The Tripod Configurations of Curves and Surfaces

Nicholas Lourie

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Chapter 1

Introduction

Given a closed curve in the plane, a tripod configuration is a choice of one point in the interior of the curve, and three points on the curve, such that the normal lines to the curve at those points form 120° angles and all intersect at a single point. This paper represents a survey of much of the current state of the study of tripod configurations. The purpose of this paper is to provide the most current results on the tripod configurations of curves, as well as to provide sufficient background for the interested reader to pursue the study of these configurations in greater depth.

1.1 Motivations

The study of special points related to and configurations associated with curves has yielded many interesting questions upon which to test the tools of topology, differential geometry, and other older kinds of geometries. Such inquiries have proved fruitful in the study of binormals, and more recently have begun to yield results in the study of tripod configurations.

Tripod configurations possess several natural contexts. These configurations are perhaps best thought of as solutions to a kind of constrained optimization problem, where one desires to minimize certain quantities while constraining solutions to specific curves. For example, if a point had three springs with one end attached to it and the other ends attached to three distinct points on a curve, then tripod configurations would represent the nontrivial critical points of the total force function. Similarly, if these springs obeyed a constant, as opposed to a linear, force law then the tripod points would represent equilibrium configurations of the system.

Besides these motivations, tripod configurations also have a deep connection with the classical Fermat-Torricelli point of a triangle, being a kind of generalization of this point to curves. Through this connection, tripod configurations are also then related to the Steiner Tree problem from computer science. These motivations, however, will be further explored in the

Section 2.1, in Chapter 2, where we will find them tremendously helpful in deciding upon the proper higher dimensional generalization.

1.2 Definitions

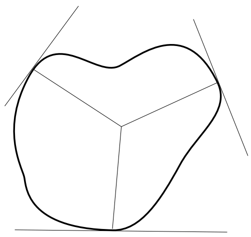
We'll now introduce the three most important definitions in the study of tripod configurations. But first, we'll need the definition of a support line.

Definition 1 (Support Line)**:** A *support line* of a curve, γ, in the plane, is a line ℓ, such that ℓ contains at least one point of γ, but γ lies entirely on one side of ℓ.

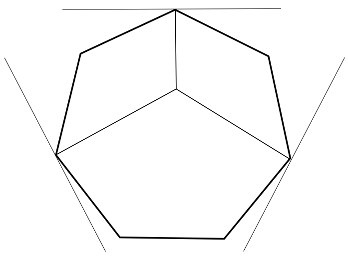
Support lines are thought of as a generalization of the notion of tangent lines. The key aspect of the definition is that a support line cannot cut a curve into two pieces.

We now introduce the definition of a tripod configuration:

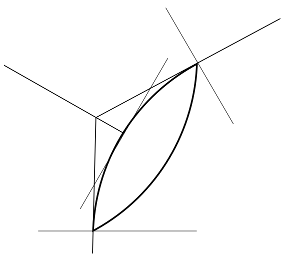
Definition 2 (Tripod Configuration)**:** Given a curve, $\gamma : S^1 \to \mathbb{R}^2$, a *tripod configuration* on that curve is a choice of three support lines such that the lines normal to the support lines passing through their points of intersection with γ all intersect at a single point and form $120°$ angles.



(a) Tripod Configuration on a Smooth Curve



(b) Tripod Configuration on a Polygon



(c) Tripod Configuration on a Piecewise Smooth Curve

Figure 1.1: Tripod Configurations on different classes of Curves

The definition I'm presenting here generalizes the existing definitions in the literature. This definition takes its inspiration from a definition formulated by Sergei Tabachnikov for the tripod configurations of a polygon, which the authors use in [5]. This more general notion allows for the study of tripod points on curves that are piecewise smooth, and unifies all the pre-existing notions under one concept. In figure 1.1, 1.1(a) illustrates a smooth curve, 1.1(b) illustrates the definition for polygons, and 1.1(c) provides an example of a tripod configuration under the general definition that does not fall into either of the previous two cases.

Note that the lines normal to the support lines are considered without respect to orientation, and that they will also necessarily form 60° angles with one another, due to the fact that they are lines and not line segments. From now on, we will also refer to the lines normal to the support lines simply as normals or normal lines.

We make two more definitions for convenience:

Definition 3 (Tripod Point)**:** Given a tripod configuration on γ, the *tripod point* for that configuration is the point, q, of intersection of the normals at p_1, p_2, and p_3.

Thus, a tripod point represents a single actual point, whereas the configuration is the entire setup.

Definition 4 (Tripod Feet)**:** The *feet* of the tripod configuration on the curve γ, are the points p_1, p_2, and p_3 on the curve.

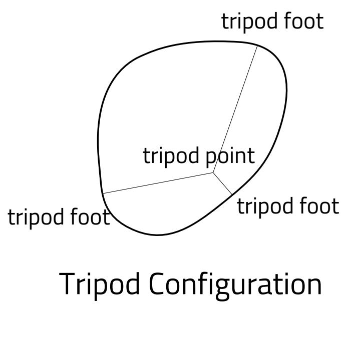


Figure 1.2: Different parts of a tripod configuration

1.3 Previous Results

In this section, we provide a summary of previous results on tripod configurations, and various citations. The most important and interesting of these results are re-proven in later sections.

The study of tripod configurations began when Sergei Tabachnikov first defined the notion in his 1995 paper, "The Four Vertex Theorem Revisited - Two Variations on the Old Theme". He initiated their study in the context of studying the four vertex theorem, and in that paper proved the following existence result:

Theorem 1. *Every C^2 convex closed curve in the plane has at least two tripod configurations. [7]*

His approach utilized a support function defined on the curve which took on zero values precisely at one of the feet of a tripod point. Because this function was the derivative of another function, it had to take on at least two zero values, proving the existence of two tripod points for the curve.

In the summer of 2013, Tabachnikov directed three undergraduates, Eric Chen, Nakul Luthra and myself, in research during an REU at the Institute for Computational and Experimental Research in Mathematics. During this time and under his suggestion, they applied a Morse theoretical approach to extend the results about tripod existence to alternative geometries. They derived the following result from this technique:

Theorem 2. *In both spherical and hyperbolic geometries, every C^2 convex curve that is a sufficiently small perturbation of a circle possesses at least two tripod configurations. [5]*

Simultaneously, they took a recent result from Lien-Yung Kao and Ai-Nung Wang bounding from below the number of tripod configurations a locally convex curve must have, and improved the lower bound to obtain the following result:

Theorem 3. *Let n be the turning number of a closed C^2 locally convex curve γ, then γ has at least $2\lceil \frac{n^2+2}{3} \rceil$ tripod configurations. [5]*

Lastly, they offered a result extending existence of tripod configurations for general C^2 curves in the plane:

Theorem 4. *Every C^2 closed curve in the plane has at least one tripod configuration. [5]*

Note that there is also a notion of tripod configuration defined for polygons in [5], and a few interesting results are obtained for it; however, this definition is not covered in this paper.

Chapter 2

Geometric and Analytic Foundations

Tripod configurations have many relations to classical geometry, and often we'll need theorems form classical geometry in order to prove the results we seek. This section develops that necessary geometric background. Moreover, as the study of tripod configurations has yielded some fruitfulness in the 2-dimensional case, people may find interest in higher dimensional analogues. When attempting to generalize a notion, it's only natural to return to its roots. Thus, in order to foreground the journey into the higher dimensional analogues of the tripod configuration, we'll begin by examining its motivation from classical geometry: the Fermat-Torricelli point, and some of the Fermat-Torricelli point's higher dimensional analogues.

2.1 Motivations

The Fermat-Torricelli point is a kind of geometric center defined for a specific class of triangles. Namely, this important point was discovered as the solution to a simple problem posed by Pierre de Fermat in a letter to Evangelista Torricelli: given three vertices in the plane, what point minimizes the sum of the distances from that point to each vertex? Torricelli afterwards resolved this problem, in what has been called the first interesting discovery associating a point to a triangle since the geometry of the Ancient Greeks. [1]

This problem is really a specific instance of the more general Euclidean Steiner Tree problem, which given n points in the plane seeks to connect them with straight line segments of minimal total length such that every point is part of the same connected component. This problem is extremely important and has wide ranging applications, though it is in general NP-hard. [2]

While ultimately what the proper higher dimensional analogue should be will remain unclear at the end of our discussion, one would imagine that such an analogue should remain faithful to the motivation coming from the Steiner tree problem, minimizing the sum of distances from $n+1$ points to a single point in $\mathbb{R}^n$. Let us now define and prove the important property of the Fermat-Torricelli point.

2.2 The Fermat-Torricelli Point

We'll begin with the obvious and appropriate definition:

Definition 5 (Fermat-Torricelli Point)**:** The *Fermat-Torricelli Point* of a triangle is the (unique) point inside of the triangle which minimizes the sums of the distances from that point to the vertices of the triangle. That is to say, F is the Fermat-Torricelli point of $\triangle ABC$ is F lies in the interior of $\triangle ABC$ and $d(A,F)+d(B,F)+d(C,F)$ is at a minimum.

In order to lay the proper context for the Fermat-Torricelli point, we must explore the notion of the isogonic centers of a triangle.[1]

Definition 6 (First Isogonic Center)**:** The *first isogonic center* of a triangle $\triangle A_1A_2A_3$ is constructed by forming an equilateral triangle at each of the sides and externally to the triangle, if we call the vertices of these equilateral triangles that were not contained in the original triangle, P_3, P_1 and P_2 respectively, then the line segments $\overline{A_1P_1}, \overline{A_2P_2}, \overline{A_3P_3}$ all intersect at a single point, I_1, and this point is known as the first isogonic center. Moreover, this point satisfies the condition:

$$\angle A_1I_1A_2 = \angle A_1I_1A_3 = \angle A_2I_1A_3 = 120^\circ$$

and is unique.

This is both a definition and a theorem, so we offer the following proof.

Proof. Consider a triangle, $\triangle A_1A_2A_3$. Then, choose the points P_1, P_2, P_3 such that we can construct equilateral triangles from the sides of $\triangle A_1A_2A_3$ as indicated in figure 2.2

First, notice that $\triangle P_2A_1A_2$ is similar to $\triangle A_3A_1P_3$ by a 60° rotation through A_1, since $\overline{A_1A_3}$ has the same length as $\overline{A_1P_2}$, and similarly for $\overline{A_1A_2}$ and $\overline{A_1P_3}$. This implies that $\angle P_3I_1A_2 = 60^\circ$, or that $\angle A_2I_1A_3 = 120^\circ$. Repeating this argument, we see that all three lines $\overline{A_1P_1}, \overline{A_2P_2}$ and $\overline{A_3P_3}$ form 120° angles with one another. Thus, I_1 must lie on the circle circumscribing $\triangle P_1A_2A_3$, since $\angle A_2P_1A_3 = 60^\circ$ implies that $\overline{A_2A_3}$ cuts a 240° arc out of this circle.

[1] Most of the material on isogonic centers to follow is adapted directly from [1]

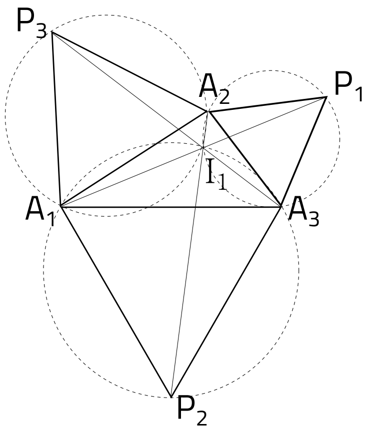


Figure 2.1: Construction of the First Isogonic Center

It is easy to see that the circles circumscribing $\triangle P_1A_2A_3$, $\triangle P_2A_1A_3$, and $\triangle P_3A_1A_2$ must intersect at a point. Moreover, since all the intersection points of the lines $\overline{A_1P_1}, \overline{A_2P_2}, \overline{A_3P_3}$ must lie on all three of these circles, the points must coincide and equal I_1.

Thus the result is proven. Moreover, I_1 is the unique point satisfying the equiangular condition because were there another point satisfying this equiangular condition, I_1', it would have to lie on all of the three aforementioned circles, but since their intersection point is unique this would imply that $I_1 = I_1'$. $\square$

Remark - To satisfy the curiosity of those wondering why it is called the first isogonic center, there is a similarly defined second isogonic center which is constructed exactly the same as the first isogonic center, except with the equilateral triangles constructed internally or facing inwards to the triangle, instead of externally. For a full treatment of the isogonic centers, see [1].

Now we stop to prove what will turn out to be a very important property of the first isogonic center, though in order to understand the property we'll need a definition, borrowed from [5].

Definition 7 (The antipedal triangle to a point)**:** Given a triangle $\triangle ABC$ and a point P, the *antipedal triangle with respect to P* is the triangle formed by the lines orthogonal to $\overline{PA}, \overline{PB}$, and $\overline{PC}$ at the points A, B, and C respectively.

Theorem 5. *Given a triangle $\triangle ABC$, an equilateral triangle T circumscribing $\triangle ABC$ is maximal if and only if it is the antipedal triangle with respect to the first isogonic center.*

The following proof is borrowed from [10].

Proof. Given a triangle $\triangle ABC$, construct a $240°$ circular arc on every side of $\triangle ABC$ such that the side is the chord defining the arc. Then any equilateral triangle circumscribing $\triangle ABC$ has its vertices lying on these circular arcs, by the fact that each vertex of an equilateral triangle forms a $60°$ angle.

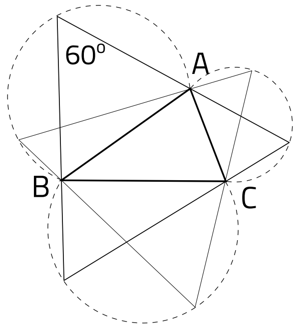


Figure 2.2: Every circumscribing equilateral triangle's vertices lie on the circular arcs

Moreover, a choice of a single side of the equilateral triangle determines the entire triangle. Therefore, finding an equilateral triangle of maximal area reduces down to finding the longest line segment whose endpoints are determined by the circular arcs.

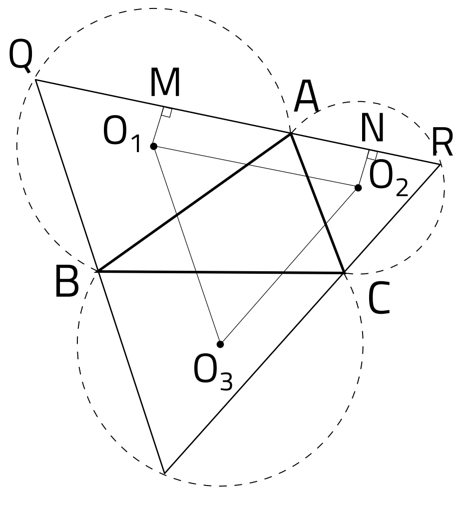


Figure 2.3:

Letting O_1, O_2, and O_3 be the centers of the circular arcs, we then take perpendicular of $\overline{QR}$ going through O_1 and O_2 and call these line segments

$\overline{O_1M}$ and $\overline{O_2N}$ respectively. See figure 2.2. $\overline{O_1M}$ must bisect $\overline{QA}$ because it is a radial line segment perpendicular to the chord, and likewise with $\overline{O_2N}$ and $\overline{RA}$.

Thus, $\overline{QR}$ is maximized when $\overline{MN}$ is, since $\overline{QR}$ is twice the length of $\overline{MN}$. Moreover, $\overline{MN}$ is shorter than $\overline{O_1O_2}$ unless they are parallel, in which case they are equal. Therefore, $\overline{QR}$ must be parallel to $\overline{O_1O_2}$ and likewise with the other sides of the maximal circumscribing equilateral triangle.

The lines perpendicular to the sides of the circumscribing equilateral triangle at A, B, and C then must form $120°$ angles with each other and thus is P is the point of intersection of two of them, P must form $120°$ angles with all the vertices of $\triangle ABC$. Thus, P must lie on the circles centered at O_1, O_2, and O_3, but we've already discussed this point of intersection and it is precisely the first isogonic center, and thus the claim is proven.

□

This theorem has an important corollary, key to the proof that every closed curve has a tripod configuration in Theorem 11.

Corollary 1. *If T is the equilateral triangle of maximal area circumscribing the triangle $\triangle ABC$, then the normals to the sides of T at A, B, and C all intersect at a point.*

One more geometric object that we will encounter in our later proofs is the *cyclic quadrilateral.* We provide a brief discussion of them here.

Definition 8 (Cyclic Quadrilateral)**:** A *cyclic quadrilateral* is a quadrilateral of four of whose vertices my be placed upon a circle.

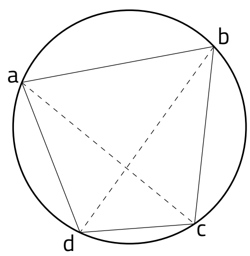


Figure 2.4: A cyclic quadrilateral

Take figure 2.2 as a typical example of a cyclic quadrilateral, then we are primarily interested in two facts about cyclic quadrilaterals that are both necessary and sufficent (thus the facts are equivalent).

1 Opposite angles are supplementary.

2 $\angle dac = \angle dbc$

Recall that an inscribed angle, such as $\angle dac$, is half of the center angle that defines the same arc length. Since the arcs defined by opposite angles sum to the whole circle, the sum of the angles must be $\frac{360°}{2} = 180°$. Likewise, since $\angle dac$ and $\angle dbc$ define the same chord, they must be equal as angles.

The content of the following theorem gives the relation between the Fermat-Torricelli point and the first isogonic center.

Theorem 6. *Let F be the point which minimizes $d(A,F)+d(B,F)+d(C,F)$ for points A, B, and C.*

If $\triangle ABC$ contains no angle greater than $120°$, F is the first isogonic center of $\triangle ABC$, otherwise F is the vertex of $\triangle ABC$ that has an angle of $120°$ or greater.

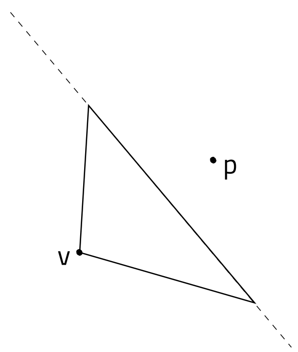
The following proof is adapted from [3]

Proof. Let $f(P) = d(A,P) + d(B,P) + d(C,P)$.

First, we'll prove that should a minimum exist, it must lie inside $\triangle ABC$ (implying that the minimum does exist, since the triangle is compact). Next, we will characterize the critical points of f inside $\triangle ABC$. Lastly, we will show that if $\triangle ABC$ contains an angle of $120°$ or greater, that vertex is the Fermat-Torricelli point, and otherwise the Fermat-Torricelli point is the first isogonic center.

The Minimum must lie inside $\triangle ABC$

Assume that P lies outside of $\triangle ABC$. Then for some vertex, v, the side of the triangle not containing v must extend to a line such that P lies on the opposite side of this line as the vertex, otherwise P would lie inside of $\triangle ABC$.



Projecting P onto the side opposite v reduces all three quantities, $d(A,P), d(B,P)$, and $d(C,P)$. Therefore, it reduces $f(P)$.

Therefore, the Fermat-Torricelli point must lie inside of the triangle. Note also, that since the triangle is compact and f is continuous, this also proves the existence of a minimum.

Characterization of the critical points of f

f is smooth everywhere except for when $P = A$, $P = B$ or $P = C$. Thus, as long as P does not lie on the vertices of $\triangle ABC$, we have the critical points exist where $\nabla f(P) = 0$.

Fixing A, B and C, notice that $\nabla f(P) = \nabla d(A, P) + \nabla d(B, P) + \nabla d(C, P)$. In general, $\nabla d(X, P)$ is the unit vector pointing away from X along the line $\overline{XP}$. Therefore, if a, b, and c are the unit vectors in the directions $\overline{AP}, \overline{BP}, \overline{CP}$, then the condition that $\nabla f(P) = 0$ is equivalent to $a+b+c = 0$. In general, the sum of three unit vectors in the plane can only be zero if they form $120°$ angles with each other.

Finding the Fermat-Torricelli Point

Assume $\triangle ABC$ has no angles of $120°$ or greater.

For contradiction, assume without loss of generality that A is the Fermat-Torricelli point. Then consider the directional derivative along the bisector of $\angle BAC$. This cannot possibly be positive if $A < 120°$. This proves that the vertex cannot be a Fermat-Torricelli point since moving inwards slightly would reduce the value.

Since the Fermat-Torricelli point must be in the interior of the triangle and must satisfy the equiangular condition, then by the uniqueness of the first isogonic center, the first isogonic center must be the Fermat-Torricelli point.

If $\triangle ABC$ has an angle of $120°$ or greater, then the minimum cannot be achieved in the interior and so, since the extrema of a convex function on a convex set must occur at the extreme points of the convex set, the Fermat-Torricelli point must occur at one of the vertices. Since the longest side of the triangle is opposite the $120°$ or greater vertex, then obviously this vertex must minimize the sum of the distances function. $\square$

This provides the main results from classical geometry on the Fermat-Torricelli point. Relating them back to tripod configurations, these results show that for a given tripod configuration, the tripod point actually is the Fermat-Torricelli point of the feet, provided that the feet do not form a very obtuse triangle. This motivation also hints at the relationship between tripod points and the sum distances function that may be used to study them. Importantly, the idea is that the sum of the distnaces from a point inside a curve to 3 points on the curve provides a the foundation for a Morse Theoretical approach to proving the existence of tripod configurations, which is developed in [5]. One appeal of this approach is that it should generalize in a fairly straightforward manner to an arbitrary number of dimensions, though there do exist some technical difficulties[2]. Lastly, this key relationship with classical geometry plays an important role in the proof that every closed curve in the plane has a tripod configuration. Now, we'll turn a generalization of the Fermat-Torricelli point from the literature.

2.3 Solid Angle and the Fermat-Toricelli Point of Tetrahedra

First, we'll explore the most geometric generalization of the Fermat-Toricelli point. This generalization of the Fermat-Toricelli point to tetrahedra goes back to Lorenz Lindelöf, who in 1867 solved the problem of minimizing the sum of the distances from 4 points in $\mathbb{R}^3$ to a point P. [3][8]

In order to understand this generalization, we must first understand the notion of solid angle, once armed with that understanding we review a modern proof that this minimum does indeed satisfy an equiangular property similar to that of the Fermat-Toricelli point in the plane.

2.3.1 Solid Angles

Using radians, we think of angles in terms of the ratio of the circumference of a circle that two lines or, more generally, the projection of some object, takes up on the unit circle. If one imagines living in the plane, then the size of a connected object would be the angle made in one's vision by the endpoints of that object. Thus, this is the natural perspective to take when generalizing angles to $\mathbb{R}^3$. This inspires the following definition:

Definition 9 (Solid Angle)**:** Let $S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$ be the unit sphere, $\mu : S^2 \to \mathbb{R}_{\geq 0}$ be the measure of the surface area of a subset of S^2. Then the *solid angle* of a submanifold $M \subset \mathbb{R}^3$ is the measure, μ, of the radial projection of M onto the unit sphere.

The notion of the solid angle of an object makes rigorous the idea of how large something appears in our field of vision. Intuitively, solid angle corresponds to how much of our view an object takes up.

The key distinction between the three and two dimensional cases for angles is that while the projection of any connected object onto the circle can be described with two lines, the projection of an object onto the sphere cannot. Thus, we cannot always think of a solid angle as being defined by 3 vectors, as certain shapes will create much more complex projections onto S^2.

[2] which is why we chose not to explore the Morse Theoretical approach in this paper, seeing great technical impediments to furthering or generalizing it. See the discussion at the end of section 2.4

That said, the aforementioned more general notion provides an obvious definition for the measure of the angle that 3 vectors make in $\mathbb{R}^3$:

Definition 10 (Solid Angle of Three Vectors)**:** Given three vectors $x, y, z \in \mathbb{R}^3$, the solid angle between them from point p, (p, abc) is the measure, μ, of the spherical triangle made by the intersection points of the rays $\overline{pa}, \overline{pb}, \overline{pc}$ with the unit sphere at p.

It is also worth noting, that in spherical coordinates there exists a very simple formula for computing the solid angle of a surface, name, letting $sr(S)$ denote the solid angle of the surface S:

$$sr(S) = \iint_S \sin\theta d\theta d\phi$$

See [11] to read more about solid angles.

2.3.2 Fermat-Torricelli Point of Tetrahedra

The minimization problem from which we derive the definition of the Fermat-Torricelli point of a triangle, has an obvious generalization to tetrahedra:

Definition 11 (Fermat-Torricelli Point of a Tetrahedron)**:** The *Fermat-Torricelli point of a tetrahedron* defined by four points, $A, B, C, D \in \mathbb{R}^3$ in general position, is the point inside the tetrahedron which minimizes the sum, $f(P) = d(A, P) + d(B, P) + d(C, P) + d(D, P)$.

Since we have motivated our generalization using the minimization definition of the Fermat-Torricelli point, it is only natural to ask if our equiangular condition will be satisfied as well in three dimensions. The answer to this question is the content of the next theorem:[3]

Theorem 7. *Let $\Delta ABCD$ be a tetrahedron, and P be the point minimizing the function:*

$$f(P) = d(P, A) + d(P, B) + d(P, C) + d(P, D)$$

Then if $\Delta ABCD$ has a vertex of solid angle π or greater, that vertex is P, otherwise every pair of line segments $\overline{PA}, \overline{PB}, \overline{PC}$, and $\overline{PD}$ form the same angles and every triple has solid angle π.

Proof. Let A, B, C, and D be four points in $\mathbb{R}^3$, and let

$$f(P) = d(P, A) + d(P, B) + d(P, C) + d(P, D)$$

[3]For the original proof and a more complete treatment, see [3]. You may compare this generalization to the one in [4]

First, we attempt to find the point minimizing the sum of these distances. We prove a lemma.

Lemma 1. *The point minimizing f must lie inside the tetrahedron $\Delta ABCD$.*

Proof. Assume the minimum, P, lies outside of $\Delta ABCD$. Then there must exist some side of $\Delta ABCD$ that when extended to a plane separates P and some vertex of $\Delta ABCD$ on opposite sides. WLOG, assume this vertex is A and the side is the plane, BCD.

The orthogonal projection of P onto BCD reduces all three quantities $d(P, A), d(P, B), d(P, C)$ and $d(P, D)$. Therefore, P cannot be the minimum of f, a contradiction. This implies that the minimum must exist inside (including the boundary) of $\Delta ABCD$. □

Since $\Delta ABCD$ is compact and f is continuous, this implies the existence of a minimum and that the minimum must lie inside the tetrahedron.

The minimum can only occur where either $\nabla f = 0$ or where ∇f does not exist, namely A, B, C, or D.

Again, the components of the gradient are unit vectors pointing in the directions, $\overline{PA}, \overline{PB}, \overline{PC}$, and $\overline{PD}$. Thus, denoting these unit vectors as $\mathbf{a}, \mathbf{b}, \mathbf{c}$, and $\mathbf{d}$ respectively, the minimum condition where ∇f exists is:

$$\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = 0$$

Thus, the key to understanding the Fermat-Torricelli points of a tetrahedron is simply understanding when four unit vectors may add to zero in $\mathbb{R}^3$. The following lemma describes when this occurs:

Lemma 2. *If four unit vectors, $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in \mathbb{R}^3$, add to the zero vector, then any three of the unit vectors must form a solid angle of π if the four vectors are in general position. Moreover, it is always the case that if we measure the angle of any two of the vectors in the plane that they form, this is equal to the angle formed by the other two vectors.*

Proof. First observe that:

$$\begin{aligned}
\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = 0 &\implies \\
\mathbf{a} + \mathbf{b} &= -(\mathbf{c} + \mathbf{d}) \\
(\mathbf{a} + \mathbf{b}) \bullet (\mathbf{a} + \mathbf{b}) &= -(\mathbf{c} + \mathbf{d}) \bullet -(\mathbf{c} + \mathbf{d}) \\
2 + 2\mathbf{a} \bullet \mathbf{b} &= 2 + 2\mathbf{c} \bullet \mathbf{d} \\
\mathbf{a} \bullet \mathbf{b} &= \mathbf{c} \bullet \mathbf{d}
\end{aligned}$$

We can do this with all pairs to obtain that:

$$\mathbf{a} \bullet \mathbf{b} = \mathbf{c} \bullet \mathbf{d}, \;\; \mathbf{a} \bullet \mathbf{c} = \mathbf{b} \bullet \mathbf{d}, \;\; \mathbf{a} \bullet \mathbf{d} = \mathbf{b} \bullet \mathbf{c}$$

Which proves that any two vectors has the same angle as the other two vectors; however, this is *not* to say that all the possible pairs of vectors for the same angle, only complementary pairs[5].

We may then calculate:

$$\begin{aligned}
(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d})^2 &= 0 \\
4 + 2(\mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c} + \mathbf{a} \bullet \mathbf{d} + \mathbf{b} \bullet \mathbf{c} + \mathbf{b} \bullet \mathbf{d} + \mathbf{c} \bullet \mathbf{d}) &= 0 \\
4 + 4(\mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c} + \mathbf{b} \bullet \mathbf{c}) &= 0 \\
\mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c} + \mathbf{b} \bullet \mathbf{c} &= -1
\end{aligned}$$

When the vectors $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ are in general position, this condition then means that the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ form a solid angle of π. Of course, this condition then holds for all of the triples of vectors. Thus, the four vectors together divide the sphere up into equal are triangular regions.

□

Now we'll prove that the vertices cannot be minima of the function if they have a solid angle less than π.

Since $(A, BCD) < \pi$, we have that $1 + \mathbf{b} \bullet \mathbf{c} + \mathbf{b} \bullet \mathbf{d} + \mathbf{c} \bullet \mathbf{d} > 0$. Thus:

$$|\mathbf{b}+\mathbf{c}+\mathbf{d}|^2 = 3 + 2(\mathbf{b} \bullet \mathbf{c} + \mathbf{b} \bullet \mathbf{d} + \mathbf{c} \bullet \mathbf{d}) > \mathbf{b} \bullet \mathbf{c} + \mathbf{b} \bullet \mathbf{d} + \mathbf{c} \bullet \mathbf{d} > 1$$

Thus we have $|\mathbf{b}+\mathbf{c}+\mathbf{d}| > 1$. The directional derivative of f at A along $\mathbf{b}+\mathbf{c}+\mathbf{d}$ is then, if we introduce the unit vector $\mathbf{e} = \frac{\mathbf{b}+\mathbf{c}+\mathbf{d}}{t}$:

$$\nabla f \bullet (\mathbf{b}+\mathbf{c}+\mathbf{d}) = 1 + \mathbf{e} \bullet (-\mathbf{b}-\mathbf{c}-\mathbf{d}) = 1 - t < 0$$

Thus, A could not possibly be a minimum.

This discussion then gives us two cases. In the first case, no vertex of $ABCD$ has solid angle greater than or equal π, and thus the minimum must occur in the interior of the tetrahedron where the vectors $\overline{PA}, \overline{PB}, \overline{PC}$, and $\overline{PD}$ satisfy the conditions given by lemma 2. Otherwise, if there is a vertex with solid angle greater than or equal to π, then no interior point can satisfy the angular condition, and thus that vertex with a solid angle of π is the only eligible point to be the minimum. □

[5]see the discussion following this proof

For people familiar with chemistry and the VESPR theory of chemical bonding, the tetrahedral molecular geometry is an example of a shape that the line segments $\overline{PA}, \overline{PB}, \overline{PC}$, and $\overline{PD}$ could form. Methane, or CH_4, is one example of a molecule with this geometric structure. Someone who recalls this theory, might also recall that the angles formed by these ligands are $\cos^{-1}(-\frac{1}{3}) \approx 109.5°$.

This relationship to chemistry, of course, is no coincidence, as such molecules take this shape because it spreads the ligands as far apart as possible. It is tempting to wonder if the minimization of the sum of distances to the vertices of the tetrahedron is not somehow related to this notion of maximal separation of the ligands, and tripod configurations on surfaces could have some connection to chemical models.

Of course, this shape is not the only one that such a minimization could take at the Fermat-Torricelli point. Intuitively, one could always place unit vectors in pairs, and for whatever angle you want between the unit vectors, and then pair the other 2 unit vectors at the same angle, placing them with opposite orientation to the first pair. See figure 2.5 for an illustration. In fact, our lemma basically says that this is the only way for the vectors to sum to zero.

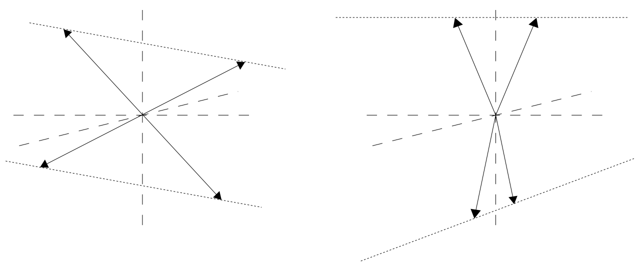


Figure 2.5: Four unit vectors summing to zero in two configurations

2.4 Tetrapod Configurations

As we saw in the previous section, the geometric motivation for tripod configurations, namely the Fermat-Torricelli point, becomes more complex when examining it in higher dimensions. Namely, while some equiangular conditions hold on the Fermat-Torricelli point, perfect symmetry does not hold.

Because of this complication, one might expect that we should weaken the definition for the three dimensional case. This motivates the following definition:

Definition 12 (Tetrapod Configuration)**:** Given a closed surface, $\sigma \subset \mathbb{R}^3$, a tetrapod configuration of of σ is a choice of four lines orthogonal to supporting planes of σ at the points of intersection with σ, such that the lines all coincide at a single point and every triple of vectors forms a solid angle of π.

It is important to remark that in the proof of Theorem 7, we established that for a given surface σ, the function

$$f : \sigma^4 \times \mathbb{R}^3 \to \mathbb{R}$$

defined by, letting $u, x, y, z \in \sigma$ and $p \in \mathbb{R}^3$

$$(u, x, y, z, p) \mapsto d(u, p) + d(x, p) + d(y, p) + d(z, p)$$

has critical points exactly at the tetrapod configurations of σ. Thus, one could imagine using a Morse Theoretical approach to prove existence of tetrapod configurations similar to that taken in [5]; however, such an approach will need to be more complex than the treatment in [5]. Unlike in the two dimensional case, one runs into the existence of dense submanifolds of critical points in the 3-dimensional case when adapting the approach used in [5].

The other approach one might take to proving the existence of tetrapod configurations would be to adapt the geometric methods presented in this papers proof of Theorem 11. In Chapter 5, we'll outline the difficulties in trying to adapt these geometric methods to the three dimensional case. Because of the complications in the geometry of the tetrahedron introduced by having a third dimension, it is the belief of the author that tetrapod configurations do not exist as generally as do tripod configurations.

But for now, we return to tripod configurations in the plane.

Chapter 3

Tripod Configurations in the Plane

3.1 Tripod Configurations for Closed Convex Curves

In this section, we'll present the original result proved by Sergei Tabachnikov when he defined the notion of a tripod configuration. This result establishes the existence of at least 2 tripod configurations for closed convex curves in the plane. Quite importantly, it establishes the relationship between tripod configurations and the area of the circumscribing equilateral triangle of the curve. Namely, we'll see that the points of contact between the largest and smallest area triangles and the curve are precisely the feet of a tripod configuration.

This property will play a key role in the following section, where we'll explore curves with dense tripod configurations. We now present the first result:

Theorem 8 (Convex Tripod Configuration Theorem)**.** *Every C^2 closed curve has at least two tripod configurations.*

Proof. Let $\gamma : S^1 \to \mathbb{R}^2$ be a closed convex curve. We'll use the variable, s to represent arc length, and α to represent the angle made by $\gamma'(s)$ with a fixed direction. It is a standard result in differential geometry that a convex curve may be parametrized by arc length or angle, thus we'll consider γ to be parametrized by both arc length and angle, respectively denoted by $\gamma(s)$ and $\gamma(\alpha)$.

Define $\ell(\alpha)$ to be the line normal to γ at α. Define a support function $p : S^1 \to \mathbb{R}$ by choosing some point O and defining $p(s) = d(\ell(\alpha), O)$, the signed distance between the line $\ell(\alpha)$ and the point O.

Recall that if we denote the tangent and normal to γ at s by $t(s)$ and $n(s)$ respectively, then $t(s) = \gamma'(s)$ and $n(s) = \frac{\gamma''}{\|\gamma''\|}$. Letting $v_1 \wedge v_2$ denote the determinant of vectors v_1, v_2, we have:

$$p(\alpha) = \gamma(\alpha) \wedge n(\alpha)$$

Notice that p is actually derivative of a function, q:

$$q(\alpha) = \gamma(\alpha) \wedge t(\alpha) \implies p(\alpha) = \frac{dq}{d\alpha} = \frac{\gamma \wedge \gamma''}{\|\gamma''\|}$$

There are convenient geometric interpretations of these two functions, which then make the result immediate. Namely, the tangent lines at $\alpha, \alpha + \frac{2\pi}{3}$, and $\alpha + \frac{4\pi}{3}$ form a large equilateral triangle, and the normal lines form a smaller equilateral triangle. Then if a_n is the area of the smaller triangle defined by the normals and A_t is the area of the larger triangle defined by the tangents, we have:

$$a_n = \frac{1}{\sqrt{3}} \left[p\left(\alpha\right) + p\left(\alpha + \frac{2\pi}{3}\right) + p\left(\alpha + \frac{4\pi}{3}\right) \right]^2$$

$$A_t = \frac{1}{\sqrt{3}} \left[q\left(\alpha\right) + q\left(\alpha + \frac{2\pi}{3}\right) + q\left(\alpha + \frac{4\pi}{3}\right) \right]^2$$

This is because the sum of the distances from a point inside an equilateral triangle to the sides of the equilateral triangle is the height of the triangle. Moreover since if h is the height and b is the side length of an equilateral triangle, then $b = \frac{2}{\sqrt{3}}h$ and so $a = \frac{1}{2}bh = \frac{1}{\sqrt{3}}h^2$.

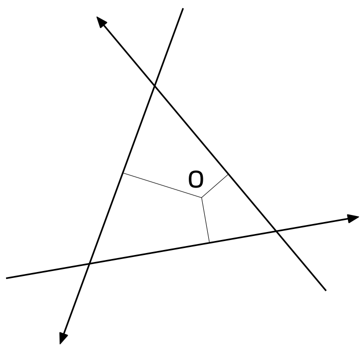


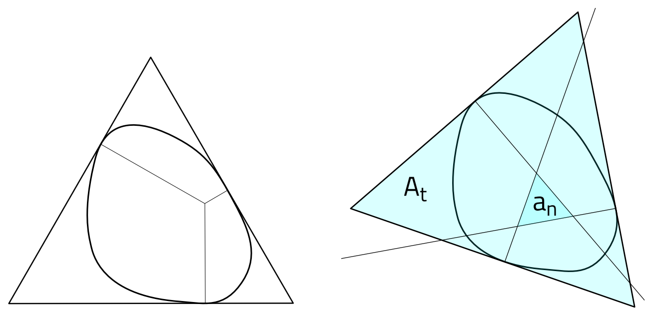
Figure 3.1: The sum of the signed distances' relation to area of triangle

Since $p = \frac{dq}{d\alpha}$, the above equations imply that A_t achieves a maximum precisely when $a_n = 0$. The size of A_t varies continuously around γ, and γ is compact, therefore there must exist some $\alpha_{max}, \alpha_{min}$ at which A_t achieves a maximum and a minimum respectively. At these points, we must correspondingly have that $a_n = 0$, which precisely says that the normals

$$n(\alpha)\,,\ n\left(\alpha+\frac{2\pi}{3}\right),\ n\left(\alpha+\frac{4\pi}{3}\right)$$

all intersect at a single point when $\alpha = \alpha_i$ or α_j. Since these normals form 120° angles by construction, this precisely says that there exist at least two tripod configurations for γ. $\square$

It is crucially important to note the relation established between the equilateral triangle that the tangents make, and the triangle formed by the normals. In particular, tripod configurations exist exactly when the triangle formed by these tangents, that is to say the circumscribing equilateral triangle, has maximal or minimal area.



(a) Tripods exist when the circumscribing equilateral triangle has maximal or minimal area

(b) illustration of A_t and a_n

Figure 3.2: The Tangent and Normal Equilateral Triangles

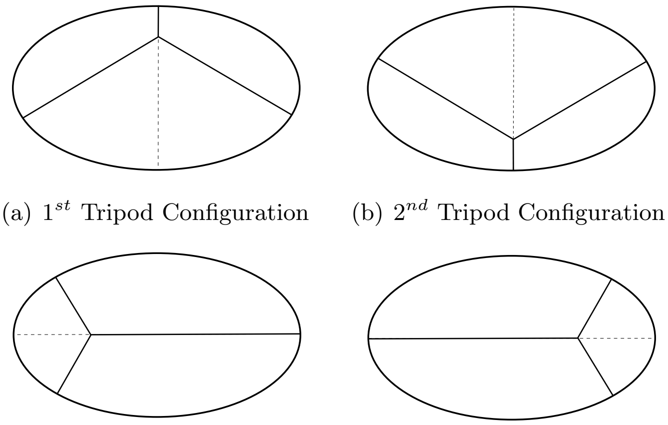
This relationship between circumscribing equilateral triangles and tripod configurations will play a key role when we seek to prove that every C^2 closed curve in the plane has a tripod configuration, and when we examine curves which have dense tripod configurations.

It may be tempting to try and generalize this method of proof to alternative geometries, the complication however comes from the support function. Because of the presence of spherical excess and analogous phenomena in non-planar geometries, the areas of the triangles, A_t and a_n posses complications and it is unclear exactly how to adapt these methods.

Of course, due to the necessity of the ability to parametrize the curve by angle in this method of proof, it cannot be easily adapted for general non-convex curves; however, one may still parametrize a curve by the angle of the tangent line if that curve is not convex but instead only locally convex, and we review results exploring exactly this possibility in the next section.

I shall make one final remark before we begin the next section. It is not known whether the preceeding bound is sharp for C^2 convex curves.

Conducting computational experiments with families of ellipses, it seems to be the case that every ellipse has four tripod configurations, two for each diameter or binormal of the curve. The diameters are normals of the curve which intersect the curve perpendicularly at both points of intersection.



(a) 1^{st} Tripod Configuration (b) 2^{nd} Tripod Configuration

(c) 3^{rd} Tripod Configuration (d) 4^{th} Tripod Configuration

Figure 3.3: Tripod Configurations of the Ellipse

A rigorous proof of this fact for ellipses, as well as an example proving whether or not the bound of two tripod configurations for any given curve is sharp, remain open questions to be solved; however, if tripod configurations do appear in pairs for every binormal (as they obviously do when the curve is symmetric about the binormal), then because binormals always appear in pairs, the number of tripod configurations would always be divisible by 4. The Morse theoretical approach to bounding the number of tripod configurations on a curve from below strongly suggests that there is indeed some deep relationship between binormals and tripod configurations. Exploring this relationship would again be a fruitful area for potential study.

3.2 Tripod Configurations for Locally Convex Curves

The previous proof suggests that one could potentially relate the number of tripod configurations of a curve to the rotation number of the curve, that is, the number of times the tangent vector to the curve spins in a complete circle. For locally convex curves, the proof provides a very clear approach of forming this relation. This approach was first taken up by Kao and Wang in [6], and then improved in [5]. We present this second result, as it fully elucidates the relation between rotation number and tripod configurations:

Theorem 9. *Let γ be a C^2 closed locally convex curve with rotation number*

n, then γ has at least $2n^2$ tripod configurations.

Proof. Given a closed locally convex C^2 plane curve γ, parametrize it by the angle that the tangent vector makes with a fixed direction. We'll denote this parametrization again by $\gamma(\alpha)$. The proof of Theorem 8 shows that there exists a tripod configuration whenever

$$p(\alpha) + p\left(\alpha + \frac{2\pi}{3}\right) + p\left(\alpha + \frac{4\pi}{3}\right) = 0$$

More generally, this motivates us to define the family of functions:

$$P_{i,j,k}(\alpha) = p\left(\alpha + i\frac{2\pi}{3}\right) + p\left(\alpha + j\frac{2\pi}{3}\right) + p\left(\alpha + k\frac{2\pi}{3}\right)$$

where $i, j, k \in \{0, 1, \ldots, n-1\}$ and $i \equiv 0$, $j \equiv 1$, $k \equiv 2 \mod 3$. This condition on the moduli of i, j, and k guarantees that the normals will intersect at $120°$ angles. From our previous proof, we can see that these functions equal zero precisely when there is a tripod configuration.

We want to identify two functions, $P_{i,j,k}$ and $P_{i',j',k'}$, when the triple of normal angles $\{x+i, x+j, x+k\}$ can be rotated into the triple $\{x'+i', x'+j', x'+k'\}$. Since rotation corresponds to a change in the x or x' variable, this condition is equivalent to the differences of all the components all being equal, or:

$$(x+i) - (x'+i') = (x+j) - (x'+j') = (x+k) - (x'+k')$$

Since we're only considering triples $\{i, j, k\}$ we can reduce this to the condition:

$$i - i' = j - j' = k - k'$$

In other words, when $\{i, j, k\} = \{i'+m, j'+m, k'+m\} \mod 3n$, where again n is the rotation number of the curve, and for some $m \in \mathbb{Z}$. Each of these equivalence classes of functions then identifies at least two distinct distinct tripod configurations.

Thus, we turn ourselves to counting the number of equivalence classes $\{i, j, k\}$. WLOG, we may assume that $i = 0$, since every other equivalence class may be obtained as an integer translation of these representative elements.

Thus, we want to know when $j - j' = k - k' = i - i'$ in this class, but $i - i' = 0$ for all these representatives, therefore, $j = j'$ and $k = k'$. This proves uniqueness. Then, we must simply count the number of choices for j and k, but there are exactly n choices for j and n choices for k. Therefore, there are n^2 choices of elements $\{0, j, k\}$. Since every choice provides a unique function that yields two distinct tripod configurations, that means the γ has $2n^2$ distinct tripod configurations.

$\square$

As with our previous theorem, it remains an open question whether or not these bounds are sharp. We now, in our final section, move on to the most general current result about tripod configurations for curves in the plane.

3.3 Tripod Configurations for General Closed Curves

We at last turn our attention to general curves in the plane. In this section, we'll see that every curve in the plane contains at least one tripod configuration. The original proof of this theorem is due to [5]. Their proof utilizes the ideas from the proof of Theorem 8, particularly the relationship proven between the maximal area circumscribing equilateral triangles and tripod configurations on C^2 curves. Using this concept, they show that all C^2 curves in the plane have at least one tripod configuration; however, it turns out that the analytic techniques utilized in the proof of Theorem 8 are unnecessary, and this lemma about the maximal circumscribing equilateral triangle can actually be replaced with a purely geometric version which holds for any closed curve in the plane.

Thus, while in [5] they were able to show the following:

Theorem 10. *Every C^2 closed curve in the plane has at least one tripod configuration*

We will extend their ideas, removing the differentiability condition and proving:

Theorem 11. *Every closed curve in the plane has a tripod configuration*

Proof. Let γ be a closed curve in the plane.

Lemma 3. *If T is a maximal equilateral triangle circumscribing γ and intersecting γ at points A, B, C, then T is also a maximal equilateral triangle circumscribing $\triangle ABC$.*

Proof. By $A(X)$, denote the area of the figure X.

Let T_{max} be a maximal equilateral triangle circumscribing γ and intersecting γ at points A, B, C. Let T be the maximum equilateral triangle circumscribing $\triangle ABC$. Then since A, B, and C are all points on the curve

γ, either T is a triangle circumscribing γ, or at least one of the sides of T intersects γ at more than one point, in which case by translating these sides outwards, we can form an equilateral triangle T' that both circumscribes γ and contains T in its interior. Thus, T' has area greater than or equal to the area of T, and T' is a circumscribing triangle of γ.

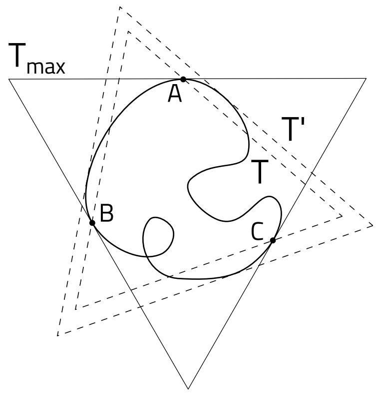


Figure 3.4: T_{max} is contained in an equilateral triangle circumscribing γ

Therefore, we either have that T is a circumscribing triangle of γ and thus:

$$A(T) \leq A(T_{max})$$

since T_{max} has maximal area among the equilateral triangles circumscribing γ. Or, we have T is contained in T' and therefore,

$$A(T) \leq A(T') \leq A(T_{max}) \implies A(T) \leq A(T_{max})$$

Similarly, since T has maximal area among the equilateral triangles circumscribing $\triangle ABC$, we have:

$$A(T_{max}) \leq A(T)$$

$$\therefore A(T_{max}) = A(T)$$

□

Recall that any triangle has a unique equilateral triangle of maximal area circumscribing it, and that is the antipodal triangle of the first isogonic center. Then, the previous theorem implies that any triangle of maximal area circumscribing a plane figure is precisely the antipodal triangle of the first isogonic center of its points of intersection with the curve. Since the sides of the circumscribing equilateral triangle are all support lines of γ, this statement precisely says that the points of intersection are the feet of a tripod configuration. □

Thus, it is the case that every closed curve in the plane has a tripod configuration as a very simple geometric fact, regardless of differentiability. Even fractal curves and nowhere differentiable curves such as the Koch Snowflake must have at least one tripod configuration in the general sense.

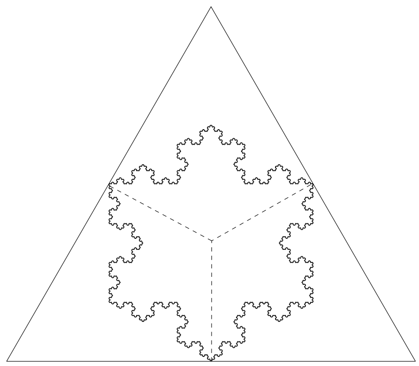


Figure 3.5: A Tripod configuration of the Koch Snowflake

By analogy to the case for convex tripods, one may hope that there is another tripod configuration for any general curve corresponding to the circumscribing equilateral triangle of minimal area, although how such a relationship or proof would be teased out is not obvious.

As a final note, it seems highly probably that for C^1 curves, tripod configurations always come in pairs. We'll formally state this conjecture in the final section, but it's unclear whether or not the general case should also have this parity condition.

For the better behaved case of C^2 locally convex curves, the tripod configurations should generically appear in pairs because they are the critical points of a function defined on the circle. The Morse theoretical results confirm this fact for tripod configurations in alternative geometries on curves sufficiently similar to a circle[1]. We will also provide some discussion on the parity of tripod configurations for the general C^1 case in the plane as well in Chapter 5, where we'll outline the approach for a proof–though technical details remain unfinished.

If the general case were to have a parity condition on the tripod configurations, that would certainly be very interesting, but now we turn our attention to topic besides existence, that is to say, analyzing the possibilities for dense tripod configurations in a curve.

[1]see [5]

Chapter 4

Δ-curves and Dense Tripod Configurations

The circle presents an interesting example of a curve, for the study of tripod configurations. While the circle has only a single tripod *point*, it has infinitely many tripod configurations. In fact, rather obviously, every point on the circle is a foot of some tripod configuration.

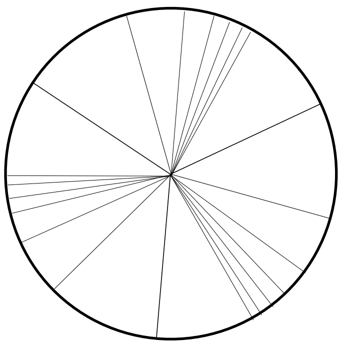


Figure 4.1: The circle has infinitely many tripod configurations

Moreover, it can rather easily be shown that the circle is the only curve with this property, which is the content of the following proposition:

Proposition 1 (Tripod Configuration Characterization of a Circle)**.** *If* $\gamma : S^1 \to \mathbb{R}^2$ *is a closed* C^1 *curve with a single tripod point,* p*, and the feet of the tripod configurations at* p *are dense in* γ*, then* γ *is a circle centered at* p*.*

This result is mostly just a straightforward application of a standard result from differential geometry saying that if all the normals of a curve intersect at a point, then the curve has constant curvature.

We'll prove the characterization of the circle in two lemmas, the first lemma proves the differential geometry result, the second lemma proves the proposition. The proposition immediately follows, but first we'll make a quick definition:

Definition 13: Given a closed curve γ, and a subset $S \subset \gamma$, then by $\mathcal{F}_S$ we'll denote the set of points $q \in S$ such that q is the foot of some tripod configuration of γ.

We now proceed with our two lemmas:

Lemma 4. *If every normal to a C^1 curve passes through a single point, then that curve has constant curvature.*

Proof. Assume every normal to a curve, γ, passes through the point P. Let P be the origin of our coordinate system, then consider that:

$$\frac{d}{ds}\left(\gamma(s) \bullet \gamma(s)\right) = \frac{d}{ds}\gamma(s) \bullet \gamma(s) + \gamma(s) \bullet \frac{d}{ds}\gamma(s) = 2\gamma(s) \bullet \frac{d}{ds}\gamma$$

But since all the normals pass through the origin, the tangent $\frac{d}{ds}\gamma$ is always orthogonal to $\gamma(s)$, therefore:

$$\frac{d}{ds}\left(\gamma(s) \bullet \gamma(s)\right) = 0$$

So in particular, $\|\gamma(s)\|$ is constant, which proves that γ is circular. $\square$

Lemma 5. *If a closed C^1 curve, γ, has only one tripod point but dense tripod configurations, then that curve is the circle.*

Proof. On the dense set of tripod feet in the curve, the normal lines all intersect at the singular tripod point. By continuity, this implies that all normal lines of the curve must intersect the tripod point. Thus, by our previous lemma, γ must have constant curvature and be the circle. $\square$

Notice that the preceding proposition also has the simple, but somewhat interesting, corollary:

Corollary 2 (Constant Curvature Corollary)**.** *If p is a tripod point of γ, and $S \subset \gamma$ is the set of feet of p, then each connected component of the closure of S has constant curvature.*

In the above corollary, we must say along each component of the closure of S because S could potentially consist of pieces of multiple concentric circles as in figure 4.

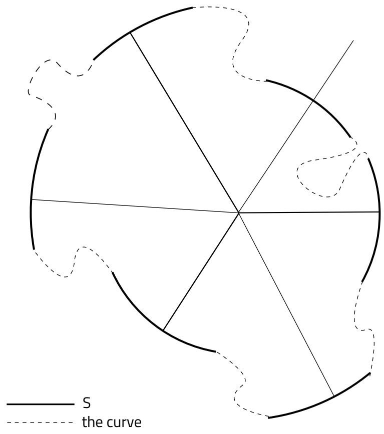


Figure 4.2: A typical example of a curve satisfying the hypothesis of corollary 2

This corollary, while immediate from the preceeding proposition, is worth mentioning because it is the first result which really makes explicit any kind of relationship between the curvature of a curve and the study of its tripod configurations. Interestingly, because of the nature of tripod configurations, this relationship between tripods and curvature has a kind of three fold symmetry, and it would be reasonable to hope that further study of tripod configurations may yield deeper interactions between these configurations and curvature. Any results of this nature would by necessity have this kind of three-fold symmetry baked in, which might lead to some interesting and surprising ideas. This example, then, motivates our further study of curves with dense sets of tripod points and feet.

4.1 Motivation and Curves of Constant Width

It is, to many, one of the great surprises of mathematics that the circle is not the only curve upon which you may role a board and keep it level. That is to say, there are many (infinitely many in fact) curves of constant width besides the circle. In order to make sense of what it means to be a curve of constant width we, of course, need the following definition:

Definition 14 (width of a curve)**:** The width of a closed curve, γ, along the direction $\hat{\imath}$, is the distance between the two support lines of γ perpendicular to $\hat{\imath}$.

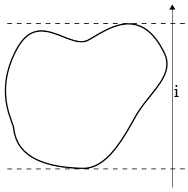


Figure 4.3: The width of a closed curve, γ

A curve of constant width, then, is defined in the obvious way. Curves of constant width have been classically well studied , though perhaps the most interesting result is *Barbier's Theorem*, which we will state here but whose proof is beyond the scope or interest of this article. This result was discovered by Joseph-Émile Barbier in 1860:

Theorem 12 (Barbier's Theorem)**.** *A given curve of constant width, h, necessarily has perimeter, πh.*

One of the simplest examples of these curves in terms of its construction and the ease with which one can prove its constant width property, is the Reuleaux triangle.[1] This triangle is constructed from an equilateral triangle by replacing each one of its sides with a $60°$ circular arc centered at the opposite vertex. Another way to think of this construction, is that one places a circle with radius equal to the side lengths of the triangle at each vertex of the triangle, and then choose the sides of the new curve to be the circular arcs whose chords are the sides of the triangle. This method of construction is illustrated in figure 4.1.

The Reuleaux triangle has constant width because any two parallel support lines will touch the curve at a vertex, and a side opposite the vertex. Because we constructed the Reuleaux triangle from these circular arcs, this necessarily means that the lines are of distance r apart, where r is the radius of these circular arcs, the side length of our original equilateral triangle.

Now, if one takes support lines perpendicular to the original support lines, then a square is obtained, and in fact it is easy to see that an equivalent definition of having constant width is that the curve can be continuously rotated within this square such that it always touches all four sides in at least one point.

Thus, we may think of curves of constant width as those which can be rotated continuously in the square in this way. This inspires a generalization of a curve of constant width, and the definition of a new kind of dimension

[1] the material in this section is adapted from [9]

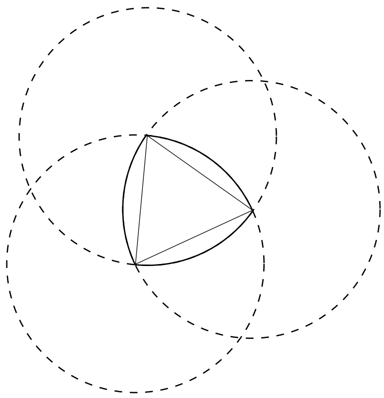
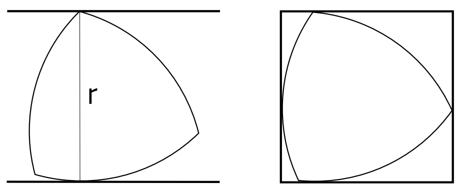


Figure 4.4: Construction of the Reuleaux Triangle



(a) The Reuleaux Triangle has constant width

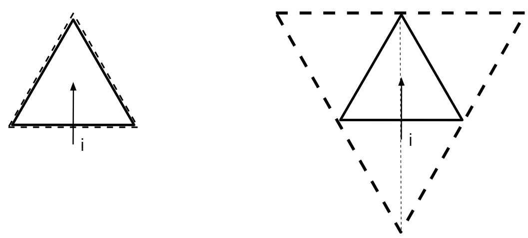
(b) The Reuleaux Triangle can rotate in a square

Figure 4.5: Properties of the Reuleaux Triangle

for a curve to have. Namely, we might ask ourselves, what kinds of curves may be continuously rotated within an equilateral triangle such that the curve always touches all three sides of the triangle in at least one point? The circle, as with curves of constant width, comes to mind as an obvious example, and in general the class of curves with this property are known as Δ-curves, or delta curves.

4.2 The Theory of Delta Curves

Taking our inspiration from curves of constant width as those curves which can be rotated in the square such that they always touch all four sides, we can also make a definition of Δ-curves as those which can be continuously rotated in the equilateral triangle such that they always touch all three sides. We can also define this class of curves in an analogous way, that is more reminiscent of the original inspiration of curves of constant width, that is to say the invariance of their notion of width as defined for a curve. Namely, we can think of Δ-curves as curves of constant height, by defining the appropriate notion:



(a) height of the triangle along i (b) height of the triangle along $-i$

Figure 4.6: Height of Equilateral Triangle along Opposite Directions

Definition 15 (height of a curve)**:** The *height of a curve*, γ, along a direction $\hat{\imath}$ is the height of the smallest equilateral triangle circumscribing γ, such that the line connecting its base to one of its vertices goes along the direction of $\hat{\imath}$ with the same orientation.

Notice that, in general, the height along $\hat{\imath}$ is not the same as the height along $-\hat{\imath}$. This fact can be seen by examining an equilateral triangle, whose height along the vector pointing inwards along one of the sides is the usual height of the triangle, but whose height along one of the vectors pointing outwards along a side is twice the usual height of the triangle.

Just as the Reuleaux triangle gives a simple first example of a curve of constant width, the curve known as a bi-angle or the lens, provides a nice starting point for studying Δ-curves. The lens is constructed by taking a 60° arc and reflecting it about the chord that defines it on the circle. Of course, in order to obtain such a 60° arc, one could use an equilateral triangle with a circle centered at one of its vertices. This method of construction is illustrated in figure 4.2.

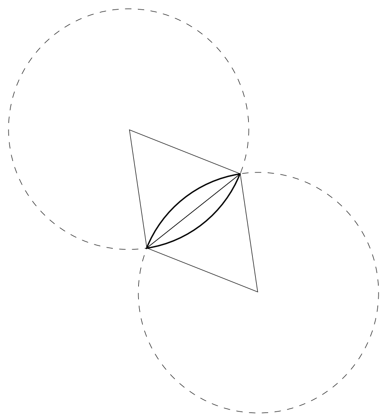


Figure 4.7: Construction of the Lens Delta Curve

Now we'll prove that this curve has constant height, and therefore can be rotated in the equilateral triangle while maintaining contact with all sides.[2]

Proposition 2. *The biangle is a Δ-curve.*

Proof. In figure 4.2, let $\triangle ABC$ be an equilateral triangle circumscribing the biangle, and O be the center of the circular arc used to construct the biangle. The key to the proof lies in showing that $\overline{AO}$ is parallel to $\overline{CB}$, and thus $\overline{OP}$ is equal in length to height of $\triangle ABC$. Thus, every circumscribing equilateral triangle of the biangle has height r, where r is the radius of the circle used to construct the biangle.

To prove that $\overline{AO}$ and $\overline{CB}$ are parallel, notice that $\angle rap = \angle rop = 60^\circ$; therefore, $raop$ is a cyclic quadrilateral, and so $\angle rao$ and $\angle opr$ are supplementary, thus $\angle rao = 120^\circ$. Since $\angle qca = 60^\circ$, this implies that $\overline{AO}$ and $\overline{CB}$ are parallel. $\square$

Thus, the biangle is truly, as claimed, a Δ-curve. The biangle is, as one would expect, not the only Δ-curve besides the circle. In fact, there are infinitely many.

We will remark here two results which will provide a key understanding of the limitations of Δ-curves, and be useful in the next section, where we will

[2]The following proof, as most of the material on Δ-curves, is adapted directly from [9]

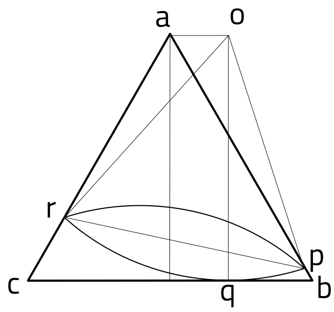


Figure 4.8: Proof that the Lens is a Δ-curve

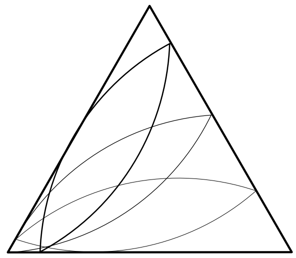


Figure 4.9: the Lens rotating inside an equilateral triangle

prove the results, when asking questions about dense tripod configurations and Δ-curves. The first result is:

Proposition 3. *Every support line of a Δ-curve intersects that curve at a single point.*

The second result is a corollary of the first,

Corollary 3. *Every Δ-curve is convex.*

In light of this corollary, another way of understanding the first proposition is that every point of a Δ-curve is an extremal point of the convex figure it encloses.

Lastly, we include what is perhaps the most interesting extent result on Δ-curves currently:

Theorem 13 (Barbier's Theorem for Δ-curves)**.** *Every Δ-curve of height h, has length $\frac{2\pi}{3}h$.*

For one of the most in-depth resources on Δ-curves, see [10].

4.3 Delta Curves and Tripod Configurations

Now that we have built up sufficient (and interesting) background, we're prepared to return to our original examination of dense tripod configurations and curves with infinitely many tripod configurations. Of course, because any closed plane curve is compact, in order to have infinitely many tripod configurations, the feet of these configurations must have at least one accumulation point. Thus, one may see that the property of having infinitely many configurations and having a dense configurations are closely related. To make rigorous this informal discussion, and this notion of dense tripod configurations, we make the following definition:

Definition 16 (Density of Tripod Configurations)**:** Given a closed curve γ, a subset $S \subset \gamma$ is said to have dense tripod configurations if $\overline{\mathcal{F}_S} = S$.

The obvious example of a closed curve with dense tripod configurations, then, is the circle; however, recall from our original proof for the existence of tripod configurations of convex curves in the plane, that if we drop three normals from a convex curve such that the normals form 120° angles with each other, then the area of the equilateral triangle they form is related to the derivative of the area of the equilateral triangle formed by the tangents at those points.

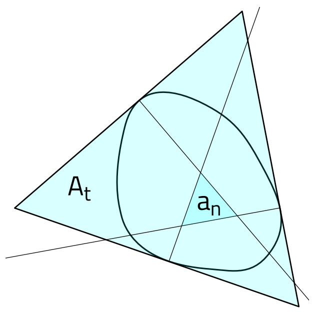


Figure 4.10: a_n is related to the derivative of A_t

This relation is such that when A_t is either a maximum or a minimum, a_n is zero, and thus the normals all intersect at a single point forming a tripod configuration. Moreover, though, A_t is constant as the triangle formed by the tangents rotates around the curve if and only if $a_n = 0$ for the entirety of that rotation. This relationship is always true on C^2 convex curves, but this is precisely to say that every C^2 convex curve with dense tripod configurations is a Δ-curve. We'll state this in a proposition:

Proposition 4. *Every C^2 closed locally convex curve with dense tripod configurations is a Δ-curve.*

This observation leads to some rather natural questions. Firstly, are there closed curves with dense tripod configurations which are not Δ-curves? Of course, such a curve would either need to be non-convex, or not C^2, as we'll see we can certainly find examples of curves with dense tripod configurations that are not C^2, but the examples we'll present turn out to be Δ-curves. This observation, then, brings us to our second important question. Since not all Δ-curves are C^2, do all Δ-curves still have dense tripod configurations?

We'll provide some discussion on the first question which will answer it in the negative, but leave many more questions open, while we'll discover that the answer to the second question is that all Δ-curves do indeed posses dense tripod configurations.

The next theorem proves that tripod configurations are dense in all Δ-curves, and will also allow us to prove the convexity properties of Δ-curves discussed in the previous section.[3]

Proposition 5. *The normals at the three points of contact between a Δ-curve and its circumscribing equilateral triangle all coincide at a point.*

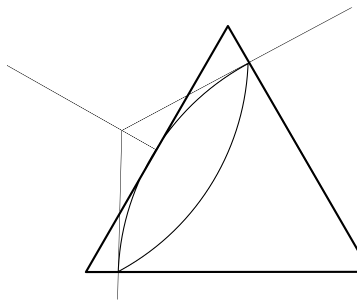


Figure 4.11: Every circumscribing equilateral triangle defines a tripod configuration

There are many ways to see this fact. Borrowing ideas from kinematics and classical mechanics, one could think of the Δ-curve as rotating inside the equilateral triangle, realize that the motion of the points of contact between the curve and the triangle must be tangent to the triangle, and thus recall that the instant center of rotation is the single point through which the perpendicular lines to each velocity vector intersect. Thus, since the

[3]much of the material in this section is adapted from [10]

perpendiculars to the curve are in particular perpendicular to the velocity vector, they must all intersect at the instant center of rotation.

Another way approach to proving this problem, would be to take a sequence of smooth curves approximating the delta curve, and to show that the proper values converge correctly; however, we do not need to work so hard as any of these, because we have actually already proven the previous proposition. Namely, recall the proof of theorem 11 in which we proved that every closed curve has a tripod point in the plane. The proposition stated that for any closed curve in the plane, every maximal circumscribing equilateral triangle defines a tripod configuration. Since every circumscribing equilateral triangle of a Δ-curve has the same area, they are all in particular maximal, thus the preceeding proposition is immediate. Moreover, by the proposition, the sides of this equilateral triangle must intersect the curve in only one point. Since every support line of the curve defines an equilateral triangle, this proves the proposition from the previous section:

Proposition 6. *Every support line of a Δ-curve intersects that curve at a single point.*

and of course, the corollary is immediate.

Corollary 4. *Every Δ-curve is convex.*

Since the Δ-curve is convex, every point has a support line passing through it, which implies that every point on a Δ-curve has a circumscribing equilateral triangle going through it and thus a tripod configuration by proposition , this implies the following result:

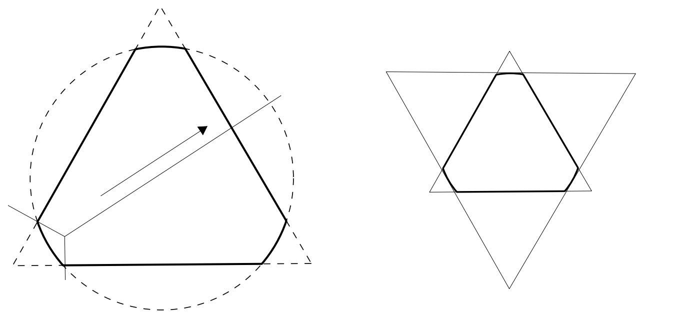
Theorem 14. *Tripod configurations are dense in every Δ-curve.*

Thus, we have our positive answer to the question, does every Δ-curve have dense tripod configurations. As an interesting remark, our characterization theorem of the circle provides the immediate corollary that every Δ-curve which is not the circle must have at least 2 tripod *points*. Moreover, unless the Δ-curve has piecewise constant curvature, it must have infinitely many tripod points. An interesting topic to explore could be what kinds of shapes the set of tripod points of a Δ-curve can make. For the Δ-biangle, the tripod points all lie on a continuous closed curve with two non-differentiable points–the same number of non-differentiable points as the Δ-biangle itself.

Now, we will return to our other motivating question for some discussion, before closing this chapter. The other question we asked was, are there any closed curves with dense tripod configurations which are not Δ-curves.

The answer turns out to be yes, there do exist curves with dense tripod configurations which are not Δ-curves. While a general theory of these curves seems to be more difficult, we provide some examples, to elucidate some of the possibilities.

Firstly, there exist C^0 convex curves with dense tripod configurations which are not Δ-curves. For example, consider an (appropriately sized) equilateral triangle and a circle, whose centers of mass are coincident. Consider the curve which forms the boundary of their intersection. Given a point initially on the boundary of the triangle, we can find a tripod configuration by seeing where it's normal line intersects the perpendicular bisector of one of the other sides, and choosing the final normal line by symmetry. Likewise, all of the points originally on the circle have tripod configurations going through the center of the circle.



(a) Tripod Configurations are dense

(b) the curve is not a Δ-curve

Figure 4.12: A C^0 curve with dense Tripod Configurations that is not a Δ-curve

Simultaneously, this curve is clearly not a Δ-curve, as can be seen by taking the initial equilateral triangle as the minimal circumscribing triangle, and examining a circumscribing triangle of the initial circle.

Thus, we have that C^0 is enough to break our theorem. If we use normals to the curve, as opposed to normals to support lines of the curve, to form our tripod configurations, then we can do even better than this. Namely, we can form a C^1, locally convex curve which is not a delta curve but has dense tripod configurations:

The curve in figure 4.3 is traversed around a large circle, half way around the small circle, then around the other large circle. Clearly this curve has continuously varying tangents, but it cannot be C^2 because the curvature changes discontinuously at the points between the large outer circles and the small inner circle.

Whether or not local convexity is a necessary condition for proposition **??** remains an open question. Of course, since all Δ-curves are convex, whether or not local convexity is necessary for proposition is equivalent

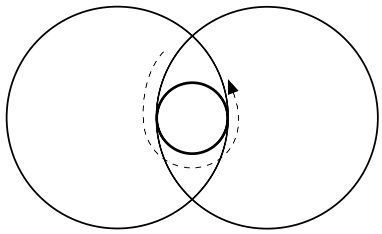


Figure 4.13: C^1 curve with Dense Tripod Configurations, but not a Δ-curve

to the question of whether or not there exist C^2 curves with dense tripod configurations which are not locally convex.

Chapter 5

Conjectures and Conclusions

In this concluding chapter, we begin by summarizing our results and some open questions for curves with dense tripod configurations, and then move on to state a conjecture, and provide the beginnings of a proof for it, which requires more technical detail than I can currently pursue.

5.1 Dense Tripod Configurations: Open Questions and Summary

Recall that we began our discussion of dense tripod configurations by observing the fact of the constant curvature corollary:

Corollary 5 (Constant Curvature Corollary)**.** *If p is a tripod point of γ, and $S \subset \gamma$ is the set of feet of p, then each connected component of the closure of S has constant curvature.*

which of course implies the tripod configuration characterization of the circle, that if γ has one tripod point and tripod configurations are dense, then γ is a circle.

In developing the theory of dense tripod configurations, we realized that every C^2 locally convex curve with dense tripod configurations is a Δ-curve, and that every Δ-curve has dense tripod configurations; however, there are C^0 and C^1 locally convex curves with dense tripod configurations. Thus, for sufficiently differentiable curves, density of tripod configurations and being a Δ-curve are analogous, but for rougher curves, the notions begin to diverge with dense tripod configuration curves being a superset of Δ-curves.

Of course, many questions were left open.

While [10] gives some methods of constructing different Δ-curves (this resource really containing most of the available literature in English on Δ-curves to date), it does not explicitly provide any examples of smooth Δ-

curves besides the circle. In order for our theorem that all C^2 curves with dense tripod configurations are Δ-curves to be meaningful, there should be more examples of C^2 Δ-curves than just the circle, thus we may ask:

Open Problem 1: Are there smooth Δ-curves besides the circle?

and more generally,

Open Problem 2: Does there exist a C^K Δ-Curve, for each $K \in \mathbb{N}$?

These questions may be approached from the perspective of dense tripod configurations, or from Δ-curves since for greater than or equal to C^2, the notions are equivalent.

A different way of counting the number of tripod configurations is counting instead the number of distinct tripod points. One could, then, imagine the number of tripod points as having certain interesting properties relating to the curves that define them. In this vein, we may also ask whether or not the condition of density of tripod configurations is necessary for our characterization theorem of the circle, namely:

Open Problem 3: Are there curves, besides the circle, which only possess a single tripod point?

And finally, while we determined that the differentiability condition is necessary for the equivalence of dense tripod configuration curves and Δ-curves, we did not answer the question of whether or not local convexity was, this leads us to our final question:

Open Problem 4: Do there exist closed curves with dense tripod configuations which are not locally convex?

5.2 Parity Conjecture for Tripod Configurations

We make the following conjecture about the parity of number of tripod configurations for a given curve.

Conjecture 1 (Tripod Parity Theorem)**.** *Almost every C^1 curve has an even number of tripod configurations.*

Recall that for convex and locally convex C^2 curves, there should always be an even number of tripod configurations, and that the Morse Theoretical work in [5] also confirms this result for curves sufficiently similar to a circle in alternative geometries. For general C^1 curves, in any geometry, the reason for the parity to be even comes from the following (incomplete) argument.

Let $M = (\mathbb{R}^2 \times S^1)^3$, or the space of triples of flags in the plane. We think of an element of M as being a choice of three points in the plane with a direction at each point. Then given a curve γ, we have an embedding of γ into M given by:

$$\Gamma : S^1 \to M$$

by

$$\alpha \mapsto (\gamma(\alpha), \gamma'(\alpha), \gamma\left(\alpha + \frac{2\pi}{3}\right), \gamma'\left(\alpha + \frac{2\pi}{3}\right), \gamma\left(\alpha + \frac{4\pi}{3}\right), \gamma'\left(\alpha + \frac{4\pi}{3}\right))$$

Interestingly, though unimportantly to this argument, Γ has nontrivial homology. The first homotopy group of M is $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, since M is a deformation retract of the 3 torus, and Γ is of the homotopy class $(1, 1, 1)$. More to the point, though, we can define an subset, $S \subset M$ by taking the set of points in M such that the lines formed by the flags all intersect at a single point. We consider here parallel lines to intersect at infinity. Then S is nearly a manifold, except for some singularities when two of the lines coincide, and a couple of other points.

S intersects Γ, precisely when γ has a tripod configuration. Now, finally notice that the homotopy class of Γ has an intersection number modulo 2 of 0 with S, this can be seen by taking three flags and rotating them around in a circle such that they never all coincide at a point.

Since intersection number modulo 2 is a constant for all curves in general position in a homotopy class, this implies that every curve should have an even number of tripod configurations. In order for the argument to work, however, the singularities of S must be dealt with, and the property of general position for Γ must be related back to a condition on the curve γ.

Should these technical details be resolved, then it will have been proven that tripod configurations come in pairs for any C^1 curve–in spherical, hyperbolic, non-hyperbolic and planar geometry since the proof is purely topological.

5.3 Complications with Tetrapod Configurations

As already discussed in section 2.4, taking a Morse theoretical approach to proving the existence of tetrapod configurations would likely run into some technical complications fairly quickly. Namely, the approach from [5] requires the use of Morse theory on manifolds with boundary. Using these tools, one can relate the number of critical points of the sum distances function that occur on the boundary of a parameter space, to the number of critical points that occur inside the manifold. The critical points on the boundary turn out to be easy to count in the two dimensional case, and the critical points inside the manifold are precisely the tripod configurations. In three dimensions though, one cannot (in any obvious way) avoid running into continuous sets of critical points, as opposed to some discrete number that one could count. This, in particular, means that one would need some kind of Morse-Bott theory on manifolds with boundary. Currently, I'm unaware if such a theory has been developed, but in any event, it would require a good deal more technical effort.

Adapting a geometric approach to prove the existence of tetrapod configurations also runs into issues. For the benefit of the interested reader that may consider pursuing this kind of approach, we'll outline its beginnings here, and provide further discussion when we reach the step which does not translate from the two to three dimensional case.

Recall our proof of theorem 11. We'll begin adapting the argument here: Let $\sigma \subset \mathbb{R}^2$ be a closed surface.

Lemma 6. *Let T be a maximum circumscribing regular tetrahedron of σ, intersecting σ at points A, B, C, and D. T is also the maximal regular tetrahedron circumscribing $\Delta ABCD$.*

The proof works almost identically to the two dimensional case:

Proof. Let T' be a regular tetrahedron circumscribing $\Delta ABCD$. T' is either circumscribes σ, since each of its sides intersects σ in at least one point, or we can move each side outwards until it intersects σ at only one point, and thus T' is contained in a regular tetrahedron circumscribing σ.

Therefore, we always have that T' has volume less than or equal to that of some regular tetrahedron circumscribing σ, and thus has volume less than or equal to that of T. $\square$

The next (somewhat surprising!) lemma is the beginning at an attempt to show that a maximal circumscribing regular tetrahedron, T, of a different reference tetrahedron $\Delta ABCD$, must have be antipedal to a higher dimensional analogue of the first isogonic center.

Lemma 7. *Given a regular tetrahedron $T \subset \mathbb{R}^3$, and a point P in the interior of the tetrahedron, the sum of the distances from P to each side of T is the altitude of T.*

Proof. Let $T = \Delta ABCD$ and let a be the area of each side of T (since all have equal area), and $d(P, ABC), d(P, ACD), d(P, ABD)$ and $d(P, BCD)$

denote the distance from P to the respectives sides of T. Split T into the tetrahedra

$$\Delta ABCP, \Delta ABPD, \Delta APCD, \text{ and } \Delta PBCD$$

Their volumes are

$$\frac{1}{3}ad(P, ABC), \frac{1}{3}ad(P, ACD), \frac{1}{3}ad(P, ABD), \text{ and } \frac{1}{3}ad(P, BCD)$$

Moreover, the sum of their volumes is the volume of T, so letting h be the height of T we have:

$$\frac{1}{3}a\left(d(P, ABC) + d(P, ACD) + d(P, ABD) + d(P, BCD)\right) = \frac{1}{3}ah$$

Therefore:

$$d(P, ABC) + d(P, ACD) + d(P, ABD) + d(P, BCD) = h$$

□

Let us define the tetrahedron antipedal to a point:

Definition 17 (Tetrahedron Antipedal to a Point)**:** Given a tetrahedron $\Delta ABCD$, the tetrahedron antipedal to a point P in the interior of $\Delta ABCD$ is formed by taking the planes orthogonal to the line segments $\overline{PA}$, $\overline{PB}$, $\overline{PC}$, and $\overline{PD}$ at the points A, B, C and D respectively.

This last lemma shows where the two and three dimensional cases diverge.

Lemma 8. *Let $\Delta ABCD$ be a tetrahedron with a point in its interior, P, such that the line segments $\overline{PA}, \overline{PB}, \overline{PC}$ and $\overline{PD}$ form angles of $\cos^{-1}(-\frac{1}{3})$ with each other, pairwise (in otherwords, they may all be placed perpendicularly to a different edge of a regular tetrahedron).*

Then the tetrahedron antipedal P is the maximal circumscribing regular tetrahedron of $\Delta ABCD$.

The proof is very simple.

Proof. Let T be a regular tetrahedron circumscribing $\Delta ABCD$. Consider the vertex A of $\Delta ABCD$. Then $d(P, A)$ is greater than or equal to the minimum distance between P and the side of T intersecting A, since A is on that side. This argument may be repeated for all the vertices of $\Delta ABCD$, showing that the sum of distances from P to the vertices of $\Delta ABCD$ is greater than the sum of the distances from P to the sides of T. But this is precisely to say, by our previous lemma, that the height of the regular tetrahedron antipedal to P is greater than or equal to the height of T.

Thus, the lemma is proven. $\square$

Of course, if one could guarantee that the tetrahedron formed by the intersection points of a maximal circumscribing tetrahedron and a surface had a point satisfying the hypotheses of Lemma 8, then it would be immediate that the surface has a tetrapod point in the strongest sense (that is, with all the normals being equiangular in the strongest possible sense), but herein lies the difficulty between adapting these methods. To explore this difficulty, we offer the following open question:

Open Problem 5: When does a tetrahedron posses a point such that the angles between the line segments connecting that point to the vertices of the tetrahedron are all equal?

Essentially, which tetrahedra possess higher dimensional analogues of the first isogonic center of a triangle?

It should be noted that any approach trying to use circumscribing regular tetrahedra would only be able to find these tetrapod configurations satisfying the strongest equiangular condition. These kinds of tetrapods, I imagine, should be extremely rare, as opposed to the tetrapods defined in 2.4, which I believe should be much more common. While it seems unlikely that every sufficiently differentiable surface contains a tetrapod in the strongest sense, the one we have used seems reasonable to expect of a reasonably nice surface.

5.4 Special Thanks

I would like to give great thanks to Eric Chen, who worked with me over the summer of 2013 on researching tripod configurations, and without whom none of the results we obtained over the summer, or results found in this paper that have been created since, would have been possible.

I would also like to thank Sergei Tabachnikov who has been an important adviser to Eric and myself throughout our work on tripod configurations.

Lastly, I would like to give a rather large thank you to my thesis adviser, Richard E. Schwartz, whose support was crucial to the development of this thesis.

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