Calculating Cobordism Rings

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Abstract

The notion of (unoriented, oriented, ...) cobordism yields an equivalence relation on closed manifolds, and can be used to construct generalized (co)homology theories. In 1954, Thom [15] determined the structure of the unoriented cobordism ring by reducing the problem to a question in stable homotopy theory. We provide an exposition of this approach, introducing basic concepts in cobordism theory and covering the computations of the unoriented and complex cobordism rings.

1 Introduction

In the construction of singular homology of spaces, two singular n-chains yield the same homology class if and only if their difference is the boundary of a singular $(n+1)$-chain. The analogous relation for closed manifolds is known as cobordism: two closed n-manifolds are said to be *cobordant* if their disjoint union is the boundary of a compact $(n+1)$-manifold. (For our purposes, all manifolds are assumed to be smooth manifolds.)

The set of cobordism classes of closed manifolds forms a graded ring (with disjoint union as addition and Cartesian product as multiplication), known as the *unoriented cobordism ring*. We can also define various additional flavors of cobordism, by considering oriented manifolds, weakly almost complex manifolds, etc. In full generality, given a fibration $f : B \to BO$, we have a notion of (B, f)-cobordism and can define the corresponding cobordism ring. The structure of these rings is known in many cases, including the unoriented ($BO \to BO$), oriented ($BSO \to BO$), and complex ($BU \to BO$) cases. We consider only the unoriented and complex cases, and avoid the full generality of (B, f)-cobordism.

As with singular homology, the notion of (B, f)-cobordism can be used to define a generalized (co)homology theory, in which the homology of a point is the (B, f)-cobordism ring. The Brown representability theorem tells us that every generalized (co)homology theory is represented by a spectrum, and in fact, the Thom-Pontryagin Theorem says that the generalized (co)homology theory of (B, f)-cobordism is represented by a *Thom spectrum* consisting of Thom spaces. As a result, letting E denote the appropriate Thom spectrum, the (B, f)-cobordism ring is $\pi_*(E)$. Therefore, the calculation of cobordism rings corresponds exactly to the calculation of homotopy groups of Thom spectra.

We demonstrate the calculation of unoriented and complex cobordism rings, or equivalently $\pi_*(MO)$ and $\pi_*(MU)$, following the approaches in [4] and [9]. To do so, we determine the structure of $H^*(MO;\mathbb{Z}/2)$ and $H^*(MU;\mathbb{Z}/p)$ as modules over the Steenrod algebras $\mathcal{A}_2$ and $\mathcal{A}_p$, and show how the Adams spectral sequence can be used to deduce information about stable homotopy groups from information about cohomology as a module over the Steenrod algebra. Ultimately we show that the unoriented cobordism ring is $\Omega_{unoriented} \cong \mathbb{Z}/2[w_k : k \neq 2^\ell - 1]$ and the complex cobordism ring is $\Omega_{complex} \cong \mathbb{Z}[x_{2k} : k \in \mathbb{N}]$.

The exposition is split into 11 sections. Section 2 begins our treatment of cobordism, explicitly defining the generalized homology theory of cobordism and the cobordism ring. In Section 3, we discuss the connection between generalized (co)homology theories and spectra, which can be summarized by the fact that every spectrum yields an generalized (co)homology theory and every generalized (co)homology theory comes from a spectrum. Our goal then is to determine a spectrum which yields the homology theory of cobordism, and to describe this spectrum we need to discuss Thom spaces of vector bundles. In Section 4, we review basic facts about vector bundles and introduce the Thom isomorphism, a key tool. Section 5 covers the Thom-Pontryagin isomorphism, which is at the heart of the computation of cobordism rings, as it expresses cobordism rings as homotopy rings of Thom spectra. In particular, the unoriented cobordism homology theory corresponds to a spectrum MO, the complex cobordism homology theory corresponds to a spectrum MU, and we have isomorphisms of cobordism rings $\mathfrak{R}_* \cong \pi_*(MO)$ and $\mathfrak{R}_*^{complex} \cong \pi_*(MU)$.

From here, we use techniques in stable homotopy theory to compute $\pi_*(MO)$ and $\pi_*(MU)$. In Section 6, we give some motivation for the computation of these homotopy groups via the computation of $H^*(MO;\mathbb{Z}/2)$ and $H^*(MU;\mathbb{Z}/p)$, as well as their structures over certain algebras known as the Steenrod algebras. In Section 7, we discuss these Steenrod algebras $\mathcal{A}_p$. A crucial fact about the Steenrod algebras, which simplifies computation, is that they form a bialgebra (in fact, a Hopf algebra) and their dual algebras have a simple commutative structure. In computations, determining the structure of an $\mathcal{A}_p$-module M is equivalent to determining the structure of M_* as an $\mathcal{A}_p^*$-comodule, which is typically easier due to the simple description of $\mathcal{A}_p^*$.

Finally, the remaining sections delve into stable homotopy theory to compute $\mathfrak{R}_* \cong \pi_*(MO)$ and $\mathfrak{R}_*^{complex} \cong \pi_*(MU)$. Section 8 gives an outline of the computation and uses the Thom Isomorphism theorem to compute the homology and cohomology of MO with mod 2 coefficients, as well as the mod p homology and cohomology of MU. Since the mod 2 Steenrod algebra $\mathcal{A}_2$ acts on $H^*(MO;\mathbb{Z}/2)$, we can describe $H^*(MO;\mathbb{Z}/2)$ as an $\mathcal{A}_2$-module; for each prime p, we also consider $H^*(MU;\mathbb{Z}/p)$ as an $\mathcal{A}_p$-module. Since we are working over a field, homology is the graded dual of cohomology, so $H_*(MO;\mathbb{Z}/2)$ is an $\mathcal{A}_2^*$-comodule and $H_*(MU;\mathbb{Z}/p)$ is an $\mathcal{A}_p^*$-comodule. We prove in Section 9 that the comodule structure of $H_*(MO;\mathbb{Z}/2)$ is "cofree," implying that $H^*(MO;\mathbb{Z}/2)$ is a free $\mathcal{A}_2$-module. In Section 10, we find that the comodule structure of $H_*(MU;\mathbb{Z}/p)$ over $\mathcal{A}_p^*$ is not cofree over $\mathcal{A}_p$, but we find a description of the comodule, and then point out how the Adams spectral sequence can be used to compute the homotopy groups of MU, i.e. the complex cobordism ring.

2 Cobordism

Definition 2.1. *Two closed manifolds M and N are said to be **cobordant** if there exists a compact manifold with boundary W such $\partial W \cong M \amalg N$.*

In particular, M is said to be **null-cobordant** if M is cobordant with $\emptyset$, which is equivalent to M being the boundary of a compact manifold.

Proposition 2.1. *Cobordism is an equivalence relation on the class of closed manifolds.*

Proof. Cobordism is reflexive, since for any closed manifold M, $\partial(M \times I) \cong M \amalg M$. Clearly cobordism is symmetric. The fact that cobordism is transitive relies on the "collaring theorem", proven in [3]. $\square$

Definition 2.2. *The n^{th} **cobordism group** $\mathfrak{N}_n$ is defined as the set of cobordism classes of closed n-manifolds, with group operation given by disjoint union and identity $[\emptyset]$.*

The group operation is well-defined, since

$$[M] = [N] \implies \partial W \cong M \amalg N \implies \partial(W \amalg (P \times I)) \cong (M \amalg P) \amalg (N \amalg P) \implies [M \amalg P] = [N \amalg P].$$

Associativity is obvious, and $[\emptyset]$ is an identity since $M \amalg \emptyset \cong M$. The inverse of $[M]$ is $[M]$, since

$$\partial(M \times I) \cong (M \amalg M) \amalg \emptyset \implies [M \amalg M] = [\emptyset].$$

Commutativity is also clear. Thus, $\mathfrak{N}_n$ is in fact a $\mathbb{Z}/2$-module.

The cobordism relation yields a way of defining homology in a way that seems somewhat more "natural" than singular homology, although it turns out to differ from singular homology. When we define singular homology of spaces, we use simplices for our model, and have an algebraic definition for the boundary of a singular simplex (and also the boundary of a singular chain). Recall that a singular n-simplex is a continuous map $\Delta^n \to X$, the n^{th} singular chain group is $C_n(X) = \mathbb{Z}[\mathrm{Sing}_n(X)]$, and the boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ is defined on singular n-simplices by

$$\partial_n \sigma = \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_n]}.$$

Another approach for defining homology is in terms of compact manifolds, with ∂_n the ordinary boundary of manifolds. That is, define a **singular n-manifold** as a pair (M, f) where M is a compact n-manifold and $f : M \to X$ is continuous.

Let $C_n^{cob}(X)$ be the set of isomorphism classes of singular n-manifolds, which is a commutative monoid under the disjoint union operation, having $\emptyset$ as the identity. Let $\partial_n : C_n^{cob}(X) \to C_{n-1}^{cob}(X)$ be the boundary operator:

$$\partial_n((M, f)) = (\partial M, f|_{\partial M})$$

Note that

$$\cdots \xrightarrow{\partial_{n+1}} C_n^{cob}(X) \xrightarrow{\partial_n} C_{n-1}^{cob}(X) \to \cdots \xrightarrow{\partial_2} C_1^{cob}(X) \xrightarrow{\partial_1} C_0^{cob}(X)$$

is a chain complex of monoids, and we can define sets

$$H_n^{cob}(X) = \frac{\ker(\delta_n)}{\mathrm{im}(\delta_{n+1})} = \frac{\{\text{singular } n\text{-manifolds without boundary}\}}{[(A, h|_A)] \sim [(B, h|_B)] \text{ for } (A \amalg B, h|_{A \amalg B}) = \partial(P, h)}$$

These are actually groups, by the same reasoning that $\mathfrak{N}_n$ are groups. We can also define relative homology groups $H_n^{cob}(X, A)$, for $A \subset X$, as the homology of the chain complex of monoids

$$\dots \xrightarrow{\overline{\partial_{n+1}}} C_n^{cob}(X)/C_n^{cob}(A) \xrightarrow{\overline{\partial_n}} C_{n-1}^{cob}(X)/C_{n-1}^{cob}(A) \to \dots \xrightarrow{\overline{\partial_2}} C_1^{cob}(X)/C_1^{cob}(A) \xrightarrow{\overline{\partial_1}} C_0^{cob}(X)/C_0^{cob}(A),$$

and also define reduced homology $\tilde{H}_n^{cob}(X) = H_n^{cob}(X, \emptyset)$. Note that if $X = *$ is a point, then a singular n-manifold is precisely an n-manifold with its unique map to $*$, and so $H_n^{cob}(*) = \mathfrak{N}_n$, implying $\mathfrak{N}_* = H_*^{cob}(*)$.

Definition 2.3. *The (unoriented)* ***cobordism ring*** *is defined as a graded $\mathbb{Z}/2$-module by*

$$\mathfrak{N}_* = \bigoplus_{n=0}^{\infty} \mathfrak{N}_n.$$

With multiplication given by $[M][N] = [M \times N]$, this becomes a graded $\mathbb{Z}/2$-algebra.

The multiplication operation is well-defined, since

$$\begin{aligned} [M] = [M'] &\implies \partial W \cong M \amalg M' \\ &\implies \partial(W \times N) \cong (M \amalg M') \times N = (M \times N) \amalg (M' \times N) \\ &\implies [M \times N] = [M' \times N] \end{aligned}$$

This is obviously associative and commutative, with an identity $[*]$ (here and from now on, $*$ denotes a point). The grading structure is respected, since $\dim(M \times N) = \dim(M) + \dim(N)$. Finally, this multiplication is compatible with addition since, for example,

$$[M]([N] + [P]) = [M \times (N \amalg P)] = [(M \times N) \amalg (M \times P)] = [M][N] + [M][P].$$

We can also define a notion of oriented cobordism, in which two oriented manifolds are cobordant if together they form the oriented boundary for an oriented compact manifold. A similar notion can be defined for stably almost complex manifolds (yielding complex cobordism), and for any of a variety of additional structures placed on a manifold (i.e., (B, f) structures). There is a careful treatment of general (B, f)-cobordism in [14], but we will only care about the unoriented and complex cobordism rings $\mathfrak{N}_*$ and $\mathfrak{N}_*^{complex}$.

3 Generalized Homology and Spectra

In order to compute $\mathfrak{N}_* = H_*^{cob}(*)$, we turn to a discussion of generalized (co)homology theories in general. An (extraordinary) generalized (co)homology theory resembles an ordinary (co)homology theory, except that the dimension axiom is omitted (so the (co)homology of a point may be nontrivial). We use the definition given in [12], Chapter 13:

Definition 3.1. *A (reduced)* ***generalized homology theory*** *on* $\mathbf{hCW}_*$ *(the homotopy category of pointed CW complexes) is a sequence of functors* $h_n : \mathbf{hCW}_* \to \mathbf{Ab}$ *with natural isomorphisms* $\sigma : h_n \to h_{n+1} \circ \Sigma$ *(here* $\Sigma : \mathbf{hCW}_* \to \mathbf{hCW}_*$ *is the suspension functor) such that:*

- *If $i : A \overset{i}{\hookrightarrow} X$ is the inclusion of a subcomplex and $q : X \to X/A$ is the quotient map, then the sequence*
$$h_n(A) \xrightarrow{h_n(i)} h_n(X) \xrightarrow{h_n(q)} h_n(X/A)$$
is exact.

- *For any pointed spaces $\{X_j\}$, the maps induced by $\alpha_i : X_i \to \bigvee_j X_j$ combine to form an isomorphism*
$$\bigoplus_j h_n(X_j) \xrightarrow{\oplus_j h_n(\alpha_j)} h_n\left(\bigvee_j X_j\right)$$
for all n.

Definition 3.2. *A **spectrum** is a sequence of pointed CW complexes $\{X_n\}$ together with maps $\Sigma X_n \to X_{n+1}$. (Here we assume that a pointed CW complex has its basepoint as a 0-cell.) A **CW-spectrum** is a spectrum where the maps $\Sigma X_n \to X_{n+1}$ are inclusions of subcomplexes.*

For example, every CW complex X has a **suspension spectrum** $\Sigma^\infty X$, consisting of spaces $\Sigma^n X$ with identity structure maps. A more interesting example is as follows:

Definition 3.3. *An **Eilenberg-Maclane space**, or a K(G,n), is a (for our purposes, pointed CW) space X such that $\pi_n X \cong G$ and $\pi_k X = 0$ for all $k \neq n$.*

Proposition 3.1 (Proposition 4.30 in [2])**.** *For every abelian group G and every $n > 0$, there exists a unique $K(G,n)$ up to homotopy equivalence.*

For this reason we can often think about $K(G,n)$ as a single object, so long as we are working in the homotopy category $\mathbf{hTop}_*$.

Given an abelian group G, we construct an **Eilenberg-Maclane spectrum** corresponding to G (denoted HG) such that the n^{th} space X_n is a $K(G,n)$. Since ΩX_n is a $K(G,n-1)$, there exists a homotopy equivalence $f_n : X_{n-1} \to \Omega X_n$, yielding $g_n : \Sigma X_{n-1} \to X_n$ by the loop-suspension adjunction. These maps g_n are the structure maps of the spectrum.

From here, we limit ourselves to CW-spectra when defining maps between spectra, for simplicity.

Definition 3.4. *A **function of spectra of degree** d between X and Y is a sequence of functions $f_n : X_n \to Y_{n-d}$ which commute with the structure maps of X and Y.*

Definition 3.5. *A **subspectrum** of a spectrum X is a sequence $\{Y_n\}$ such that $Y_n \subset X_n$ is a subcomplex (containing the basepoint) and the structure map $f_n : \Sigma X_n \to X_{n+1}$ satisfies $f_n(\Sigma Y_n) \subset Y_{n+1}$.*

Definition 3.6. *A **cofinal subspectrum** of X is a subspectrum $\{Y_n\}$ such that, for every X_n and every cell e of X_n, eventually e is in Y_m.*

Definition 3.7. *A* ***map of spectra of degree*** *d from X to Y is an equivalence class of functions of degree d from cofinal subspectra to Y, where two such functions are equivalent if they agree on their intersection. A* ***homotopy of maps of spectra*** *is a map $X \wedge I^+ \to Y$ where $I^+ = [0,1] \amalg *$. The set of homotopy classes of maps of spectra $X \to Y$ of degree d is denoted $[X, Y]_d$, and the set of homotopy classes of maps of spectra $X \to Y$ (of any degree) is denoted $[X, Y]$.*

Note that, letting $\mathbb{S} = \Sigma^\infty \mathbb{S}^0$ be the sphere spectrum, the k^{th} stable homotopy group of spheres is $[\mathbb{S}, \mathbb{S}]_k$. We denote the category of spectra with homotopy classes of maps as morphisms by **hSpectra**, and will always work in this category (rather than the category with ordinary maps as morphisms).

Proposition 3.2 ([11], p. 68-69)**.** *Every spectrum X yields a generalized homology theory, given by*

$$H_n(Y) = [\mathbb{S}, X \wedge Y]_n = \varinjlim_k \pi_{n+k}(X_k \wedge Y),$$

and a generalized cohomology theory given by

$$H^n(Y) = [Y, X]_{-n}.$$

For example, let $X = HG$ be an Eilenberg-Maclane spectrum; then

$$H^n(Y; G) = [Y, K(G, n)].$$

Theorem 3.1 (Brown Representability, Theorem 4E.1 in [2])**.** *Every generalized homology theory is represented by a spectrum. That is, given a generalized homology theory $\{H_n^{gen}(-) : n \geq 0\}$, there exists a spectrum E such that $E_n(-) = [\mathbb{S}, E \wedge -]_n$ is naturally isomorphic to $H_n^{gen}(-)$.*

The Brown Representability theorem tells us that there exists some spectrum E with $E_n(-) \cong H_n^{cob}(-)$. In particular,

$$\mathfrak{R}_* = H_*^{cob}(*) \cong E_*(*) = [\mathbb{S}, E] = \pi_*(E).$$

Thus, in order to compute $\mathfrak{R}_*$ we determine a spectrum representing the generalized homology theory of cobordism, and then compute its homotopy groups. We will find such a spectrum, but in order to construct it, we will first need to discuss vector bundles and their Thom spaces.

4 Vector Bundles

We review basic material from [7]:

Definition 4.1. *A* ***vector bundle of rank*** *n is a continuous map $\pi : E \to B$ such that for every $b \in B$, there exists an open neighborhood $U \ni b$ and a homeomorphism $f : \pi^{-1}(U) \xrightarrow{\cong} U \times \mathbb{R}^n$ such that for $x \in U$, f takes $\pi^{-1}(x)$ onto $\{x\} \times \mathbb{R}^n$ linearly.*

Definition 4.2. *A* ***bundle map*** *between bundles $\pi_1 : E_1 \to B_1$ and $\pi_2 : E_2 \to B_2$ is a continuous map $f : E_1 \to E_2$ such that f takes $\pi_1^{-1}(b)$ onto $\pi_2^{-1}(b)$ as a linear isomorphism.*

Definition 4.3. *The* ***Grassmannian*** *of n-subspaces in $\mathbb{R}^{n+k}$ is the space*

$$Gr_n(\mathbb{R}^{n+k}) = \{V \subset \mathbb{R}^{n+k} : \dim(V) = n\},$$

topologized as a quotient space of

$$V_n^0(\mathbb{R}^{n+k}) = \{(v_1, \ldots, v_n) \subset (\mathbb{R}^{n+k})^n : v_1, \ldots, v_n \textit{ linearly independent}\} \subset (\mathbb{R}^{n+k})^n.$$

There is an obvious chain of inclusions

$$\mathrm{Gr}_n(\mathbb{R}^n) \subset \mathrm{Gr}_n(\mathbb{R}^{n+1}) \subset \cdots$$

The direct limit is an infinite Grassmannian, denoted

$$BO(n) = \{V \subset \mathbb{R}^\infty : \dim(V) = n\},$$

where $\mathbb{R}^\infty$ consists of infinite tuples with finitely many nonzero elements, and $BO(n)$ is topologized such that $X \subset BO(n)$ is open (resp. closed) if and only if $X \cap \mathrm{Gr}_n(\mathbb{R}^{n+k})$ is open (resp. closed) for all k.

Definition 4.4. *The* ***tautological bundle*** *over $Gr_n(\mathbb{R}^{n+k})$ consists of a total space*

$$E(\gamma_n^{n+k}) = \{(V, v) : V \in Gr_n(\mathbb{R}^{n+k}), v \in V\} \subset Gr_n(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k}$$

together with a projection $(V, v) \mapsto V$. Similarly, the tautological bundle over $BO(n)$ consists of

$$E(\gamma_n) = \{(V, v) : V \in BO(n), v \in V\} \subset BO(n) \times \mathbb{R}^\infty$$

together with a projection $(V, v) \mapsto V$.

Proposition 4.1 (Lemma 5.2 in [7])**.** *These are vector bundles.*

The bundles γ_n are called "universal bundles", for the following reason:

Proposition 4.2 (Theorems 5.6 and 5.7 in [7])**.** *Let $\xi : E \to B$ be a rank-n bundle where B is paracompact. Then there exists a bundle map from ξ to γ_n, and any two bundle maps from ξ to γ_n are bundle-homotopic.*

5 Thom-Pontryagin Isomorphism

The goal of this section is to find a ring spectrum representing the generalized (co)homology theory of cobordism, and to express $\mathfrak{N}_*$ as its homotopy ring.

Recall that the classifying space $BO(k)$ is the infinite Grassmanian consisting of k-planes in $\mathbb{R}^\infty$. There is a bundle $\gamma^k : E(\gamma^k) \to BO(k)$, where

$$E(\gamma^k) = \bigcup_{V \in BO(k)} \{(V, v) : v \in V\}$$

and $\gamma^k(V, v) = V$. This is called the **universal bundle of rank** k, since every bundle ξ of rank k over a paracompact base is a pullback $\xi \cong f^*\gamma^k$ for some continuous map $f : B(\xi) \to BO(k)$. Moreover, this map f is unique up to homotopy.

Definition 5.1. *Given a bundle ξ with an inner product, its **Thom space** is $Th(\xi) = D(\xi)/S(\xi)$ where $D(\xi)$ is the **disk bundle** consisting of all $e \in E(\xi)$ with $\|e\| \leq 1$, and $S(\xi)$ is the **sphere bundle** consisting of all $e \in E(\xi)$ with $\|e\| = 1$.*

Definition 5.2. *The Thom space of the universal rank-k bundle is denoted $MO(k) = Th(\gamma^k)$.*

Theorem 5.1 (Thom-Pontryagin Isomorphism, Theorem IV.8 in [15])**.** *For each n, we have an isomorphism*

$$\mathfrak{N}_n \cong \varinjlim_k \pi_{n+k}(MO(k)) = \pi_n(MO).$$

Together these form an isomorphism $\mathfrak{N}_* \xrightarrow{\cong} \pi_*(MO)$.

Without proving the Thom-Pontryagin isomorphism, we give a basic description, following [4]. The idea is to construct, for each closed manifold M, a homotopy class of some $MO(k)$. Given M of dimension n, embed M in some $\mathbb{R}^{n+k}$ (which is possible by the Whitney Embedding Theorem). Let $\nu = T\mathbb{R}^{n+k}/TM$ be the normal bundle of the embedding. By the tubular neighborhood theorem ([3], Theorem 5.1), $E(\nu)$ can be realized as an open set in $\mathbb{R}^{n+k}$, with the zero section of M identical to the embedding of M in $\mathbb{R}^{n+k}$.

Since ν is a rank-k bundle over M, we have a map $f : M \to BO(k)$ such that $\nu \cong f^*\gamma^k$. This means that f is covered by a bundle map $\beta : \nu \to \gamma^k$. This gives a map $g : \mathrm{Th}(\nu) \to \mathrm{Th}(\gamma^k) = MO(k)$, defined by $g(e) = \beta(e)$ for $e \in E(\nu)$ and $g(\infty_\nu) = \infty_{\gamma^k}$. Also, the inclusion $E(\nu) \hookrightarrow \mathbb{R}^{n+k}$ yields a map from $\mathbb{S}^{n+k} = \mathbb{R}^{n+k} \cup \{\infty\}$ to $\mathrm{Th}(\nu)$, which fixes elements of $E(\nu)$ and sends everything else to $\infty \in \mathrm{Th}(\nu)$. Composing the two maps yields a new map

$$\mathbb{S}^{n+k} \to \mathrm{Th}(\nu) \to MO(k)$$

which determines an element of $\pi_{n+k}(MO(k))$. By certain facts about embeddings, we end up with a unique element of

$$\varinjlim_k \pi_{n+k}(MO(k))$$

which depends only on the cobordism class of M. Thus, we have a map from $\mathfrak{N}_*$ to the direct limit of $\pi_{n+k}(MO(k))$, which turns out to be a homomorphism. In fact, it also turns out to be an isomorphism.

For any k, $\gamma^k \oplus \varepsilon^1$ is a rank-$(k+1)$ bundle over $BO(k)$ so there exists a bundle map $\gamma^k \oplus \varepsilon^1 \to \gamma^{k+1}$. Let e_1 be the rank-1 bundle over a point. Since the Thom functor takes products to smash products, we have a map

$$\Sigma MO(k) = \mathbb{S}^1 \wedge MO(k) = \mathrm{Th}(e_1) \wedge \mathrm{Th}(\gamma^k) = \mathrm{Th}(e_1 \times \gamma^k) = \mathrm{Th}(\varepsilon^1 \oplus \gamma^k) \to \mathrm{Th}(\gamma^{k+1}) = MO(k+1).$$

This gives us a spectrum MO, and the Thom-Pontryagin isomorphism says that

$$\mathfrak{N}_* \cong \pi_*(MO)$$

as graded abelian groups. Moreover, there is an analogous Thom-Pontryagin isomorphism

$$\mathfrak{N}_*^{complex} \cong \pi_*(MU),$$

where MU consists of spaces $MU(k) = \mathrm{Th}(BU(k))$ and maps

$$\Sigma^2 MU(k) = \mathbb{S}^2 \wedge MU(k) = \mathrm{Th}(e_{1,\mathbb{C}}) \wedge \mathrm{Th}(\gamma_{\mathbb{C}}^k) = \mathrm{Th}(e_{1,\mathbb{C}} \times \gamma_{\mathbb{C}}^k) = \mathrm{Th}(\varepsilon_{\mathbb{C}}^1 \oplus \gamma_{\mathbb{C}}^k) \to \mathrm{Th}(\gamma_{\mathbb{C}}^{k+1}) = MU(k+1).$$

Moreover, $\pi_*(MO)$ and $\pi_*(MU)$ have ring structures, and these are isomorphisms of rings. The spectrum MO yields the homology theory of unoriented cobordism, while MU yields the homology theory of complex cobordism.

6 Homotopy groups of spectra

In the last section, we expressed $\mathfrak{N}_*$ as $\pi_*(MO)$; thus, all that remains is to calculate $\pi_*(MO)$. In this section we give a further discussion of spectra and their homotopy groups, and describe a strategy for computing homotopy.

First, we consider some categorical aspects of spectra. Recall that we have a notion of smash product (denoted $\wedge$) for pointed spaces, so that the homotopy category of pointed spaces $\mathbf{hTop}_*$ is a monoidal category (with unit $\mathbb{S}^0$). Similarly, the stable homotopy category **hSpectra** (i.e. the category of spectra with homotopy classes of maps as morphisms) has a smash product $\wedge$ and forms a monoidal category (with unit $\mathbb{S} = \Sigma^\infty \mathbb{S}^0$, the sphere spectrum). Note that for spaces we have that $[\Sigma X, Y]$ is naturally a group (e.g. $\pi_1(Y)$ for $X = \mathbb{S}^0$) and $[\Sigma^2 X, Y]$ is naturally an abelian group, so for spectra we have that $[E, F] = [\Sigma^2 E, \Sigma^2 F]$ is an abelian group ([11], II.1 p. 39).

For any ring R, the Eilenberg-Maclane spectrum HR comes equipped with morphisms $i_r \in [\mathbb{S}, HR]$ (corresponding to $r \in H^*(\mathbb{S}; R) \cong R$) for each $r \in R$, as well as a morphism

$$m \in [HR \wedge HR, HR] \cong H^*(HR \wedge HR; R)$$

The morphisms $i_1 : \mathbb{S} \to HR$ and $m : HR \wedge HR \to HR$ turn HR into a monoid object in **hSpectra**, also known as a **ring spectrum**. Moreover, $H^*(E; R) = [E, HR]$ and $H_*(E; R) = [\mathbb{S}, E \wedge HR]$ are naturally R-modules via

$$R \times [E, HR] \to [E, HR], \qquad (r, E \xrightarrow{f} HR) \mapsto (E \xrightarrow{f} HR \xrightarrow{\beta_r} HR)$$

and

$$R \times [\mathbb{S}, E \wedge HR] \to [\mathbb{S}, E \wedge HR], \qquad (r, \mathbb{S} \xrightarrow{f} E \wedge HR) \mapsto (\mathbb{S} \xrightarrow{f} E \wedge HR \xrightarrow{\mathrm{Id} \wedge \beta_r} E \wedge HR)$$

where $\beta_r : HR \to HR$ is the composition

$$HR \xrightarrow{\cong} HR \wedge \mathbb{S} \xrightarrow{\mathrm{Id} \wedge i_r} HR \wedge HR \xrightarrow{m} HR.$$

Our spectra MO and MU are also ring spectra, with maps $MO \wedge MO \to MO$ and $MU \wedge MU \to MU$ coming from the operation of bundle product.

Let E be a spectrum, and recall that $\pi_*(E) = [\mathbb{S}, E]$. Recall that mod p cohomology of a spectrum is represented by $H\mathbb{Z}/p$, so

$$H^*(E) \cong [E, H\mathbb{Z}/p].$$

This is usually not too difficult to compute, and one might hope that it would yield some information about $[\mathbb{S}, E]$, since we know that

$$[\mathbb{S}, H\mathbb{Z}/p] = \pi_*(H\mathbb{Z}/p) \cong \mathbb{Z}/p.$$

Note that $[E, H\mathbb{Z}/p]$ is a $[H\mathbb{Z}/p, H\mathbb{Z}/p]$-module in a natural way. If we compute not only $[E, H\mathbb{Z}/p] = H^*(E; \mathbb{Z}/p)$ as an abelian group but also its structure as a module over $[H\mathbb{Z}/p, H\mathbb{Z}/p]$, then we have a better chance at being able to deduce $\pi_*(E)$. In fact, we really can determine $\pi_*(E)$ if $H^*(E; \mathbb{Z}/p)$ is a free module and $\pi_*(E)$ is both bounded below and killed by multiplication by p. When we choose a basis $\{w_i\}$ for $H^*(E; \mathbb{Z}/p) = [E, H\mathbb{Z}/p]$ and combine to get a map

$$f : E \to \prod_i \Sigma^{|w_i|} H\mathbb{Z}/p \simeq \bigvee_i \Sigma^{|w_i|} H\mathbb{Z}/p, \tag{1}$$

this map is a cohomology isomorphism, and is thus a homology isomorphism. We then adapt the argument in [5]: let C be the homotopy cofiber of f. Since f is a homology isomorphism, C has trivial homology (and it is also bounded below in homotopy). If C is $(n-1)$-connected, then the Hurewicz map $\pi_n(C) \to H_n(C; \mathbb{Z})$ is an isomorphism, so $\pi_n(C) \otimes \mathbb{Z}/p \to H_n(C; \mathbb{Z}/p)$ is also an isomorphism. But $H_n(C; \mathbb{Z}/p) = 0$, so $\pi_n(C) \otimes \mathbb{Z}/p = 0$ and (using our knowledge of the fact that $\pi_*(E)$ is killed by p) we deduce that $\pi_n(C) = 0$. Thus, C has trivial homotopy groups so f is an isomorphism on homotopy groups, yielding the structure of $\pi_*(E)$.

As it turns out, this situation will occur for MO (note that $\pi_*(MO)$ is killed by 2, since the unoriented cobordism ring is a $\mathbb{Z}/2$-algebra). It will not turn out this nicely for MU; however, we can still use information about cohomology as a module over $[H\mathbb{Z}/p, H\mathbb{Z}/p]$ as part of a spectral sequence to determine $\pi_*(MU)$.

7 The Steenrod algebra

We now turn to a discussion of $[H\mathbb{Z}/2, H\mathbb{Z}/2]$, as well as $[H\mathbb{Z}/p, H\mathbb{Z}/p]$ for $p > 2$ prime. From the previous section, we know that $[H\mathbb{Z}/p, H\mathbb{Z}/p]$ is a $\mathbb{Z}/p$-module.

Lemma 7.1 (Special case of Theorem 2.2.4 in [10], the Yoneda Lemma)**.** *Let X and Y be objects of a category $\mathcal{C}$. Then there is a natural isomorphism*

$$Hom(X, Y) \cong Nat(Hom(-, Y), Hom(-, X))$$

*where $Nat(-,-) : \mathcal{C} \times \mathcal{C} \to$ **Set** is the functor of natural transformations.*

As a consequence of the Yoneda lemma, we can identify $[H\mathbb{Z}/p, H\mathbb{Z}/p]$ with the natural transformations from the contravariant functor $[-, H\mathbb{Z}/p] \cong H^*(-; \mathbb{Z}/p) :$ **hSpectra** $\to$ **Set** to itself. We call such a natural transformation a (mod p) **stable cohomology operation**. The set of mod p stable cohomology operations $[H\mathbb{Z}/p, H\mathbb{Z}/p]$ forms an algebra known as the mod p Steenrod algebra, denoted $\mathcal{A}_p$. Moreover, in the $p = 2$ case this algebra is generated by a relatively simple family of stable cohomology operations known as the **Steenrod squares**

$$\mathrm{Sq}^i : H^*(-; \mathbb{Z}/2) \to H^{*+i}(-; \mathbb{Z}/2).$$

for $i \geq 0$. In the $p > 2$ case, the mod p Steenrod algebra is generated by the **Steenrod powers**

$$P^i : H^*(-; \mathbb{Z}/p) \to H^{*+2i(p-1)}(-; \mathbb{Z}/p)$$

and the Bockstein map

$$\beta : H^*(-; \mathbb{Z}/p) \to H^{*+1}(-; \mathbb{Z}/p).$$

The Steenrod squares have a simple axiomatic characterization, as stated and proved in [13]:

Theorem 7.1. *The Steenrod squares are uniquely defined by the following:*

1. $Sq^0 : H^n(X; \mathbb{Z}/2) \to H^n(X; \mathbb{Z}/2)$ *is the identity.*
2. $Sq^n : H^n(X; \mathbb{Z}/2) \to H^{2n}(X; \mathbb{Z}/2)$ *is the cup square,* $x \mapsto x^2$.
3. $Sq^k : H^n(X; \mathbb{Z}/2) \to H^{n+k}(X; \mathbb{Z}/2)$ *is zero for* $k > n$.
4. *The Cartan formula holds:*

$$Sq^k(xy) = \sum_{i+j=k} (Sq^i x)(Sq^j y)$$

Proposition 7.1 ([2], p. 489)**.** *The Steenrod squares are stable operations, i.e. the diagram*

$$\begin{array}{ccc} H^n(X; \mathbb{Z}/2) & \xrightarrow{Sq^i} & H^{n+i}(X; \mathbb{Z}/2) \\ \downarrow & & \downarrow \\ H^{n+1}(\Sigma X; \mathbb{Z}/2) & \xrightarrow{Sq^i} & H^{n+i+1}(X; \mathbb{Z}/2) \end{array}$$

commutes where the left and right arrows are the suspension isomorphisms.

Consequently, the Steenrod squares define maps $\mathrm{Sq}^i : H^*(-; \mathbb{Z}/2) \to H^{*+i}(-; \mathbb{Z}/2)$ in the category of spectra.

Note that the Steenrod algebra is noncommutative; for example, letting $x \in H^1(\mathbb{R}P^\infty)$ be a generator,

$$\mathrm{Sq}^2\mathrm{Sq}^1 x = \mathrm{Sq}^2(x^2) = (x^2)^2 = x^4 \neq 0 = \mathrm{Sq}^1 0 = \mathrm{Sq}^1\mathrm{Sq}^2 x$$

and so $\mathrm{Sq}^2\mathrm{Sq}^1 \neq \mathrm{Sq}^1\mathrm{Sq}^2$.

Proposition 7.2 (p. 496 in [2])**.** *The algebra $\mathcal{A}_2$ satisfies the* ***Adem relations****:*

$$Sq^i Sq^j = \sum_{k=0}^{\lfloor i/2 \rfloor} \binom{j-k-1}{i-2k} Sq^{i+j-k} Sq^k.$$

While the Adem relations make it possible to work directly with $\mathcal{A}_2$, it turns out that the graded $\mathbb{Z}/2$-dual of $\mathcal{A}_2$ has an algebra structure that is simpler to work with. This comes from the fact that, as shown in [6], $\mathcal{A}_2$ has a comultiplication $\mathcal{A}_2 \to \mathcal{A}_2 \otimes \mathcal{A}_2$ which turns it into a coalgebra (and more strongly, a **bialgebra** and even a **Hopf algebra**):

Definition 7.1. *An F-**coalgebra** is a vector space A over F with linear maps $\Delta : A \to A \otimes A$ (the **comultiplication**) and $\varepsilon : A \to F$ (the **counit**) satisfying coassociativity and coidentity axioms. That is, it is a module object in the opposite category of the monoidal category of F-vector spaces (equipped with the notion of tensor product).*

Milnor proved in [6] that the Steenrod algebra $\mathcal{A}_2$ is a Hopf algebra with comultiplication $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ satisfying

$$\Delta(\mathrm{Sq}^k) = \sum_{i+j=k} \mathrm{Sq}^i \otimes \mathrm{Sq}^j$$

and counit given by $\varepsilon(\mathrm{Sq}^k) = 1$. In fact he proved more generally that $\mathcal{A}_p$ is a Hopf algebra, and consequently, the graded duals of $\mathcal{A}_p$ are Hopf algebras (in particular, algebras and coalgebras) as well.

Theorem 7.2 (Stated as Theorem 3.1.1 in [9], due to [6])**.** *As a graded algebra,*

$$\mathcal{A}_2^* = \mathbb{Z}/2[\xi_1, \xi_2, \xi_3, \dots]$$

where $|\xi_i| = 2^i - 1$. For $p > 2$,

$$\mathcal{A}_p^* = \mathbb{Z}/p[\xi_1, \xi_2, \xi_3, \dots] \otimes E(\tau_0, \tau_1, \tau_2, \dots)$$

where $|\xi_i| = 2(p^i - 1)$, $|\tau_i| = 2p^i - 1$, and $E(\tau_0, \tau_1, \tau_2, \dots)$ is the exterior algebra on the τ_i. The comultiplication maps $\mathcal{A}_p^* \to \mathcal{A}_p^* \otimes \mathcal{A}_p^*$ are given by

$$\Delta\xi_i = \sum_{j=0}^{i} \xi_{i-j}^{2^j} \otimes \xi_j$$

for $p = 2$ where $\xi_0 = 1$, and by

$$\Delta\xi_i = \sum_{j=0}^{i} \xi_{i-j}^{p^j} \otimes \xi_j$$

for $p > 2$ where $\xi_0 = 1$ and

$$\Delta\tau_i = \tau_i \otimes 1 + \sum_{j=0}^{i} \xi_{i-j}^{p^j} \otimes \tau_j.$$

8 (Co)homology

Note that $\pi_*(MO)$ and $\pi_*(MU)$ have natural ring structures, and more generally, $\pi_*(E)$ has a natural ring structure when E is a ring spectrum. This structure is given as follows:

$$[\mathbb{S}, E] \otimes [\mathbb{S}, E] \to [\mathbb{S}, E], \quad (f \otimes g) \mapsto (\mathbb{S} \xrightarrow{f \wedge g} E \wedge E \to E)$$

Similarly, $H_*(E;R)$ has a natural ring structure when E is a ring spectrum. This structure is given as follows:

$$[\mathbb{S}, E \wedge HR] \otimes [\mathbb{S}, E \wedge HR] \to [\mathbb{S}, E \wedge HR], \quad (f,g) \mapsto (\mathbb{S} \xrightarrow{f \wedge g} E \wedge HR \wedge E \wedge HR \cong E \wedge E \wedge HR \wedge HR \xrightarrow{m_E \wedge m_{HR}} E \wedge HR)$$

The computation of $\pi_*(MO)$ and $\pi_*(MU)$ goes as follows:

1. Compute $H_*(BO;\mathbb{Z}/2)$ and $H_*(BU;\mathbb{Z}/p)$ as rings.
2. Show that $H_*(MO;\mathbb{Z}/2) \cong H_*(BO;\mathbb{Z}/2)$ and $H_*(MU;\mathbb{Z}/p) \cong H_*(BU;\mathbb{Z}/p)$ as rings.
3. Show that $H_*(MO;\mathbb{Z}/2)$ is cofree as a comodule over $\mathcal{A}_2^*$, and therefore $H^*(MO;\mathbb{Z}/2)$ is free as a module over $\mathcal{A}_2$. Determine the (not quite cofree) structure of $H_*(MU;\mathbb{Z}/p)$ as a comodule over $\mathcal{A}_p^*$.
4. Apply the same idea we discussed earlier: using the fact that $H^*(H\mathbb{Z}/2) = \mathcal{A}_2$, construct a map of spectra from MO to a wedge of $H\mathbb{Z}/2$ which is an isomorphism on (co)homology, and then conclude that it is an isomorphism on homotopy groups. Apply the Adams spectral sequence to determine $\pi_*(MU)$.

Since we are working over a field, the universal coefficient theorem tells us that $H_*(MG(k);\mathbb{Z}/p)$ is the graded vector space dual of $H^*(MG(k);\mathbb{Z}/p)$, and so $H_*(MG;\mathbb{Z}/p)$ is the graded vector space dual of $H^*(MG;\mathbb{Z}/p)$. (Homology classes are dual elements via the cap pairing $H^i(MG(k)) \otimes H_i(MG(k)) \to \mathbb{F}_p$.)

Theorem 8.1 (Theorem 7.1 in [7])**.** *As rings,*

$$H^*(BO(n);\mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, \dots, w_n]$$

where $w_i \in H^i(BO(n))$ is the i^{th} Stiefel-Whitney class of $BO(n)$.

Proof. Consider the map $BO(1)^n \to BO(n)$ lying under $\gamma^1 \times \cdots \times \gamma^1 \to \gamma^n$. Note that, as vector spaces,

$$H^*(BO(1)^n;\mathbb{Z}/2) \cong H^*(BO(1);\mathbb{Z}/2)^{\otimes n} \cong \mathbb{Z}/2[x]^{\otimes n} = \mathbb{Z}/2[x_1] \otimes \cdots \otimes \mathbb{Z}/2[x_n]$$

using the Kunneth formula. Since the Stiefel-Whitney classes are natural, the induced map on cohomology $H^*(BO(n);\mathbb{Z}/2) \to H^*(BO(1)^n;\mathbb{Z}/2)$ takes $w_i(\gamma^n)$ to $w_i(\gamma^1 \times \cdots \times \gamma^1)$. By the Cartan formula, the total Stiefel-Whitney class is multiplicative and so

$$w(\gamma^1 \times \cdots \times \gamma^1) = w(\gamma^1)^n = (1+x_1)\cdots(1+x_n) = \sum_{i=0}^{n} \sigma_i(x_1, \dots, x_n)$$

implying that the induced map on cohomology takes

$$w_i(\gamma^n) \mapsto w_i(\gamma^1 \times \cdots \times \gamma^1) = \sigma_i(x_1, \dots, x_n) \in H^i(BO(1)^n).$$

If the $w_i(\gamma^n)$ are algebraically dependent, then so are the $\sigma_i(x_1, \dots, x_n)$. But the elementary symmetric polynomials are algebraically independent, so the $w_i(\gamma^n)$ are as well, implying that the kernel is trivial and so the map on cohomology is injective. The image of the map on cohomology consists of the symmetric polynomials in $x_1, \dots, x_n$. $\square$

Theorem 8.2 (Theorem 14.5 in [7]). *As rings,*

$$H^*(BU(n);\mathbb{Z}) \cong \mathbb{Z}[c_1, \dots, c_n]$$

where $c_i \in H^{2i}(BU(n);\mathbb{Z})$ *is the* i^{th} *Chern class of* $BU(n)$.

It follows that, for p prime,

$$H^*(BU(n);\mathbb{Z}/p) \cong H^*(BU(n);\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/p \cong \mathbb{Z}/p[c_1, \dots, c_n].$$

Consider again the map $BO(1)^n \to BO(n)$. Note that, since we are working over a field, the graded dual map of the induced map on cohomology (i.e., the induced map on homology) is the quotient by symmetric polynomials

$$H_*(BO(1)^n;\mathbb{Z}/2) \cong \mathbb{Z}/2[y_0, y_1, \dots]^{\otimes n} \to \mathbb{Z}/2[y_0, y_1, \dots]^{\otimes n}_{\Sigma} = H_*(BO(n);\mathbb{Z}/2)$$

where $M^{\otimes n}_{\Sigma}$ is the n^{th} symmetric power of M, y_i (in degree i) is dual to $x^i \in H^i(BO(1);\mathbb{Z}/2)$ and

$$\mathbb{Z}/2[y_0, y_1, \dots]^{\otimes n} = \mathbb{Z}/2[y_{i_1} \cdots y_{i_n} : i_1 \leq \cdots \leq i_n].$$

Since homology commutes with direct limits ([12] Theorem 5.4.1),

$$H_*(BO;\mathbb{Z}/2) \cong \varinjlim H_*(BO(n);\mathbb{Z}/2) \cong \varinjlim \mathbb{Z}/2[y_{i_1} \cdots y_{i_n} : i_1 \leq \cdots \leq i_n] \cong \mathbb{Z}/2[y_0, y_1, y_2, \dots]$$

as graded vector spaces, where the last vector space is free on the y_i (of degree i), i.e. it is the vector space structure of the polynomial graded ring with variables $\{y_i\}$. In fact, $H_*(BO;\mathbb{Z}/2)$ has a ring structure coming from the homology cross product and $BO(k) \times BO(\ell) \to BO(k+\ell)$, and

$$H_*(BO) \cong \mathbb{Z}/2[y_1, y_2, \dots]$$

as rings where $y_0 = 1$.

Similarly, consider the map $BU(1)^n \to BU(n)$ lying under $\gamma^1_{\mathbb{C}} \times \cdots \times \gamma^1_{\mathbb{C}} \to \gamma^n_{\mathbb{C}}$. Its induced map on mod p homology is the quotient by symmetric polynomials

$$H_*(BU(1)^n;\mathbb{Z}/p) \cong H_*(BU(1);\mathbb{Z}/p)^{\otimes n} = \mathbb{Z}/p[d_0, d_1, d_2, \dots]^{\otimes n} \to \mathbb{Z}/p[d_0, d_1, d_2, \dots]^{\otimes n}_{\Sigma} \cong H_*(BU(n);\mathbb{Z}/p)$$

where $d_i \in H_{2i}(BU(1);\mathbb{Z}/p)$ is dual to $c_1^i \in H^{2i}(BU(1);\mathbb{Z}/p)$. Since homology commutes with direct limits,

$$H_*(BU;\mathbb{Z}/p) \cong \varinjlim_n H_*(BU(n);\mathbb{Z}/p) \cong \varinjlim_n \mathbb{Z}/p[d_{i_1} \cdots d_{i_n} : i_1 \leq \cdots \leq i_n] \cong \mathbb{Z}/p[d_1, d_2, \dots]$$

as rings.

Using our knowledge of the homology of BO and BU, we can compute the homology of MO and MU using the Thom isomorphism theorems:

Theorem 8.3 (Thom Isomorphism Theorem, Theorem 8.1 in [7])**.** *Let $\xi : E \to B$ be a rank-n real vector bundle. Then there exists a **Thom class** $u \in H^n(Th(\xi); \mathbb{Z}/2)$ such that the cup product map*

$$x \mapsto x \cup u : \quad H^k(B; \mathbb{Z}/2) \to H^{n+k}(Th(\xi); \mathbb{Z}/2)$$

is an isomorphism for each k.

Theorem 8.4 (Thom Isomorphism Theorem version 2, Theorem 9.1 in [7])**.** *Let $\xi : E \to B$ be a rank-n oriented real vector bundle. Then there exists a **Thom class** $u \in H^n(Th(\xi); \mathbb{Z})$ such that the cup product map*

$$x \mapsto x \cup u : \quad H^k(B; \mathbb{Z}) \to H^{n+k}(Th(\xi); \mathbb{Z})$$

is an isomorphism for each k.

Corollary 8.1. *If $\xi : E \to B$ is a rank-n complex vector bundle, then there is a Thom class $u \in H^{2n}(Th(\xi); \mathbb{Z})$ such that the cup product*

$$x \mapsto x \cup u : \quad H^k(B; \mathbb{Z}) \to H^{2n+k}(Th(\xi); \mathbb{Z})$$

is an isomorphism for each k.

Proof. This follows from Lemma 14.1 in [7], which says that ξ has a canonical orientation on its underlying rank-$2n$ real bundle. □

The dualization of these Thom isomorphisms tells us that

$$H_*(MO(n); \mathbb{Z}/2) \cong H_{*+n}(BO(n); \mathbb{Z}/2), \qquad H_*(MU(n); \mathbb{Z}) \cong H_{*+2n}(BU(n); \mathbb{Z})$$

as abelian groups. Since the Thom isomorphisms commute with the structure maps, we have Thom isomorphisms

$$H_*(BO; \mathbb{Z}/2) \to H_*(MO; \mathbb{Z}/2), \qquad H_*(BU; \mathbb{Z}) \to H_*(MU; \mathbb{Z})$$

which take

$$H_i(BO(n); \mathbb{Z}/2) \to H_{i+n}(MO(n); \mathbb{Z}/2) \to H_i(MO; \mathbb{Z}/2), \qquad H_i(BU(n); \mathbb{Z}) \to H_{i+2n}(MU(n); \mathbb{Z}) \to H_i(MU; \mathbb{Z}).$$

These turn out to be not just additive isomorphisms, but ring isomorphisms as well. Hence, as a graded $\mathbb{Z}/2$-algebra,

$$H_*(MO; \mathbb{Z}/2) \cong H_*(BO; \mathbb{Z}/2) = \mathbb{Z}/2[y_1, y_2, \dots]$$

where $y_i \in H_i(BO(1))$ is the dual of $x^i \in H^i(BO(1))$ (here, $y_0 = 1$). Let $\tilde{y}_i \in H_{i+1}(MO(1))$ be the elements corresponding to y_i. Note that

$$S\gamma^1 = \{(V, v) : V \in \mathbb{R}P^\infty, v \in V \subset \mathbb{R}^\infty, \|v\| = 1\} \cong \mathbb{S}^\infty$$

which is contractible. Quotienting by a contractible subcomplex of a CW complex is a homotopy equivalence, so $MO(1) = D\gamma^1 / S\gamma^1 \simeq D\gamma^1$. Moreover, $D\gamma^1$ deformation retracts to the zero section

$\cong \mathbb{R}P^\infty$ so it is homotopy equivalent to $\mathbb{R}P^\infty$. Thus, $MO(1) \simeq \mathbb{R}P^\infty$. Let $\tilde{x}$ be a generator of $H^1(MO(1))$; then since $\tilde{y}_i \in H_{i+1}(MO(1))$ and $H_{i+1}(MO(1)) \cong \mathbb{Z}/2$ with nonzero element dual to $\tilde{x}^{i+1}$, it follows that $\tilde{y}_i$ is dual to $\tilde{x}^{i+1}$. We can write

$$H_*(MO;\mathbb{Z}/2) = \mathbb{Z}/2[\tilde{y}_1, \tilde{y}_2, \dots]$$

as a graded $\mathbb{Z}/2$-algebra freely generated by the $\tilde{y}_i$ (each in degree i), so as a graded $\mathbb{Z}/2$-vector space,

$$H^*(MO;\mathbb{Z}/2) = \mathbb{Z}/2[w_1, w_2, \dots]$$

where the right side is the additive structure of the graded polynomial ring on w_i, and w_i is dual to $\tilde{y}_i$.

We also have

$$H_*(MU;\mathbb{Z}) \cong H_*(BU;\mathbb{Z}) \implies H_*(MU;\mathbb{Z}/p) \cong H_*(BU;\mathbb{Z}/p) = \mathbb{Z}/p[d_1, d_2, \dots]$$

as rings, where $d_i \in H_{2i}(BU(1);\mathbb{Z}/p)$ is dual to $c_1^i \in H^{2i}(BU(1);\mathbb{Z}/p)$. Let $\tilde{d}_i \in H_{2i+2}(MU(1);\mathbb{Z}/p)$ be the elements corresponding to $d_i \in H_{2i}(BU(1);\mathbb{Z}/p)$ under Thom isomorphism. Note that $MU(1)$ is homotopy equivalent to $BU(1) = \mathbb{C}P^\infty$, since

$$S\gamma^1_{\mathbb{C}} = \{(V, v) : V \in \mathbb{C}P^\infty, v \in V \subset \mathbb{C}^\infty, \|v\| = 1\} \cong \mathbb{S}^\infty$$

is contractible and thus $MU(1) = D\gamma^1_{\mathbb{C}}/S\gamma^1_{\mathbb{C}} \simeq D\gamma^1_{\mathbb{C}} \simeq \mathbb{C}P^\infty$. Let $z \in H^2(MU(1);\mathbb{Z}/p) \cong \mathbb{Z}/p$ be such that $H^*(MU(1);\mathbb{Z}/p) = \mathbb{Z}/p[z]$. Then $\tilde{d}_i \in H_{2i+2}(MU(1);\mathbb{Z}/p)$ is dual to $z^{i+1} \in H^{2i}(MU(1);\mathbb{Z}/p)$. We can write

$$H_*(MU;\mathbb{Z}/p) = \mathbb{Z}/p[\tilde{d}_1, \tilde{d}_2, \tilde{d}_3, \dots] \tag{2}$$

as a graded ring, where $|\tilde{d}_i| = 2i$.

9 Computing $\pi_*(MU)$

Theorem 9.1. *As a module over $\mathcal{A}$, $H^*(MO;\mathbb{Z}/2)$ is free with basis*

$$\{w_2, w_4, w_5, \dots\} = \{w_j : j \neq 2^i - 1\}.$$

Our goal is to prove this theorem, and we follow closely the proof given in [4]. An equivalent statement of the theorem is that

$$\mathcal{A} \otimes B \to \mathcal{A} \otimes H^*(MO;\mathbb{Z}/2) \to H^*(MO;\mathbb{Z}/2)$$

is a $\mathbb{Z}/2$-vector space isomorphism, where B is the vector space span of $\{w_j : j \neq 2^i - 1\}$; surjectivity of this map means that every element of $H^*(MO;\mathbb{Z}/2)$ is of the form $\sum a_j w_j$ for $a_j \in \mathcal{A}$, and injectivity means that the w_j are linearly independent over $\mathcal{A}$. Dualizing, another equivalent statement is that

$$H_*(MO;\mathbb{Z}/2) \to \mathcal{A}^* \otimes H_*(MO;\mathbb{Z}/2) \to \mathcal{A}^* \otimes B^*$$

is an isomorphism of vector spaces, i.e. that $H_*(MO;\mathbb{Z}/2)$ is "cofree" as a comodule over $\mathcal{A}^*$. Here B^* is the graded dual of B, and $(\mathcal{A}\otimes B)^* \cong \mathcal{A}^*\otimes B^*$ since this isomorphism holds on each (finite-dimensional) graded piece.

To prove that $H_*(MO;\mathbb{Z}/2)\to \mathcal{A}^*\otimes B^*$ is a vector space isomorphism, we prove that it is in fact an isomorphism of $\mathbb{Z}/2$-algebras. Since this is a ring homomorphism, and since $H_*(MO;\mathbb{Z}/2)$ is generated (as a ring) by the $\tilde{y}_i$, it is determined by the image of the $\tilde{y}_i$. Now we only have to consider the map

$$H_*(MO(1);\mathbb{Z}/2) \xrightarrow{\phi} H_*(MO(1);\mathbb{Z}/2)\otimes_{\mathbb{Z}/2}\mathcal{A}^*$$

and see where each $\tilde{y}_i$ is sent.

Note that, since the $\tilde{y}_i$ form a basis for $H_*(MO(1);\mathbb{Z}/2)$ and the ξ_i (dual to $\mathrm{Sq}^{2^{i-1}}\cdots\mathrm{Sq}^2\mathrm{Sq}^1$) form a basis for $\mathcal{A}_2^*$,

$$\phi(\tilde{y}_i)=\sum_{j,k} c_{jk}\tilde{y}_j\otimes\xi_k$$

where $c_{jk}\in\mathbb{Z}/2$, and $c_{jk}=1$ if and only if

$$\mathrm{Sq}^{2^{k-1}}\cdots\mathrm{Sq}^2\mathrm{Sq}^1\tilde{x}^{j+1}=\tilde{x}^{i+1}.$$

Proposition 9.1. *If $i_1,\dots,i_n>0$, then*

$$Sq^{i_1}\cdots Sq^{i_n}\tilde{x}$$

is $\tilde{x}^{2^n}$ if $(i_1,\dots,i_n)=(2^{n-1},\dots,2,1)$, or 0 otherwise.

Proof. The proof is by induction on n. For the base case, suppose $n=1$; then since $\tilde{x}\in H^1(MO(1);\mathbb{Z}/2)$, it follows $\mathrm{Sq}^{i_1}\tilde{x}$ is either $\tilde{x}^2$ if $i_1=1$, or 0 if $i_1>1$.

Now suppose the claim holds for $n-1$ and

$$\mathrm{Sq}^{i_1}\cdots\mathrm{Sq}^{i_n}\tilde{x}\neq 0.$$

Then

$$\mathrm{Sq}^{i_2}\cdots\mathrm{Sq}^{i_n}\tilde{x}\neq 0 \implies (i_2,\dots,i_n)=(2^{n-2},\dots,2,1),\ \mathrm{Sq}^{i_2}\cdots\mathrm{Sq}^{i_n}\tilde{x}=\tilde{x}^{2^{n-1}}$$

and

$$\mathrm{Sq}^{i_1}\cdots\mathrm{Sq}^{i_n}\tilde{x}=\mathrm{Sq}^{i_1}\tilde{x}^{2^{n-1}}.$$

Since the total Steenrod square $\mathrm{Sq}: H^*(\mathbb{R}P^\infty)\to H^*(\mathbb{R}P^\infty)$ is a ring homomorphism by the Cartan formula,

$$\mathrm{Sq}(\tilde{x}^{2^{n-1}})=\mathrm{Sq}(\tilde{x})^{2^{n-1}}=(\tilde{x}+\tilde{x}^2)^{2^{n-1}}=\tilde{x}^{2^{n-1}}+\tilde{x}^{2^n}$$

and thus it must be the case that $i_1=2^{n-1}$ and

$$\mathrm{Sq}^{i_1}\cdots\mathrm{Sq}^{i_n}\tilde{x}=\mathrm{Sq}^{2^{n-1}}\cdots\mathrm{Sq}^2\mathrm{Sq}^1\tilde{x}=\tilde{x}^{2^n}$$

as claimed. $\square$

As a result,
$$\mathrm{Sq}^{2^{k-1}} \cdots \mathrm{Sq}^2\mathrm{Sq}^1\tilde{x} = \tilde{x}^{i+1}$$
if and only if $i+1 = 2^k$, so $c_{0k} = 1$ if and only if $i+1 = 2^k$. That is,
$$\phi(\tilde{y}_i) = \begin{cases} 1 \otimes \xi_k + \tilde{y}_i \otimes 1 + \dots & i+1 = 2^k \\ \tilde{y}_i \otimes 1 + \dots & i+1 \neq 2^k \end{cases}$$
since $\tilde{y}_0 = 1$.

Now consider the map of $\mathbb{Z}/2$-algebras
$$H_*(MO;\mathbb{Z}/2) \xrightarrow{\psi} B^* \otimes \mathcal{A}^* = H_*(MO;\mathbb{Z}/2)/\langle \tilde{y}_j : j = 2^k - 1\rangle \otimes \mathcal{A}^* = \mathbb{Z}/2[\tilde{y}_j : j \neq 2^k - 1] \otimes \mathbb{Z}/2[\xi_1, \xi_2, \dots].$$
We have shown that
$$\psi(\tilde{y}_i) = \begin{cases} 1 \otimes \xi_k + \dots & i+1 = 2^k \\ \tilde{y}_i \otimes 1 + \dots & i+1 \neq 2^k \end{cases}$$
since $\tilde{y}_i \otimes 1$ is quotiented out when $i+1 = 2^k$. Note that the codomain
$$\mathbb{Z}/2[\tilde{y}_j : j \neq 2^k - 1] \otimes \mathbb{Z}/2[\xi_1, \xi_2, \dots]$$
is a graded polynomial algebra on the $\tilde{y}_j \otimes 1$ (of degree j) and $1 \otimes \xi_i$ (of degree $2^i - 1$) since the tensor product of polynomial algebras is a polynomial algebra.

Proposition 9.2. *The map ψ is an isomorphism of graded $\mathbb{Z}/2$-algebras.*

Proof. We claim by induction on n that each element of the codomain of degree $\leq n$ is hit by ψ. For $n = 0$, this is trivial since $\psi(1) = 1 \otimes 1$. Now suppose the claim is true for $n-1$. If $n \neq 2^k - 1$, then $\tilde{y}_n \otimes 1$ is of degree n, and
$$\psi(\tilde{y}_n) = \tilde{y}_n \otimes 1 + \dots$$
where the $\dots$ include elements $\tilde{y}_j \otimes \xi_\ell$ for $\deg(\tilde{y}_j), \deg(\xi_\ell) < n$. For $\tilde{y}_j \otimes \xi_\ell$ with $\deg(\tilde{y}_j), \deg(\xi_\ell) < n$, we can use the inductive hypothesis to see that
$$\tilde{y}_j = \psi(a),\ \xi_\ell = \psi(b),\ \psi(\tilde{y}_j \otimes \xi_\ell) = \psi(a)\psi(b) = \psi(ab)$$
so each decomposable element in the ellipses is in the image of ψ, hence by additivity of ψ,
$$\tilde{y}_n \otimes 1 \in \mathrm{im}(\psi).$$
Similarly, if $n = 2^k - 1$, then $1 \otimes \xi_k \in \mathrm{im}(\psi)$ by the same argument. Thus all generators of degree $\leq n$ are in the image, so every element of degree $\leq n$ is in the image. This proves the inductive claim, and so ψ is surjective.

Note that ψ is a graded map of graded $\mathbb{Z}/2$-vector spaces, i.e. it takes elements of degree n to elements of degree n. (This comes from the fact that $\psi(\tilde{y}_i)$ has degree i.) In order to prove that ψ is injective, it suffices to show that it is injective on each graded piece. Note that the n^{th} graded piece of the domain consists of homogenous degree-n polynomials in the $\tilde{y}_i$ (along with 0), so the degree-n monomials, of which there are

$$N = |\{(i_1, \dots, i_r) : \deg(\tilde{y}_1^{i_1} \tilde{y}_2^{i_2} \cdots \tilde{y}_r^{i_r}) = i_1 + 2i_2 + \cdots + ri_r = n\}|,$$

form a basis. Note that the codomain is isomorphic to the domain as a graded algebra as $\tilde{y}_j$ corresponds to itself for $j \neq 2^k - 1$, and ξ_k corresponds to $\tilde{y}_j$ for $j = 2^k - 1$. Thus, the n^{th} graded piece of the codomain has dimension N as well. A surjective map of vector spaces of the same (finite) dimension is injective, so ψ is injective on each graded piece and is thus injective. This proves that ψ is injective and surjective, hence is an isomorphism. $\square$

This proves Theorem 7.1, and so $H^*(MO; \mathbb{Z}/2)$ is free on $\{w_j : j \neq 2^i - 1\}$. As we discussed in Section 6, this is enough to show that

$$\mathfrak{N}_* \cong \pi_*(MO; \mathbb{Z}/2) \cong \mathbb{Z}/2[w_j : j \neq 2^i - 1].$$

10 Module Structure of $H^*(MU; \mathbb{Z}/p)$ over $\mathcal{A}_p$

Theorem 10.1 (Lemmas 3.1.6, 3.1.7 in [9]). *Let $P^* \subset \mathcal{A}_p^*$ be defined as either $P^* = \mathbb{Z}/2[\xi_1^2, \xi_2^2, \dots]$ for $p = 2$ or as $P^* = \mathbb{Z}/p[\xi_1, \xi_2, \dots]$ for $p > 2$. Then P^* is a subcoalgebra of $\mathcal{A}_p^*$, and as a comodule over $\mathcal{A}_p^*$, $H_*(MU; \mathbb{Z}/p) = P^* \otimes \mathbb{Z}/p[\tilde{d}_i : i \neq p^k - 1]$.*

Proof. We follow the argument in [9]. First, we need to show that P^* in fact a coalgebra. In the $p = 2$ case, we have $P^* = \mathbb{Z}/2[\xi_1^2, \xi_2^2, \dots]$, and the comultiplication $\Delta : \mathcal{A}_2^* \to \mathcal{A}_2^* \otimes \mathcal{A}_2^*$ satisfies

$$\Delta(\xi_i^2) = (\Delta(\xi_i))^2 = \left(\sum_{j=0}^{i} \xi_{i-j}^{2^j} \otimes \xi_j \right)^2 = \sum_{j=0}^{i} (\xi_{i-j}^{2^j} \otimes \xi_j)^2 = \sum_{j=0}^{i} \xi_{i-j}^{2^{j+1}} \otimes \xi_j^2 \in P^* \otimes P^*$$

using the fact that $\mathcal{A}_2^*$ has characteristic 2. This implies that Δ restricts to a comultiplication $\Delta_{P^*} : P^* \to P^* \otimes P^*$. In the $p > 2$ case, we have $P^* = \mathbb{Z}/p[\xi_1, \xi_2, \dots]$ and the comultiplication $\Delta : \mathcal{A}_p^* \to \mathcal{A}_p^* \otimes \mathcal{A}_p^*$ satisfies

$$\Delta(\xi_i) = \sum_{j=0}^{i} \xi_{i-j}^{p^j} \otimes \xi_j \in P^* \otimes P^*,$$

so that Δ restricts to a comultiplication $\Delta_{P^*} : P^* \to P^* \otimes P^*$. In either case, we have proven that P^* is a subcoalgebra of $\mathcal{A}_p^*$.

Next we claim that $H_*(MU; \mathbb{Z}/p)$ is a comodule over P^*, i.e. the structure map

$$\phi : H_*(MU; \mathbb{Z}/p) \to \mathcal{A}_p^* \otimes H_*(MU; \mathbb{Z}/p)$$

has its image in $P^* \otimes H_*(MU; \mathbb{Z}/p)$. Recall, by Equation 2, that

$$H_*(MU; \mathbb{Z}/p) = \mathbb{Z}/p[\tilde{d}_1, \tilde{d}_2, \tilde{d}_3, \dots]$$

where $|\tilde{d}_i| = 2i$. In particular, $H_i(MU;\mathbb{Z}/p) = 0$ for i odd. Let $\{m_i\}$ be an additive basis for $H_*(MU;\mathbb{Z}/p)$, such as the set of finite products $\tilde{d}_{i_1}\cdots\tilde{d}_{i_k}$. Let

$$\phi(m_n) = \sum_i a_{i,n} \otimes m_i$$

where $a_{i,n} \in \mathcal{A}_p^*$. Since ϕ is a graded map and since each m_i has even degree, it follows that each $a_{i,n}$ has even degree. Note that since $H_*(MU;\mathbb{Z}/p)$ is a comodule over $\mathcal{A}_p^*$, we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{A}_p^* \otimes \mathcal{A}_p^* \otimes H_*(MU;\mathbb{Z}/p) & \mathcal{A}_p^* \otimes H_*(MU;\mathbb{Z}/p) \arrow[l, "\Delta\otimes 1"'] \\
\mathcal{A}_p^* \otimes H_*(MU;\mathbb{Z}/p) \arrow[u, "1\otimes\phi"] & H_*(MU;\mathbb{Z}/p) \arrow[l, "\phi"'] \arrow[u, "\phi"']
\end{tikzcd}$$

Therefore,

$$\begin{aligned}
(1\otimes\phi)(\phi(m_n)) &= \sum_i a_{i,n} \otimes \phi(m_i) \\
&= \sum_i a_{i,n} \otimes \sum_j a_{j,i} \otimes m_j \\
&= \sum_j \left(\sum_i a_{i,n} \otimes a_{j,i}\right) \otimes m_j \\
&= (\Delta\otimes 1)(\phi(m_n)) \\
&= \sum_j \Delta(a_{j,n}) \otimes m_j.
\end{aligned}$$

This tells us that $\Delta(a_{j,n})$ can be written as $\sum_i a_{i,n} \otimes a_{j,i}$, which consists of pieces that are purely even-dimensional.

If $p = 2$ and $i > 0$, then $\Delta(\xi_i) = \sum_{j=0}^{i} \xi_{i-j}^{2^j} \otimes \xi_j$ has an odd-dimensional piece $\xi_{i-1}^2 \otimes \xi_1$ (as $|\xi_1| = 1$ is odd). However,

$$\Delta(\xi_i^2) = (\Delta(\xi_i))^2 = \left(\sum_{j=0}^{i} \xi_{i-j}^{2^j} \otimes \xi_j\right)^2 = \sum_{j=0}^{i} \xi_{i-j}^{2^{j+1}} \otimes \xi_j^2$$

has purely even-dimensional pieces. If $p > 2$ and $i \geq 0$, then

$$\Delta(\tau_i) = \tau_i \otimes 1 + \sum_{j=0}^{i} \xi_{i-j}^{p^j} \otimes \tau_j$$

has an odd-dimensional piece $\tau_i \otimes 1$, since $|\tau_i| = 2p^i - 1$ is odd. In fact, the constraint that $\Delta(a_{j,n})$ consists of purely even-dimensional pieces tells us that $a_{j,n} \in P^*$. Therefore, $H_*(MU;\mathbb{Z}/p)$ is a comodule over P^*. By an argument similar to the one we gave in calculating the comodule structure of $H_*(MO;\mathbb{Z}/2)$ over $\mathcal{A}_2^*$, it turns out that $H_*(MU;\mathbb{Z}/p)$ is cofree as a comodule over P^*, with "cobasis" $\{\tilde{d}_i : i \neq p^k - 1\}$. $\square$

Note that, unlike in the case of MO, the mod p homology of MU is not cofree as a comodule over $\mathcal{A}_p^*$ (i.e., the mod p cohomology is not free as a module over $\mathcal{A}_p$). Thus, we cannot use the same method as earlier to compute $\pi_*(MU)$. However, it is possible to do so using a spectral sequence known as the Adams spectral sequence.

Theorem 10.2 (Theorem 2.2.3 in [9]). *Let X be a spectrum. Then there is a spectral sequence with differentials $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$ such that $E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_p^*}^{s,t}(\mathbb{Z}/p, H_*(X;\mathbb{Z}/p))$, and the E_∞ page is associated to a filtration of $\pi_*(X) \otimes \mathbb{Z}/p$.*

We refer to [9] (Section 3.1) for the remainder of the calculation. In our case, the E_2 page equals the E_∞ page, and [9] proves first that $\pi_*(MU) \otimes \mathbb{Z}/p = 0$ for all p, and subsequently that the global structure $\pi_*(MU)$ is simply $\mathbb{Z}[x_1, x_2, \dots]$ with $|x_i| = 2i$.

11 Conclusion

We have seen how cobordism rings (in this case, unoriented and complex cobordism rings) can be computed by reducing to the computation of π_* of a Thom spectrum, via the Thom-Pontryagin isomorphism. Many other flavors of cobordism exist, perhaps most notably oriented cobordism, but also including spin cobordism, symplectic cobordism, etc.; see [14] for a survey. While some cobordism rings are known, some are not; an important example is the case of framed bordism, as the framed bordism ring is isomorphic to the stable homotopy groups of spheres.

There are a couple natural questions about cobordism that we have not yet answered, including the natural question: How can we tell in practice whether two manifolds are cobordant? As it turns out, this is possible by means of characteristic classes. Given a closed n-manifold M and nonnegative integers $i_1, \dots, i_k$ with $i_1 + 2i_2 + \cdots + ki_k = n$, we have an associated **Stiefel-Whitney number** in $\mathbb{Z}/2$ which we obtain by pairing the mod 2 fundamental class $[M] \in H_n(M)$ with $w_1(M)^{i_1} \cdots w_k(M)^{i_k} \in H^n(M)$. Two manifolds are cobordant (in the unoriented sense) if and only if they have the same Stiefel-Whitney numbers for all possible tuples $(i_1, \dots, i_k)$ (see Corollary 4.11 in [7]); in particular, the nonvanishing of any Stiefel-Whitney number of M implies that M is not null-cobordant, i.e. it is not the boundary of a compact manifold. The real projective plane $\mathbb{R}P^2$ is one such example (corresponding to w_2 in $\mathfrak{N}_* \cong \mathbb{Z}/2[w_2, w_4, w_5, \dots]$).

Another natural question is that of determining reasonable descriptions of generators for cobordism rings. It turns out that there are reasonably simple representatives of generators for the unoriented cobordism ring; in 1956, Dold proved that there is a set of generators consisting of $\mathbb{R}P^n$ for n even, and for $n \neq 2^k - 1$ odd a **Dold manifold** of the form $P(a,b) = (\mathbb{S}^a \times \mathbb{C}P^b)/((x,[y]) \sim (-x,[\overline{y}]))$; see [1].

Finally, while unoriented cobordism may be more geometrically natural than complex cobordism, complex cobordism is theoretically significant. As one example, Quillen's theorem [8] establishes a surprising connection between the complex cobordism ring and the theory of formal group laws. Given a ring R, a **formal group law** on R is an element $f(x,y) \in R[[x,y]]$ such that $f(x,y) = x + y + \cdots$ and $f(f(x,y),z) = f(x, f(y,z))$; a commutative formal group law also satisfies $f(x,y) = f(y,x)$. Consider the multiplication map $m : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$. This map induces a map on the cohomology theory of cobordism, i.e. $MU^*(\mathbb{C}P^\infty) \to MU^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$. By [9], Theorems 1.3.2 and 1.3.3, there is some $x \in MU^2(\mathbb{C}P^\infty)$ such that $MU^*(\mathbb{C}P^\infty) = MU^*(*)[[x]]$ and $MU^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = MU^*(*)[[x \otimes 1, 1 \otimes x]]$, and $m^*(x) \in MU^*(*)[[x \otimes 1, 1 \otimes x]]$ is a formal group law on $MU^*(*)$. Quillen's theorem states that the complex cobordism ring $MU^*(*)$, together with this formal group law, is isomorphic to the Lazard ring carrying the universal formal group law.

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