) \longleftarrow \Omega_c^*(U \coprod V) \underset{\partial_V^*}{\overset{\partial_U^*}{\leftleftarrows}} \Omega_c^*(U \cap V)$$

As before, we take the difference of the two maps. The right map simply adds the inclusion of the two forms.

$$0 \longleftarrow \Omega_c^*(M) \longleftarrow \Omega_c^*(U \coprod V) \xleftarrow{\partial_U^* - \partial_V^*} \Omega_c^*(U \cap V) \longleftarrow 0$$

by a similar argument, this sequence is exact. Note the most difficult step is that any form $\omega \in \Omega^*(M) = \rho_U \omega + \rho_V \omega$ and we see that $(\rho_U \omega, \rho_V \omega)$ is a form on $U \coprod V$. Therefore, we have a long exact sequence

$$\cdots \longleftarrow H_c^{n+1}(U \cap V) \xleftarrow{\delta^n} H_c^n(M) \longleftarrow H_c^n(U) \oplus H_c^n(V) \xleftarrow{\partial_U^* - \partial_V^*} H_c^n(U \cap V) \xleftarrow{\delta^{n-1}} H_c^{n-1}(M) \longleftarrow \cdots$$

5 Poincaré Duality

5.1 Duality Between Cohomologies

Recall that V^* is the dual space of V, which is the set of all linear maps $V \to \mathbb{R}$, and that in the case that V is finite dimensional, we have $\dim V^* = \dim V$.

Definition 5.1.1. We say a bilinear function of real vector spaces $f : V \times W \to \mathbb{R}$ is non-degenerate if it satisfies the following two conditions

- if for a fixed $v \in V$ and all $w \in W$, $f(v, w) = 0$, then $v = 0$
- if for a fixed $w \in W$ and all $v \in V$, $f(v, w) = 0$, then $w = 0$

This is equivalent to the statement that f defines injections $f(-,-) : V \to W^* : v \mapsto f(v,-)$ and $f(-,-) : W \to V^* : w \mapsto f(-,w)$.

Proposition 5.1.2. If V, W are finite dimensional, then f defines an isomorphism $V \cong W^*$ and $W \cong V^*$.

Theorem 5.1.3 (Poincaré Duality)**.** We wish to show the result

$$(H_c^k(M))^* \cong H^{n-k}(M)$$

Proof. Now, we know by Stokes' theorem that integration descends to cohomology. Then we wish to show that when M is orientable and has a finite good cover, then for any $\omega \in H^k(M)$ and $\eta \in H_c^{n-k}(M)$.

$$\int : H^k(M) \otimes_{\mathbb{R}} H_c^{n-k}(M) \to \mathbb{R}$$

$$\omega\,\eta \to \int_M \omega\,\eta$$

is non-degenerate. Upon showing this, we will have the result desired. We pair the two Mayer-Vietoris sequences via a tensor, and integrate the result. We aim to show that the diagram commutes, up to a possible sign change:

$$\begin{tikzcd}
\cdots \arrow[r] & H^q(U \cup V) \arrow[r] & H^q(U) \oplus H^q(V) \arrow[r] & H^q(U \cap V) \arrow[r, "\delta^*"] & H^{q+1}(U \cup V) \arrow[r] & \cdots \\
& \bigotimes & \bigotimes & \bigotimes & \bigotimes & \\
\cdots & H_c^{n-q}(U \cup V) \arrow[l] \arrow[d, "\int_{U \cup V}"] & H_c^{n-q}(U) \oplus H_c^{n-q}(V) \arrow[l] \arrow[d, "\int_U + \int_V"] & H_c^{n-q}(U \cap V) \arrow[l] \arrow[d, "\int_{U \cap V}"] & H_c^{n-q-1}(U \cup V) \arrow[l, "\delta_*"] \arrow[d, "\int_{U \cup V}"] & \cdots \arrow[l] \\
& \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} &
\end{tikzcd}$$

In other words, we have to show that for any $\omega \in H^q(U \cap V)$ and $\tau \in H_c^{n-q-1}(U \cup V)$ the following is true:

$$\int_{U \cap V} \omega \wedge \delta_* \tau = \pm \int_{U \cup V} \delta^* \omega \wedge \tau$$

We first check the first square. We recall the map $H^q(U \cup V) \to H^q(U) \oplus H^q(V)$ is simply restriction of forms, and the map $H_c^{n-q}(U) \oplus H_c^{n-q}(V) \to H_c^{n-q}(U \cup V)$ is the sum of the two forms. So we choose any $\omega \in H^q(U \cup V)$ and $(\tau_1, \tau_2) \in H_c^{n-q}(U) \oplus H_c^{n-q}(V)$, then we see that

$$\int_{U \cup V} \omega \wedge (\tau_1 + \tau_2) = \int_{U \cup V} \omega \wedge \tau_1 + \int_{U \cup V} \omega \wedge \tau_2 = \int_U \omega|_U \wedge \tau_1 + \int_V \omega|_V \wedge \tau_2$$

The last equality comes from the fact that the support of τ_1 and τ_2 are entirely within U and V, respectively. Thus, $\omega \wedge \tau_1 = \omega|_U \wedge \tau_1$ and $\omega \wedge \tau_2 = \omega|_V \wedge \tau_2$.

We can check the middle square. We recall the map $H^q(U) \oplus H^q(V) \to H^q(U \cap V)$ is the difference of the restrictions, and the map $H_c^{n-q}(U \cap V) \to H_c^{n-q}(U) \oplus H_c^{n-q}(V)$ is the signed embedding. Then we take any $(\omega_1, \omega_2) \in H^q(U) \oplus H^q(V)$ and $\tau \in H_c^{n-q}(U \cap V)$ and we see that

$$\int_U \omega_1 \wedge \tau + \int_V \omega_2 \wedge (-\tau) = \int_{U\cap V} \omega_1|_{U\cap V} \wedge \tau + \int_{U\cap V} (-\omega_2|_{U\cap V}) \wedge \tau = \int_{U\cap V} (\omega_1|_{U\cap V} - \omega_2|_{U\cap V}) \wedge \tau$$

We then check the last square commutes. We recall that for any $\omega \in H^q(U \cap V)$ we recall that $\delta^*\omega|_V = d(\rho_U \omega)$. Note that this implies the support is in $U \cap V$ Similarly, for any $\tau \in H_c^{n-q-1}(U \cup V)$, we have that when $\delta_*\tau$ is extended to U it equals $-d(\rho_U \tau)$. Since both ω and τ are closed, we have $d(\rho_U \omega) = d(\rho_U)\omega$ and $d(\rho_U \tau) = d(\rho_U)\tau$. Then we see

$$\int_{U\cup V} \delta^*\omega \wedge \tau = \int_{U\cap V} d(\rho_U)\omega \wedge \tau = (-1)^{\deg \omega + 1} \int_{U\cap V} \omega \wedge -d(\rho_U \tau) = (-1)^{\deg \omega + 1} \int_{U\cap V} \omega \wedge \delta_*\tau$$

This proves that the diagram commutes up to a sign. Therefore, we have the following diagram (which also commutes up to sign:

$$\begin{tikzcd}
{} \arrow[r] & H^q(U \cup V) \arrow[r] \arrow[d] & H^q(U) \oplus H^q(V) \arrow[r] \arrow[d] & H^q(U \cap V) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & H_c^q(U \cap V)^* \arrow[r] & H_c^q(U)^* \oplus H_c^q(V)^* \arrow[r] & H_c^q(U \cap V)^* \arrow[r] & {}
\end{tikzcd}$$

We have shown that the arrows are isomorphisms in the case of $\mathbb{R}^n$. Since we have let M be covered by a good cover, we can check inductively that this induces an isomorphism on $H^q(U \cup V) \to H_c^{n-q}(U \cup V)^*$. Indeed, we apply

Lemma 5.1.4 (Five lemma)**.** If in the following diagram, the two rows are exact and a, b, d, e are all isomorphisms, then so is c:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "a"] & B \arrow[r] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & D \arrow[r] \arrow[d, "d"] & E \arrow[d, "e"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

Proved in [Mas91, Lemma 7.1]. This proves that we get an isomorphism on the union. So we get the result that $H^q(M) \cong H_c^{n-q}(M)^*$ for any orientable manifold with a finite good cover. This result can be improved to a manifold with any good cover, not necessarily finite [War71, p. 14, 198]. $\square$

Example 5.1.5 (Torus)**.** We see that the torus, denoted $\mathbb{T}$, is an orientable, compact, and connected 2 manifold. This immediately tells us that $\mathbb{R} \cong H^0(\mathbb{T}) \cong H_c^2(\mathbb{T}) \cong H^2(\mathbb{T})$. Then we see the torus can be covered by two open cylinders, denoted here as U, V. Their intersection is diffeomorphic to the disjoint union of two open cylinders. Note however, that an open cylinder is diffeomorphic to the punctured plane. Therefore $H^*(U) \cong H^*(\mathbb{R} - \{(0,0)\})$. Then we can compute the cohomology of $\mathbb{T}$:

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathbb{T}) \arrow[r] & H^0(U) \bigoplus H^0(V) \arrow[r] & H^0(U \cap V) \arrow[dll] & \\
& H^1(\mathbb{T}) \arrow[r] & H^1(U) \bigoplus H^1(V) \arrow[r] & H^1(U \cap V) \arrow[dll] & \\
& H^2(\mathbb{T}) \arrow[r] & H^2(U) \bigoplus H^2(V) \arrow[r] & H^2(U \cap V) \arrow[r] & 0
\end{tikzcd}$$

Then we can calculate:

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \longrightarrow H^1(\mathbb{T}) \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R} \longrightarrow 0$$

We now see that $H^1(\mathbb{T}) \cong \mathbb{R}^2$

Example 5.1.6 ($H^*(\mathbb{T}_g)$)**.** We let $\mathbb{T}_g$ denote the torus with genus g. If we align $\mathbb{T}_g$ to be symmetric over the xy plane in $\mathbb{R}^3$, we can cover it with two sheets $U := \{(x,y,z) : z > -\varepsilon\} \cap \mathbb{T}_g$ and $V := \{(x,y,z) : z < \varepsilon\} \cap \mathbb{T}_g$ for some $\varepsilon > 0$ sufficiently small so that $U \cong V \simeq \mathbb{R}^2 - \{x_1, \ldots, x_g\}$ which we denote X^n, as before. We see that $U \cap V$ deformation retracts to $g+1$ disjoint copies of S^1.

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathbb{T}_g) \arrow[r] & H^0(X^g)^2 \arrow[r] & H^0(S^1)^{g+1} \arrow[dll] \\
& H^1(\mathbb{T}_g) \arrow[r] & H^1(X^g)^2 \arrow[r] & H^1(S^1)^{g+1} \arrow[dll] \\
& H^2(\mathbb{T}_g) \arrow[r] & H^2(X^g)^2 \arrow[r] & H^2(S^1)^{g+1} \arrow[r] & 0
\end{tikzcd}$$

Since $\mathbb{T}_g$ is compact and orientable, we see by the same argument for $\mathbb{T}$ that $\mathbb{R} \cong H^0(\mathbb{T}_g) \cong H^2_c(\mathbb{T}_g)^* \cong H^2_c(\mathbb{T}_g) \cong H^2(\mathbb{T}_g)$. We recall $H^0(X^n) \cong \mathbb{R}$, $H^1(X^g) \cong \mathbb{R}^g$, and $H^2(X^g) \cong 0$.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r] & \mathbb{R}^2 \arrow[r] & \mathbb{R}^{g+1} \arrow[dll] \\
& H^1(\mathbb{T}_g) \arrow[r] & \mathbb{R}^{2g} \arrow[r] & \mathbb{R}^{g+1} \arrow[dll] \\
& \mathbb{R} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

We simplify to

$$0 \longrightarrow \mathbb{R}^g \longrightarrow H^1(\mathbb{T}_g) \longrightarrow \mathbb{R}^{2g} \longrightarrow \mathbb{R}^g \longrightarrow 0$$

By a quick counting argument we see that $H^1(\mathbb{T}_g) \cong \mathbb{R}^{2g}$.

5.2 Non-orientable Manifolds

For any connected manifold M, there exists a manifold $\hat{M}$ and a projection $p : \hat{M} \to M$ that is a double cover called the orientation cover. A point on $\hat{M}$ consists of a point on M and a local orientation, which is simply an orientation on a neighborhood of the point. If M is orientable, we see that $\hat{M}$ is the disjoint union of M with itself. If M is non-orientable, then $\hat{M}$ is connected, as there exists a path in M from any point to itself which flips the local orientation. Moreover, $\hat{M}$ is orientable as shown in [Lee12, Proposition 15.40]. There is a smooth automorphism $f : \hat{M} \to \hat{M}$ that sends points to their opposite local orientation. This map is orientation reversing. We see that any form $\omega \in \Omega^*(\hat{M})$ can be identified with an form on $\Omega^*(M)$ if $\omega = f^*\omega$.

Proposition 5.2.1. If a non-orientable manifold M has $\dim(M) = n$, then $H^n(M) \cong 0$ and $H^n_c(M) \cong 0$.

Proof. Let ω be an arbitrary closed n-form on an n-dimensional non-oriented manifold M. We see that $p^*\omega = f * p^*\omega$ so $p^*\omega$ is invariant under reversing orientation. Therefore,

$$\int_{[\hat{M}]} p^*\omega = \int_{[\hat{M}]} f^*p^*\omega = \int_{-[\hat{M}]} p^*\omega = -\int_{[\hat{M}]} p^*\omega$$

Therefore, $\int_{\hat{M}} p^*\omega = 0$. Therefore, $p^*\omega$ is exact. We take $\eta \in \Omega^{n-1}(\hat{M})$ such that $d\eta = p^*(\omega)$. We construct

$$\eta' = \frac{\eta + f^*\eta}{2}$$

So we see that $f^*\eta' = \eta'$ since $(f^*)^2$ is the identity. Moreover, we recall that $df^* = f^*d$. Then we can calculate:

$$d\eta' = \frac{d\eta + df^*\eta}{2} = \frac{p^*\omega + f^*d\eta}{2} = \frac{p^*\omega + f^*p^*\omega}{2} = p^*\omega$$

Therefore, we see that $\eta' = p^*\eta''$ for some $\eta'' \in \Omega^{n-1}(M)$ and therefore $d\eta'' = \omega$. Thus, every closed top form is exact. Therefore, $H^n(M) \cong 0$. Moreover, since $\hat{M} \to M$ is a finite cover, either with 1 sheet or 2, it must be a proper map. Thus, you can also pull back compactly supported forms, and we see that $H^n_c(M) \cong 0$ as well. $\square$

Example 5.2.2 (Möbius Strip part 2)**.** Let M denote the Möbius strip. Then as before we may cover it with two open sets U, V such that $U \cong V \cong \mathbb{R}^2$ and $U \cap V \cong R^2 \sqcup \mathbb{R}^2$. Then, we know using Mayer-Vietoris:

$$\begin{tikzcd}
0 \arrow[r] & H^0_c(U \cap V) \arrow[r] & H^0_c(U) \oplus H^0_c(V) \arrow[r] & H^0_c(M) \arrow[dll] & \\
& H^1_c(U \cap V) \arrow[r] & H^1_c(U) \oplus H^1_c(V) \arrow[r] & H^1_c(M) \arrow[dll] & \\
& H^2_c(U \cap V) \arrow[r] & H^2_c(U) \oplus H^2_c(V) \arrow[r] & H^2_c(M) \arrow[r] & 0
\end{tikzcd}$$

We evaluate, now knowing that $H^2_c(M) \cong 0$:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & H^0_c(M) \arrow[dll] & \\
& 0 \arrow[r] & 0 \arrow[r] & H^1_c(M) \arrow[dll] & \\
& \mathbb{R}^2 \arrow[r] & \mathbb{R}^2 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Therefore, we see that

$$H^i_c(M) \cong 0$$

for every i.

Example 5.2.3 ($\mathbb{RP}^n$)**.** We define $\mathbb{RP}^n$ as the lines going through the origin in $\mathbb{R}^{n+1}$. This is equivalent to identifying antipodal points on S^n. This is also equivalent to the set of nonzero points in $\mathbb{R}^{n+1}$ and any two points are equivalent if they differ by a nonzero scalar $\lambda \in \mathbb{R}$. We often write points using the square brackets (such as $[x_0, \ldots, x_n]$) to make it clear the point is in $\mathbb{RP}^n$. In set notation, this is written:

$$\mathbb{RP}^n := \frac{\mathbb{R}^{n+1} - \{0\}}{\sim}$$

where $(x_0, \ldots, x_n) \sim (y_0, \ldots, y_n)$ if there exists some $\lambda \in \mathbb{R} - \{0\}$ such that for every i, $x_i = \lambda y_i$. These equivalence classes are denoted $[x_0, \ldots, x_n]$ where $(x_0, \ldots, x_n)$ is some representative of the equivalence class. There is a natural cover of $\mathbb{RP}^n$ by the sets U_i defined

$$U_i := \{[x_0, \ldots, x_n] \in \mathbb{RP}^n : x_i = 1\}$$

We note that if the $x_i = 1$, we never have that all coordinates are 0. Moreover, fixing x_i to be 1, means we get unique elements of $\mathbb{RP}^n$ for each $(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ since we cannot scale it. Thus, $U_i \cong R^n$. Moreover, since every element $x = [x_0, \ldots, x_n] \in \mathbb{RP}^n$ needs $x_j \neq 0$ for some $0 \leq j \leq n$, we see that $x \in U_j$. Thus, the U_i form a smooth atlas for $\mathbb{RP}^n$. We see that the natural map $p : S^n \to \mathbb{RP}^n$ is surjective and smooth, and every $x \in \mathbb{RP}^n$ has exactly two elements that map to it. Thus, $\mathbb{RP}^n$ is connected and compact. Therefore:

$$H^0(\mathbb{RP}^n) \cong \mathbb{R}$$

We check to see when $\mathbb{RP}^n$ is orientable. We see that map $S^n \to S^n$ which sends points to their antipodal points, is simply $n+1$ flips over the n dimensional hyperplanes spanned by n axes in $\mathbb{R}^{n+1}$. We see that each of these is orientation reversing. Thus, this map is orientation preserving if and only if n is odd. Therefore, $\mathbb{RP}^n$ is orientable if n is odd. We can then see that, for n odd:

$$H^0(\mathbb{RP}^n) \cong H^n_c(\mathbb{RP}^n) \cong H^n(\mathbb{RP}^n) \cong \mathbb{R}$$

If n is even, then $\mathbb{RP}^n$ is non-orientable, so we saw that

$$H^n(\mathbb{RP}^n) \cong 0$$

Now we cover $\mathbb{RP}^n$ with two open sets: $U := U_0$ and $V := \mathbb{RP}^n - ([1, 0, \ldots, 0])$. We know that $U \cong \mathbb{R}^n$ and therefore $U \cap V \cong \mathbb{R}^n - (0) \cong S^{n-1}$. Next we see that any element $[x_0, \ldots, x_n] \in V$ has the property that not all $x_1, \ldots, x_n$ can be zero. Therefore, $V \cong \mathbb{RP}^{n-1} \times \mathbb{R}$. Lastly, we notice that $\mathbb{RP}^1 \cong S^1$ by simply halving/doubling the angle. Now we examine using Mayer-Vietoris, for $n > 1$, and $0 < i < n - 1$:

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathbb{RP}^n) \arrow[r] & H^0(\mathbb{R}^n) \oplus H^0(\mathbb{RP}^{n-1}) \arrow[r] & H^0(S^{n-1}) \arrow[dll, "\cdots" description] & \\
 & H^i(\mathbb{RP}^n) \arrow[r] & H^i(\mathbb{R}^n) \oplus H^i(\mathbb{RP}^{n-1}) \arrow[r] & H^i(S^{n-1}) \arrow[dll, "\cdots" description] & \\
 & H^{n-1}(\mathbb{RP}^n) \arrow[r] & H^{n-1}(\mathbb{R}^n) \oplus H^{n-1}(\mathbb{RP}^{n-1}) \arrow[r] & H^{n-1}(S^{n-1}) \arrow[dll] & \\
 & H^n(\mathbb{RP}^n) \arrow[r] & H^n(\mathbb{R}^n) \oplus H^n(\mathbb{RP}^{n-1}) \arrow[r] & H^n(S^{n-1}) \arrow[r] & 0
\end{tikzcd}$$

Since we know many of these values, we can see this reduces to

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r] & \mathbb{R}^2 \arrow[r] & \mathbb{R} \arrow[dll, "\cdots" description] \\
 & H^i(\mathbb{RP}^n) \arrow[r] & H^i(\mathbb{RP}^{n-1}) \arrow[r] & 0 \arrow[dll, "\cdots" description] \\
 & H^{n-1}(\mathbb{RP}^n) \arrow[r] & H^{n-1}(\mathbb{RP}^{n-1}) \arrow[r] & \mathbb{R} \arrow[dll] \\
 & H^n(\mathbb{RP}^n) \arrow[r] & 0 &
\end{tikzcd}$$

Then we see that $H^i(\mathbb{RP}^n) \cong H^i(\mathbb{RP}^n)$ for $0 < i < n - 1$. More importantly it gives us the following, for any n:

$$0 \longrightarrow H^{n-1}(\mathbb{RP}^n) \longrightarrow H^{n-1}(\mathbb{RP}^{n-1}) \longrightarrow \mathbb{R} \longrightarrow H^n(\mathbb{RP}^n) \longrightarrow 0$$

When n is odd, this reduces even further:

$$0 \longrightarrow H^{n-1}(\mathbb{RP}^n) \longrightarrow 0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \longrightarrow 0$$

Similarly, for n even:

$$0 \longrightarrow H^{n-1}(\mathbb{RP}^n) \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \longrightarrow 0 \longrightarrow 0$$

So we see that regardless, $H^{n-1}(\mathbb{RP}^n) \cong 0$. Therefore, we can inductively see that for all $n \in \mathbb{N}$ and $0 < i < n$:

$$H^i(\mathbb{RP}^n) \cong \mathbb{R} \text{ if } i = 0 \text{ or } i = n \text{ and } n \text{ is odd} \qquad\qquad H^i(\mathbb{RP}^n) \cong 0 \text{ otherwise}$$

5.3 The Poincaré Dual

Suppose that we have a closed submanifold $\iota : S \to M$ for oriented manifold M where $\dim S = k$ and $\dim M = n$. Then, we know that for any compactly supported k-form ω, we can integrate:

$$\int_S \iota^* \omega$$

Since integration descends to cohomology, we see that S induces a linear map: $H^k_c(M) \to \mathbb{R}$. Therefore, S corresponds to an element of $(H^k_c(M))^*$ Thus, by Poincaré duality, there exists a cohomology class, which we denote $[\eta_S] \in H^{n-k}(M)$. Phrased another way, we have for any compact k-form ω:

$$\int_S \iota^* \omega = \int_M \omega \wedge \eta_S$$

Since pulling back along ι ignores parts of the manifold outside of S, we can find a form with support inside any open neighborhood of S. This is called the localization principle.

5.4 The Künneth Formula

Theorem 5.4.1 (Künneth)**.**

$$H^*(M \times N) \cong H^*(M) \otimes H^*(N)$$

This can be proven from Mayer-Vietoris, as done in [BT95, Eq. 5.9].

Example 5.4.2 ($H^*(\mathbb{T}^n)$)**.** We let $\mathbb{T}^n$ be higher dimensional tori. We define explicitly that $\mathbb{T}^n := S^1 \times \cdots \times S^1$ exactly n times. Our common notation of $\mathbb{T}$ for the two dimensional torus would be denoted $\mathbb{T}^2$. If we apply the Künneth formula:

$$\begin{aligned} H^*(\mathbb{T}^n) &\cong \bigotimes_{i=1}^n H^*(S^1) \\ H^i(\mathbb{T}^n) &\cong \bigoplus_{I \in \mathcal{I}_i} H^{I_1}(S_1) \otimes \cdots \otimes H^{I_n}(S^1) \end{aligned}$$

Where $\mathcal{I}$ is the set of all multi-indexes with n components such that $I_1, \ldots, I_n \in \mathbb{N}$ and

$$\sum_{j=1}^m I_j = i$$

We recall that $\mathbb{R} \otimes \mathbb{R} \cong \mathbb{R}$ and $\mathbb{R} \otimes 0 \cong 0$. We compute for $n = 2$:

$$\begin{aligned} H^0(\mathbb{T}^2) &\cong H^0(S^1) \otimes H^0(S^1) \\ &\cong \mathbb{R} \otimes \mathbb{R} \\ &\cong \mathbb{R} \\ H^1(\mathbb{T}^2) &\cong \big(H^1(S^1) \otimes H^0(S^1)\big) \oplus \big(H^0(S^1) \otimes H^1(S^1)\big) \\ &\cong (\mathbb{R} \otimes \mathbb{R}) \oplus (\mathbb{R} \otimes \mathbb{R}) \\ &\cong \mathbb{R}^2 \\ H^2(\mathbb{T}^2) &\cong \big(H^2(S^1) \otimes H^0(S^1)\big) \oplus \big(H^1(S^1) \otimes H^1(S^1)\big) \oplus \big(H^0(S^1) \otimes H^2(S^1)\big) \\ &\cong (0 \otimes \mathbb{R}) \oplus (\mathbb{R} \otimes \mathbb{R}) \oplus (\mathbb{R} \otimes 0) \\ &\cong \mathbb{R} \end{aligned}$$

We see the effect when $I_j > 1$, then $H^{I_1}(S^1) \otimes \cdots \otimes H^{I_n}(S^1) \cong 0$. We then see that if the multi-index I consists of elements of only $0, 1$, then $H^{I_1}(S^1) \otimes \cdots \otimes H^{I_n}(S^1) \cong \mathbb{R}$. Therefore $H^i(\mathbb{T}^n) = \mathbb{R}^m$ where m is the number of ways you can choose exactly i many 1s from n many options. This is exactly $\binom{n}{i}$. Therefore,

$$H^i(\mathbb{T}^n) \cong \mathbb{R}^m \qquad\qquad m = \binom{n}{i}$$

6 Lefschetz Fixed Point Theorem

6.1 Submanifolds of $M \times M$

Before we state and prove the main theorem, there are two vital submanifolds of $M \times M$ that are important to know. We let Δ denote the diagonal of M, and let Γ denote the graph of some fixed function $f : M \to M$. We construct these as one might expect:

$$\Delta = \{(x, y) \in M \times M : x = y\} \qquad \Gamma = \{(x, y) \in M \times M : f(x) = y\}$$

Note that both Δ and Γ are embeddings of M, by $\iota_\Delta : x \mapsto (x, x)$ and $\iota_\Gamma : x \mapsto (x, f(x))$. The last important definition we need for this section is naming the two projections. We have two projections, π, ρ

$$\begin{tikzcd} & M \times M \arrow[dl, "\pi"'] \arrow[dr, "\rho"] & \\ M & & M \end{tikzcd}$$

such that $\pi(x, y) = x$ and $\rho(x, y) = y$. Then, we can see that $\pi \circ \iota_\Delta = \rho \circ \iota_\Delta = \pi \circ \iota_\Gamma = \mathrm{id}_M$ and $\rho \circ \iota_\Gamma = f$. Since Δ is a closed and oriented submanifold, we can explicitly find η_Δ, its dual. We choose forms $\{\omega_i\}$ that are a basis of the vector space $H^*(M)$. Then there are forms $\{\tau_i\}$ that make the dual basis. That is, we have

$$\int_M \omega_i \wedge \tau_j = \delta_{i,j}$$

for all i, j. We also know, by the Künneth formula, that

$$H^*(M \times M) \cong H^*(M) \otimes H^*(M)$$

with a basis of $\{\pi^*\omega_i \wedge \rho^*\tau_j\}$. Thus, we know that

$$\eta_\Delta = \sum_{i,j} c_{i,j} \pi^*\omega_i \wedge \rho^*\tau_j$$

Now we simply must find the values of $c_{i,j}$. We construct $\pi^*\tau_l \wedge \rho^*\omega_k$ to examine η_Δ with the goal of using the using $\int_M \omega \wedge \tau = \delta$ to our advantage. To relate this form to η_Δ, we should integrate the form on Δ. We can calculate pull back the form along the diagonal

$$\int_\Delta \pi^*\tau_l \wedge \rho^*\omega_k = \int_M \iota_\Delta^*\pi^*\tau_l \wedge \iota_\Delta^*\rho^*\omega_k = \int_M \tau_l \wedge \omega_k = (-1)^{\deg \omega_k \deg \tau_l} \int_M \omega_k \wedge \tau_l = (-1)^{\deg \omega_k \deg \tau_l} \delta_{lk}$$

But by Poincaré duality, we also can compute it in the following way:

$$\begin{aligned} \int_\Delta \pi^*\tau_l \wedge \rho^*\omega_k &= \int_{M \times M} \pi^*\tau_l \wedge \rho^*\omega_k \wedge \eta_\Delta \\ &= \sum_{i,j} c_{i,j} \int_{M \times M} \pi^*\tau_l \wedge \rho^*\omega_k \wedge \pi^*\omega_i \wedge \rho^*\tau_j \\ &= \sum_{i,j} (-1)^{\deg \omega_i (\deg \tau_l + \deg \omega_k)} c_{i,j} \int_{M \times M} \pi^*\omega_i \wedge \pi^*\tau_l \wedge \rho^*\omega_k \wedge \rho^*\tau_j \\ &= \sum_{i,j} (-1)^{\deg \omega_i (\deg \tau_l + \deg \omega_k)} c_{i,j} \int_{M \times M} \pi^*\left(\omega_i \wedge \tau_l\right) \wedge \rho^*\left(\omega_k \wedge \tau_j\right) \\ &= (-1)^{\deg \omega_k (\deg \tau_l + \deg \omega_k)} c_{l,k} \end{aligned}$$

Therefore, we get that

$$(-1)^{(\deg \omega_k)^2} \delta_{l,k} = c_{l,k} = (-1)^{\deg \omega_k} \delta_{l,k}$$

We can now combine the results to get:

$$\eta_\Delta = \sum_i (-1)^{\deg \omega_i} \pi^*\omega_i \wedge \rho^*\tau_i$$

6.2 Theorem

Let M be a compact manifold, and let $f : M \to M$ be a smooth map. This induces an linear endomorphism $H^*(f) : H^*(M) \to H^*(M)$. We define

Definition 6.2.1 (The Lefschetz number)**.** The Lefschetz number of f is

$$L(f) = \sum_q (-1)^q \text{trace } H^q(f)$$

Definition 6.2.2 (Lefschetz Fixed Point Theorem)**.** If $L(f) \neq 0$, then f has no fixed points.

Proof. We prove this in two steps. First, we claim that

$$\int_\Delta \eta_\Gamma = L(f)$$

How do we see this? Well, we know by definition of Poincaré dual:

$$\int_\Delta \eta_\Gamma = \int_{M\times M} \eta_\Delta \wedge \eta_\Gamma = \int_\Gamma \eta_\Delta$$

Now we can pull back Γ along ι_Γ to integrate in M. We have

$$\int_\Gamma \eta_\Delta = \sum_i (-1)^{\deg \omega_i} \int_M \iota_\Gamma^* \pi^* \omega_i \wedge \iota_\Gamma^* \rho^* \tau_i$$

We recall that $\iota_\Gamma^* \pi^* = \text{id}$ and $\iota_\Gamma^* \rho^* = f^*$. We know that f^q is a linear endomorphism on $H^q(M)$. So we write

$$f^*(\tau_i) = \sum_k d_{i,j} \tau_j$$

for some $d_{i,j} \in \mathbb{R}$. Then we get that

$$\sum_i (-1)^{\deg \omega_i} \int_M \omega_i \wedge f^* \tau_i = \sum_i \sum_j d_{i,j} (-1)^{\deg \omega_i} \int_M \omega_i \wedge \tau_j = \sum_i d_{i,i} (-1)^{\deg \omega_i}$$

If we group the $d_{i,i}$ by the degree of ω_i, we get the trace of f^q. Thus, we get

$$\int_\Delta \eta_\Gamma = \sum_i (-1)^i \text{trace } H^i(f) = L(f)$$

We then recall that we can shrink the support of η_Γ to any open neighborhood of Γ. If and only if f has no fixed point, we have that $\Gamma \cap \Delta = \varnothing$. Then we can find open disjoint neighborhoods of Γ, Δ. Then we see that

$$\int_\Delta \eta_\Gamma = 0$$

Thus, we see that if there are no fixed points, then $L(f) = 0$. Taking the contrapositive, we see that $L(f) \neq 0$ implies f has a fixed point. Thus, we have a way to detect fixed points! □

Example 6.2.3 (Brower's fixed point theorem for smooth maps)**.** Any smooth map $f : D^n \to D^n$ has some fixed point. Since D^n is homotopic to $\mathbb{R}^n$, we see that $H^i(f)$ is the zero map for all $i > 0$ since $H^i(D^n) \cong 0$ for all $i > 0$. We recall that $H^0(M)$ are the locally constant functions. For a connected manifold, the pullback along any f must result in the same constant function. Thus, we know that $H^0(f)$ is the identity, and therefore has trace 1. We then see that $L(f) = 1$ for any smooth map, so there exists some fixed point.

Example 6.2.4 ($\mathbb{RP}^{2n}$)**.** We know that real projective space with even dimension has zero cohomology except in the zero degree. Then, as before, we have that any smooth function $f : \mathbb{RP}^{2n} \to \mathbb{RP}^{2n}$ has a fixed point.

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