The Gelfand-MacPherson Correspondence and Torus Orbit Closures in Grassmannians

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1 Introduction

Geometric invariant theory (GIT) is motivated by the desire to construct a quotient of an algebraic variety X by the action of a linear algebraic group G. We note that the naive quotient X/G is almost always non-separated due to the existence of non-closed orbits. GIT gives a way of choosing subsets of X such that the quotients are quasiprojective varieties. The Gelfand-MacPherson construction is an isomorphism of two GIT quotients: the Grassmannian $\mathrm{Gr}(2, n)$ under the action of the torus $\mathbb{G}_m^n$ and the n-fold product of projective space $(\mathbb{P}^1)^n$ under the action of SL_2 [19]. Understanding these quotients is particularly important due to connections to hypergeometric functions [18], polylogarithms [19], and combinatorial formulas for Chern and Pontryagin classes [1] (additionally, see the references in [3] for discussion of these connections).

In the first part of this thesis, we introduce the Gelfand-MacPherson isomorphism, motivated by a concrete invariant theory question. Specifically, our motivating question is to show that the ring of invariants of the diagonal action of SL_2 on $(\mathbb{P}^1)^4$ of equal degree in each set of variables is generated as a k-algebra by the polynomials (2.1)-(2.3), which we note determine the cross ratio of 4 points in $\mathbb{P}^1$, modulo the relation (2.4). In proving this result, we also present a proof of the first fundamental theorem of invariant theory for $\mathrm{SL}_2(k)$, where k is any infinite field, using the theory of standard monomials. We use the example of the Gelfand-MacPherson construction to introduce in an accessible manner some of the central ideas of GIT such as linearization of algebraic group actions and the notion of stability of points with respect to an action [7],[10],[27]. Additionally, we compute explicitly what the conditions on the weights $(d_1, \ldots, d_n)$ are for the invariant ring, or equivalently the quotient under this action, to be non-empty.

It turns out that the invariant theory of subgroups of the general linear group is closely related to the intersection theory of Grassmannians, which is the main topic in the second part of this thesis. For example, the multiplicity of each representation in a direct sum decomposition of the tensor product of irreducible representations of the general linear group is given the Littlewood-Richardson coefficients [14]. In this section, we introduce Schubert calculus and explain how the Schubert varieties generate the cohomology ring of the Grassmannian with the cup product given by intersection. We prove a formula for computing the product of arbitrary Schubert cycles in $\mathrm{Gr}(2, n)$, and additionally introduce Pieri's formula and the Littlewood-Richardson rule for intersections in an arbitrary Grassmannian. Finally, we consider the right action of the torus $\mathbb{G}_m^n$ on the Grassmannian $\mathrm{Gr}(k, n)$ and torus orbit closures for generic subspaces under this action. In particular, we explore further a formula stated by Klyachko in [23] for the decomposition of the cohomology class of the torus orbit closure in terms of the Schubert classes.

2 Invariants in $(\mathbb{P}^1)^4$

Let k be a field. We start by considering the diagonal action of $\mathrm{SL}_2(k)$ on $(\mathbb{P}^1)^4$, on which the coordinates are labelled by $(z_1, w_1; z_2, w_2; z_3, w_3; z_4, w_4)$. We now consider the ring $k[z_1, w_1, z_2, w_2, z_3, w_3, z_4, w_4]$ and want to find invariants of equal degree in each set of (z_i, w_i). We call this an invariant of degree (d, d, d, d) under this action.

Starting with the $d = 1$ case, we immediately see that the polynomials

$$(z_1w_2 - z_2w_1)(z_3w_4 - z_4w_3),\ (z_1w_3 - z_3w_1)(z_2w_4 - z_4w_2),\ (z_1w_4 - z_4w_1)(z_2w_3 - z_3w_3)$$

are invariants satisfying the equal degree condition. We label these polynomials as

$$x_{12,34} =(z_1w_2 - z_2w_1)(z_3w_4 - z_4w_3), \tag{2.1}$$
$$x_{13,24} =(z_1w_3 - z_3w_1)(z_2w_4 - z_4w_2), \tag{2.2}$$
$$x_{14,23} =(z_1w_4 - z_4w_1)(z_2w_3 - z_3w_3). \tag{2.3}$$

One notices that they satisfy the relation

$$x_{12,34} - x_{13,24} + x_{14,23} = 0. \tag{2.4}$$

A fundamental question in invariant theory, known as Hilbert's 14th problem, asks whether the ring of invariants of any linear algebraic group G acting on a finitely generated k-algebra is finitely generated. It turns out that more assumptions need to be placed on G for this to be true, which motivates the following definitions (see [4], [27] for more on algebraic groups).

Definition 1. *An algebraic group is a group object in the category of varieties over an algebraically closed field k. A linear algebraic group is an algebraic group that is an affine variety.*

By this definiton, a morphism of algebraic groups is a group homomorphism that is also a morphism of varieties. Additionally, a (rational) representation of a linear algebraic group G on a k-vector space V is a morphism of algebraic groups $\rho : G \to \mathrm{GL}(V)$.

Definition 2. *Let k be an algebraically closed field. A linear algebraic group G is linearly reductive if every rational representation of G is completely reducible.*

It turns out that the answer to Hilbert's 14th problem is positive if G is linearly reductive.

Theorem 1. *[27, Theorem 4.51] Let G be linearly reductive algebraic group acting on a finitely generated k-algebra A. Then, the ring of invariants A^G is finitely generated.*

In our case, we have that $\mathrm{SL}_2(k)$ is linearly reductive if k is algebraically closed with characteristic 0, so the above theorem gives us that A^{SL_2} is finitely generated in this case. However, it turns out that A^{SL_2} is finitely generated for any infinite field [26]. Specifically, we claim that if $|k| = \infty$, then

$$k[z_1, w_1, z_2, w_2, z_3, w_3, z_4, w_4]^{\mathrm{SL}_2(k)}_{(d,d,d,d)} = \frac{k[x_{12,34}, x_{13,24}, x_{14,23}]}{\langle x_{12,34} - x_{13,24} + x_{14,23} \rangle}.$$

2.1 First Fundamental Theorem of Invariant Theory

Theorem 2. *The algebra of invariants under the action of $\mathrm{SL}_n(k)$ defined by left multiplication on the set of $n \times m$ matrices $M = \{x_{ij}\}$ is generated by the determinants of the $n \times n$ minors of M.*

Following the approach in [26], we now prove this for $n = 2$ in a characteristic independent manner by utilizing the theory of standard monomials, which is an analog of Gröbner bases for subalgebras rather than ideals. Our reference for standard monomial theory is [29]. The general setting is as follows: Let R be an A-algebra, and let $S = \{s_1, \ldots, s_n\} \subseteq R$ be a partially ordered set.

Definition 3. *An ordered product $s_{i_1} \ldots s_{i_k}$ of elements is a **standard monomial** if the elements of the product appear in nondecreasing order with respect to the partial ordering on S. We say that R has a **standard monomial theory** for S if the standard monomials form a basis for R over A.*

We now consider the k-algebra $k[x_1, y_1, \ldots, x_n, y_n] \coloneqq k[\mathbf{x}, \mathbf{y}]$, and define the polynomials

$$f_{i,j} = x_i y_j - y_i x_j.$$

Clearly, $f_{i,i} = 0$ and $f_{i,j} = -f_{j,i}$. Additionally, the $f_{i,j}$ satisfy the Plücker type relation:

$$f_{i,j} f_{k,l} = f_{i,k} f_{j,l} - f_{i,l} f_{j,k}. \tag{2.5}$$

We now show that $k[\mathbf{x}, \mathbf{y}]^{\mathrm{SL}_2} = k[f_{i,j}]$, which is the statement of the first fundamental theorem of invariant theory for $\mathrm{SL}_2(k)$.

We now consider the algebra $k[f_{i,j}]$. Any product $f_{i_1,j_1} f_{i_2,j_2} \ldots f_{i_m,j_m}$ can be associated with the diagram

$$\begin{bmatrix} i_1 & i_2 \ldots i_m \\ j_1 & j_2 \ldots j_m \end{bmatrix}.$$

Since $f_{i,j} = -f_{j,i}$, we can replace any non-decreasing indices in the $f_{i,j}$, so we require that $i_k < j_k$ for all k in the above diagram. We define the standard monomials in $k[f_{i,j}]$ as the set of products $f_{i_1,j_1} f_{i_2,j_2} \dots f_{i_m,j_m}$ such that $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_m$ and $j_1 \leqslant i_2 \leqslant \cdots \leqslant j_m$, We define a partial ordering on the standard monomials by denoting

$$\begin{bmatrix} i_1 & i_2 \dots i_m \\ j_1 & j_2 \dots j_m \end{bmatrix} \leqslant \begin{bmatrix} u_1 & u_2 \dots u_n \\ v_1 & v_2 \dots v_n \end{bmatrix}$$

if and only if $m \leqslant n$, $i_k \geqslant u_k$, and $j_\ell \geqslant v_\ell$ for all $1 \leqslant k \leqslant m$ and $1 \leqslant \ell \leqslant n$.

Lemma 1. *The standard monomials form a k-basis of $k[f_{i,j}]$.*

Proof. The relation (2.5) implies that the standard monomials span $k[f_{i,j}]$. Additionally, if we fix the monomial ordering $x_1 \prec y_1 \prec x_2 \prec y_2 \cdots \prec x_n \prec y_n$, all of the $f_{i,j}$ have pairwise distinct leading terms and are hence linearly independent. □

Lemma 2. *Let $F_1, \dots, F_k$ be any finite collection of standard monomials, and let $p = \sum_{j=1}^{k} c_j F_j$ be a nonzero k-linear combination. If for some i, $p|_{(x_i,y_i)=(0,0)} = 0$, then for each $1 \leqslant j \leqslant k$, F_j is divisible by f_{i,r_j} for some $r_j \in \{1, \dots, n\}$.*

Proof. We assume for the sake of contradiction that there exist standard monomials $F_1, \dots, F_k$ which are not divisible by any $f_{i,r}$ for $r \in \{1, \dots, n\}$. It then follows that for each $1 \leqslant j \leqslant k$,

$$F_j = F_j|_{(x_i,y_i)=(0,0)}.$$

Then, evaluating p at $(x_i, y_i) = (0, 0)$ gives

$$0 = \sum_{j=1}^{k} c_j F_j$$

for c_j not all zero, which contradicts the fact that the F_j are linearly independent. □

One more lemma will be useful before proving the first fundamental theorem for $\mathrm{SL}_2(k)$.

Lemma 3. *Let $q \in k[\mathbf{x}, \mathbf{y}]$ be a polynomial satisfying $f_{1,2} \cdot q \in k[f_{i,j}]$. Then q is also an element of $k[f_{i,j}]$.*

Proof. We start by writing $p = f_{1,2} \cdot q$ as a k-linear combination of standard monomials, $p = \sum_{\alpha \in I} c_\alpha F_\alpha$. Our goal is to prove for all $\alpha \in I$, the standard monomial F_α is divisible by $f_{1,2}$, which will imply that $q \in k[f_{i,j}]$ if p is. We note that

$$p|_{(x_1,y_1)=(0,0)} = 0 \quad \text{and} \quad p|_{(x_2,y_2)=(0,0)} = 0,$$

since $f_{1,2}|p$, and $f_{1,2}|_{(x_k,y_k)=(0,0)} = 0$ for $k = 1, 2$. By the previous lemma, we have that for all $\alpha \in I$, F_α is divisible by $f_{1,r}$ and $f_{2,s}$ for some $r, s \in \{1, \dots, n\}$. We claim that we can choose $r = 2$.

We assume for the sake of contradiction that there exist $F_{\alpha_1}, \ldots, F_{\alpha_k}$ which are not divisible by $f_{1,2}$. Then, we can write

$$p = f_{1,2} \cdot P + \sum_{\ell=1}^{k} c_{\alpha_\ell} F_{\alpha_\ell},$$

where P is a sum of standard monomials of degree less than $\deg(p)$. We now consider the k-algebra $k[\mathbf{x}, \mathbf{y}, \lambda]$, where λ is an indeterminate. We define the substitution $*$ which maps $(x_1, y_1) \mapsto (\lambda x_2, \lambda y_2)$ and note that

$$\begin{cases} f_{1,k}^* = \lambda f_{2,k} & \text{for all } k \geqslant 2 \\ f_{i,j}^* = f_{i,j} & \text{if } i, j \neq 1. \end{cases}$$

It follows that if a polynomial is divisible by $f_{1,2}$, it is mapped to zero under this substitution. Additionally, we have that $*$ acts injectively on standard monomials not divisible by $f_{1,2}$. We now apply $*$ to $p = f_{1,2}P + \sum_{l=1}^{k} c_{\alpha_\ell} F_{\alpha_\ell}$. We have that

$$p^* = 0 = \sum_{\ell=1}^{k} c_{\alpha_\ell} F_{\alpha_\ell}^*.$$

However, $*$ is injective on the $F_{\alpha_\ell}^*$, and all of the $F_{\alpha_\ell}^*$ are non-zero, so we have reached a contradiction. □

We are finally ready to prove the first fundamental theorem for $\mathrm{SL}_2(k)$. By linearity of the SL_2 action, it suffices to consider homogeneous polynomials in $k[\mathbf{x}, \mathbf{y}]$. We consider the matrix

$$S = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in \mathrm{SL}_2(k) \tag{2.6}$$

for $t \in k^*$. If p a homogeneous degree d polynomial invariant under the action of SL_2, then, using multi-indices for the x_i, y_i, we have that

$$p = \sum_{i \in \mathbb{N}^n} c_i x^i y^{d-i} = S \cdot p = \sum_{i \in \mathbb{N}^n} c_i t^d (x^i)(t^{-1})^{d-i}(y^{d-i}) = \sum_{i \in \mathbb{N}^n} c_i x^i y^{i-d}.$$

Therefore, $d = 2i$, and we see that the polynomial has to be homogeneous degree m in both the $\{x_i\}$ and $\{y_i\}$. We now consider an arbitrary matrix $G \in \mathrm{GL}_2(k)$. We can write

$$G = G \begin{pmatrix} \det(G)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \det(G) & 0 \\ 0 & 1 \end{pmatrix},$$

and since $G \begin{pmatrix} \det(G)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(k)$, if p is invariant under the action, the action of G on p reduces to the action of the matrix

$$\tilde{G} = \begin{pmatrix} \det(G) & 0 \\ 0 & 1 \end{pmatrix}.$$

We have that

$$G \cdot p(x_i, y_i) = \tilde{G} \cdot p(x_i, y_i) = p(\det(G)x_i, y_i) = (\det G)^m p(x_i, y_i) = (ad - bc)^m p(x_i, y_i).$$

Because the equality holds on $\mathrm{GL}_2(k)$, which is a Zariski dense set of $\mathrm{Mat}_{2\times 2}(k)$, it holds on all of $\mathrm{Mat}_{2\times 2}(k)$ since $|k| = \infty$. We can therefore consider a, b, c, d as variables, and after substituting $a = -y_1, b = x_1, c = -y_2, d = x_2$, we have that

$$G \cdot p(x_i, y_i) \mapsto f_{1,2}^m \cdot p(x_i, y_i) = p(f_{1i}, f_{2i}) \in k[f_{i,j}].$$

It follows by Lemma (3) that $p(x_i, y_i) \in k[f_{i,j}]$, thereby completing the proof.

2.2 The Gelfand-MacPherson Correspondence

The first fundamental theorem of invariant theory shows that

$$k[z_1, w_1, z_2, w_2, z_3, w_3, z_4, w_4]^{\mathrm{SL}_2} = \frac{k[p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}]}{\langle p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}\rangle},$$

where the p_{ij} are the minors consisting of the i, j rows of the matrix

$$\begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ w_1 & w_2 & w_3 & w_4 \end{pmatrix}.$$

We note that this invariant ring is the homogeneous coordinate ring for the Grassmannian $\mathrm{Gr}(2,4)$ under the Plücker embedding into $\mathbb{P}^5$, and we denote this ring as $k[p_{ij}]$ from now on.

While we now know what the ring of invariants under the SL_2 action is, we are still specifically interested in the invariants of degree (d, d, d, d). Motivated by this question, we now consider two actions on the space of $2 \times n$ matrices $\mathrm{Mat}(2, n)$: the left action by SL_2 given by

$$A \cdot M \mapsto AM$$

and the right action by the torus $\mathbb{G}_m^n$, where $\mathbb{G}_m := \operatorname{Spec} k[t, s]/(ts - 1) \cong \operatorname{Spec} k[t, t^{-1}]$. The torus action is given by right multiplication:

$$M \cdot \mathrm{diag}(t_1, \ldots, t_n) = M(\mathrm{diag}(t_1, \ldots, t_n)).$$

By the first fundamental theorem of invariant theory, we have that

$$\mathrm{SL}_2 \backslash \mathrm{Mat}(2, n) \cong \mathrm{Gr}(2, n),$$

where the quotient $\mathrm{SL}_2 \backslash \mathrm{Mat}(2, n)$, is interpreted as $\mathrm{Proj}(\mathrm{Mat}(2, n)^{\mathrm{SL}_2}) = \mathrm{Proj}(k[p_{ij}])$, the projective variety associated to the $\mathbb{Z}_{\geqslant 0}$ graded ring of invariants.

We may now consider the quotient with respect to the right action of the torus on the Grassmannian,

$$(\mathrm{SL}_2 \backslash \mathrm{Mat}(2, n))/\mathbb{G}_m^n \cong \mathrm{Gr}(2, n)/\mathbb{G}_m^n.$$

Conversely, we have that

$$\mathrm{Mat}(2, n)/\mathbb{G}_m^n \cong (\mathbb{P}^1)^n,$$

and considering the leftover right action by SL_2 shows that

$$\mathrm{SL}_2 \backslash (\mathrm{Mat}(2, n)/\mathbb{G}_m^n) \cong \mathrm{SL}_2 \backslash (\mathbb{P}^1)^n.$$

We therefore have the isomorphism

$$\mathrm{SL}_2\backslash(\mathbb{P}^1)^n \cong \mathrm{Gr}(2,n)/\mathbb{G}_m^n. \tag{2.7}$$

which is known as the Gelfand-MacPherson construction [15], [20].

We note that there is some constructive imprecision surrounding the definition of these quotients (see [7],[13],[27] for more on the full definition of a GIT quotient). Although the action of $\mathbb{G}_m^n$ on $\mathrm{Gr}(k,n)$ is defined unambiguously, the lift to the homogeneous coordinate ring is ambiguous. We consider the example of $\mathrm{Gr}(2,n)$ for simplicity. If we define the action

$$p_{ij} \mapsto t_i t_j p_{ij},$$

we have that $k[p_{ij}]^{\mathbb{G}_m^n} = k$, so there are no non-trivial invariants. Therefore, we fix the character $\chi \in \mathrm{Hom}(\mathbb{G}_m^n, \mathbb{G}_m)$ given by

$$\chi(t_1,\ldots,t_n) = t_1^{d_1}\ldots t_n^{d_n}$$

and define

$$k[p_{ij}]_\chi = \{f \in k[p_{ij}] : f(t_i t_j p_{ij}) = \chi \cdot f(p_{ij})\}.$$

In other words, we are looking for covariants (also called semiinvariants) with respect to χ. We then define the graded ring

$$R = \bigoplus_{n\geqslant 0} k[p_{ij}]_{\chi^n}.$$

We note that the lift of action to the coordinate ring R depends on the choice of embedding into projective space, and additionally for choices of χ that do not simply differ by an integer multiple in the $(d_1,\ldots,d_n)$, taking $\mathrm{Proj}(R)$ may lead to different quotients.

Because of this ambiguity, for $X \hookrightarrow \mathbb{P}^n$ a projective variety, G a linearly reductive algebraic group, and $\chi \in \mathrm{Hom}(G,\mathbb{G}_m)$ a character, we define, following [27], the Proj quotient in the direction χ as

$$X/\!\!/_\chi G \coloneqq \mathrm{Proj} \bigoplus_{n\geqslant 0} R^G_{\chi^n},$$

where R is the homogeneous coordinate ring of X with respect to the chosen embedding into $\mathbb{P}^n$.

Definition 4. *A point $x \in X$ satisfying $f(x) \neq 0$ for some semiinvariant $f \in R$ with weight χ^n, $n > 0$ is called semistable with respect to χ. If no such f exists, then x is called unstable. The set of points semistable with respect to χ is the open set $X_\chi^{ss} \subseteq X$.*

We and consider the rational map $X \dashrightarrow X/\!\!/_\chi G$ given by

$$x \longmapsto (f_0(x) : f_1(x) : \cdots : f_n(x)) \in \mathbb{P}(a_0,\ldots,a_n),$$

where $f_0,\ldots,f_n \in \bigoplus_{m\geqslant 0} R^G_{\chi^m}$ are generators for the ring of semiinvariants with weights $\chi^{a_0},\ldots,\chi^{a_n}$. This rational map is called the Proj quotient map in the direction χ.

Going back to the case that $k = 2, n = 4$, the isomorphism (2.7) implies that invariants under the SL_2 action on $k[z_1, w_1, z_2, w_2, z_3, w_3, z_4, w_4]$ of degree (d_1, d_2, d_3, d_4) correspond to semiinvariants under the $\mathbb{G}_m^4$ action on $k[p_{ij}]$ with respect to the character $\chi(t_1, t_2, t_3, t_4) = t_1^{d_1} t_2^{d_2} t_3^{d_3} t_4^{d_4}$. Because we are interested in invariants of degree (d, d, d, d), we fix $\chi(t_1, t_2, t_3, t_4) = t_1^d t_2^d t_3^d t_4^d$. We note that if $f \in k[p_{ij}]$ is a semiinvariant of degree (d, d, d, d), then each monomial of f must be a semiinvariant of degree (d, d, d, d). If

$$m = \prod_{1 \leqslant i < j \leqslant 4} p_{ij}^{x_{ij}}$$

is a semiinvariant monomial of degree (d, d, d, d), then we require

$$m \mapsto t_1^d t_2^d t_3^d t_4^d m \tag{2.8}$$

under the $\mathbb{G}_m^4$ action. Now, let I_a be the number of times the index $a \in [1, 4]$ appears in the expansion of m in terms of the p_{ij}. For the condition 2.8 to be satisfied, we see that $I_a = d$ for each $a \in [1, 4]$. It follows that the semiinvariant monomials of degree (d, d, d, d) must be products of the degree $(1, 1, 1, 1)$ semiinvariants which are $p_{12}p_{34}, p_{13}p_{24}$ and $p_{14}p_{23}$. Additionally, if m is a semiinvariant monomial of degree (d, d, d, d) given by

$$m = (p_{12}p_{34})^{x_1} (p_{13}p_{24})^{x_2} (p_{14}p_{23})^{x_3},$$

then $x_1 + x_2 + x_3 = d$. Because any semiinvariant of degree (d, d, d, d) is a linear combination of semiinvariant monomials, we have that the ring of semiinvariants of degree (d, d, d, d) is generated as a k-algebra by $p_{12}p_{34}, p_{13}p_{24}$, and $p_{14}p_{23}$. This completes the proof that

$$k[z_1, w_1; z_2, w_2 : z_3, w_3; z_4, w_4]_{d,d,d,d}^{\mathrm{SL}_2} = \frac{k[x_{12}x_{34}, x_{13}x_{24}, x_{14}x_{23}]}{\langle x_{12,34} - x_{13,24} + x_{14,23} \rangle} = k[x_{1423}, x_{24,13}].$$

We note that $\mathrm{Proj}(k[x_{14}x_{23}, x_{24}x_{13}]) \cong \mathbb{P}^1$, and consider the rational map

$$(\mathbb{P}^1)^4 \dashrightarrow \mathbb{P}^1$$

$$[(z_1 : w_1), (z_2 : w_2), (z_3 : w_3), (z_4 : w_4)] \mapsto [(z_1w_4 - z_4w_1)(z_2w_3 - z_3w_4) : (z_2w_4 - z_4w_2)(z_1w_3 - z_3w_1)].$$

This map is defined on the locus where no three points coincide, which is exactly the semistable locus of $(\mathbb{P}^1)^4$ under the action of SL_2 which shows that this is the Proj quotient map.

2.3 Counting Invariants

We are specifically interested in invariants of the graded ring $k[z_1, w_1; z_2, w_2; z_3, w_3; z_4, w_4]$ of degree (d_1, d_2, d_3, d_4), for which the isomorphism tells us that we should consider covariants of the coordinate ring $\mathrm{Gr}(2, 4)$ under the $\mathbb{G}_m^n$ action with respect to the character $\chi(t_1, t_2, t_3, t_4) = t_1^{d_1} t_2^{d_2} t_3^{d_3} t_4^{d_4}$. We now attempt to answer the question: For what (d_1, d_2, d_3, d_4) are the invariant rings $(\mathbb{P}^1)^4_{d_1,d_2,d_3,d_4}$ non-constant? We start with the case $(d_1, d_2, d_3, d_4) = (d_1, d_1, d_1, d_2)$. We set $\chi(t_1, t_2, t_3, t_4) = t_1^{d_1} t_2^{d_1} t_3^{d_1} t_4^{d_2}$ and note that it suffices to consider monomials in the p_{ij} of which will generically be of the form

$$(p_{12})^{x_{12}} (p_{13})^{x_{13}} (p_{14})^{x_{14}} (p_{23})^{x_{23}} (p_{24})^{x_{24}} (p_{34})^{x_{34}}.$$

Therefore, if the monomial is covariant with respect to $t_1^{d_1}t_2^{d_1}t_3^{d_1}t_4^{d_2}$, we require that the system of equations

$$\begin{aligned} x_{12} + x_{13} + x_{14} &= d_1, \\ x_{12} + x_{24} + x_{23} &= d_1, \\ x_{13} + x_{23} + x_{34} &= d_1, \\ x_{14} + x_{24} + x_{34} &= d_2 \end{aligned} \tag{2.9}$$

to have a solution in $\mathbb{Q}^4_{\geqslant 0}$. Although the degrees have to be integers, the reason that we only require solutions to this system of equations in $\mathbb{Q}^4_{\geqslant 0}$ is because the projective variety defined by the graded ring

$$k[z_1, w_1, z_2, w_2, z_3, w_3, z_4, w_4]^{\mathrm{SL}_2(k)}_{(d_1,d_2,d_3,d_4)}$$

is isomorphic to the projective variety defined by

$$k[z_1, w_1, z_2, w_2, z_3, w_3, z_4, w_4]^{\mathrm{SL}_2(k)}_{n(d_1,d_2,d_3,d_4)}$$

for any $n \in \mathbb{N}$. To see this, suppose that $X \hookrightarrow \mathbb{P}^r$ is a projective variety with coordinate ring $R = \frac{k[x_0,\dots,x_r]}{J} = \bigoplus_{n\geqslant 0} R_n$, and define for a fixed $d \in \mathbb{N}$, the ring $R^{(d)}$ as

$$R^{(d)} = \bigoplus_{n\geqslant 0} R_{nd} = \bigoplus_{n\geqslant 0} \frac{k[t_0,\dots,t_r]_{nd}}{J \cap k[t_0,\dots,t_r]_{nd}}.$$

We consider the degree d Veronese map $\nu_d : \mathbb{P}^r \to \mathbb{P}^m$, where $m = \binom{r+d}{d} - 1$, which sends

$$[x_0 : \dots : x_r] \mapsto [x_0^d : x_0^{d-1}x_1 : \cdots : x_r^d].$$

In other words, $[x_0 : \dots : x_r]$ is sent to all possible degree d monomials in $r+1$ variables. The Veronese map is an embedding, so it is an isomorphism onto its image, which is exactly the projective variety defined by the graded ring $\bigoplus_{n\geqslant 0} k[t_0,\dots,t_r]_{nd}$. Restricting the Veronese embedding to the subvariety X shows that R and $R^{(d)}$ define isomorphic projective varieties.

Adding the first three equations of (2.9) and subtracting the last shows that

$$2x_{12} + 2x_{13} + 2x_{23} = 3d_1 - d_2.$$

Because the x_{ij} are always positive, we must have that $3d_1 - d_2 \geqslant 0$.

Prop 1. *The invariant ring* $k[z_1, w_1, z_2, w_2, z_3, w_3, z_4, w_4]^{\mathrm{SL}_2}_{d_1,d_1,d_1,d_2}$ *is non-trivial if and only if* $3d_1 - d_2 \geqslant 0$.

Proof. We claim that if $3d_1 - d_2 \geqslant 0$, then the system of equations (2.9) always has a positive rational solution which defines a non-trivial invariant. This system of equations defines the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & d_1 \\ 1 & 0 & 0 & 1 & 1 & 0 & d_1 \\ 0 & 1 & 0 & 1 & 0 & 1 & d_1 \\ 0 & 0 & 1 & 0 & 1 & 1 & d_2 \end{pmatrix},$$

which is equal to

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & \frac{1}{2}(d_1-d_2) \\ 0 & 1 & 0 & 0 & -1 & 0 & \frac{1}{2}(d_1-d_2) \\ 0 & 0 & 1 & 0 & 1 & 1 & d_2 \\ 0 & 0 & 0 & 1 & 1 & 1 & \frac{1}{2}(d_1+d_2) \end{pmatrix}$$

in reduced row echelon form (RREF). Therefore, the equations

$$\begin{aligned} x_{12} &= x_{34} + \frac{1}{2}(d_1 - d_2), \\ x_{13} &= x_{24} + \frac{1}{2}(d_1 - d_2), \\ x_{14} &= -x_{24} - x_{34} + d_2, \\ x_{23} &= -x_{24} - x_{34} + \frac{1}{2}(d_1 + d_2) \end{aligned}$$

must have a solution in $\mathbb{Q}^4_{\geqslant 0}$. In the case that $d_1 - d_2 \geqslant 0$, then setting $x_{34} = x_{24} = 0$ defines an invariant. We note that in this case we always have that $3d_1 - d_2 \geqslant 0$. In the case that $d_1 - d_2 < 0$, setting $x_{12} = x_{13} = 0$ defines an invariant given by

$$\begin{aligned} x_{12} &= 0, \\ x_{13} &= 0, \\ x_{34} &= -\frac{1}{2}(d_1 - d_2), \\ x_{24} &= -\frac{1}{2}(d_1 - d_2), \\ x_{14} &= (d_1 - d_2) + d_2 = d_1, \\ x_{23} &= (d_1 - d_2) + \frac{1}{2}(d_1 + d_2) = \frac{1}{2}(3d_1 - d_2). \end{aligned}$$

Indeed, all of these values are positive since $3d_1 - d_2 \geqslant 0$. □

We now consider the general case where $\chi(t_1, t_2, t_3, t_4) = t_1^{d_1} t_2^{d_2} t_3^{d_3} t_4^{d_4}$. For there to be an invariant in this situation, we require a positive integral solution to the system of equations

$$\begin{aligned} x_{12} + x_{13} + x_{14} &= d_1, \\ x_{12} + x_{24} + x_{23} &= d_2, \\ x_{13} + x_{23} + x_{34} &= d_3, \\ x_{14} + x_{24} + x_{34} &= d_4. \end{aligned}$$

Again, after adding the first three and subtracting the last, we see that

$$2x_{12} + 2x_{13} + 2x_{23} = d_1 + d_2 + d_3 - d_4,$$

so we require that $d_1 + d_2 + d_3 - d_4 \geqslant 0$. We can clearly add any three equations and subtract the other to obtain similar constraints, which we claim are the only requirements necessary for there to be a non-trivial invariant.

Prop 2. *The invariant ring* $k[z_1, w_1, z_2, w_2, z_3, w_3, z_4, w_4]^{\mathrm{SL}_2}_{d_1,d_2,d_3,d_4}$ *is non-trivial if and only if*

$$d_{\sigma(1)} + d_{\sigma(2)} + d_{\sigma(3)} - d_{\sigma(4)} \geqslant 0$$

for all permutations $\sigma \in \mathfrak{S}_4$*.*

Proof. The system of equations defines the following matrix in RREF form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & \frac{1}{2}(d_1 + d_2 - d_3 - d_4) \\ 0 & 1 & 0 & 0 & -1 & 0 & \frac{1}{2}(d_1 - d_2 + d_3 - d_4) \\ 0 & 0 & 1 & 0 & 1 & 1 & d_4 \\ 0 & 0 & 0 & 1 & 1 & 1 & \frac{1}{2}(-d_1 + d_2 + d_3 + d_4) \end{pmatrix}.$$

The invariants are defined by positive rational solutions to the following system of equations

$$\begin{aligned} x_{12} &= x_{34} + \frac{1}{2}(d_1 + d_2 - d_3 - d_4), \\ x_{13} &= x_{24} + \frac{1}{2}(d_1 - d_2 + d_3 - d_4), \\ x_{14} &= -x_{24} - x_{34} + d_4, \\ x_{23} &= -x_{24} - x_{34} + \frac{1}{2}(-d_1 + d_2 + d_3 + d_4). \end{aligned}$$

Finding a non-trivial invariant breaks down into four cases: In the first case, suppose $d_1 + d_2 - d_3 - d_4 \geqslant 0$ and $d_1 - d_2 + d_3 - d_4 \geqslant 0$. Then, setting $x_{34} = x_{24} = 0$ defines an invariant given by

$$\begin{aligned} x_{12} &= \frac{1}{2}(d_1 + d_2 - d_3 - d_4), \\ x_{13} &= \frac{1}{2}(d_1 - d_2 + d_3 - d_4), \\ x_{14} &= d_4, \\ x_{23} &= \frac{1}{2}(-d_1 + d_2 + d_3 + d_4). \end{aligned}$$

In the second case, suppose that $d_1 + d_2 - d_3 - d_4 \geqslant 0$ and $d_1 - d_2 + d_3 - d_4 < 0$. Then, setting $x_{34} = x_{13} = 0$ defines an invariant given by

$$\begin{aligned} x_{12} &= \frac{1}{2}(d_1 + d_2 - d_3 - d_4), \\ x_{24} &= -\frac{1}{2}(d_1 - d_2 + d_3 - d_4), \\ x_{14} &= \frac{1}{2}(d_1 - d_2 + d_3 - d_4) + d_4 = \frac{1}{2}(d_1 + d_3 + d_4 - d_2), \\ x_{23} &= \frac{1}{2}(d_1 - d_2 + d_3 - d_4) + \frac{1}{2}(-d_1 + d_2 + d_3 + d_4) = d_3, \end{aligned}$$

where we recall that $d_1 + d_3 + d_4 - d_2$ is always greater than zero. In the third case, suppose that $d_1 + d_2 - d_3 - d_4 < 0$ and $d_1 - d_2 + d_3 - d_4 \geqslant 0$. Setting $x_{24} = x_{12} = 0$ defines an invariant given by

$$\begin{aligned} x_{34} &= -\frac{1}{2}(d_1 + d_2 - d_3 - d_4), \\ x_{13} &= -\frac{1}{2}(d_1 - d_2 + d_3 - d_4), \\ x_{14} &= \frac{1}{2}(d_1 + d_2 - d_3 - d_4) + d_4 = \frac{1}{2}(d_1 + d_2 + d_4 - d_3), \\ x_{23} &= \frac{1}{2}(d_1 + d_2 - d_3 - d_4) + \frac{1}{2}(-d_1 + d_2 + d_3 + d_4) = d_2, \end{aligned}$$

which is again an invariant as $d_1 + d_2 - d_3 - d_4 \geqslant 0$. In the fourth and final case, suppose that $d_1 + d_2 - d_3 - d_4 < 0$ and $d_1 - d_2 + d_3 - d_4 < 0$. Then, setting $x_{12} = x_{13} = 0$ defines an invariant given by

$$\begin{aligned} x_{34} &= -\frac{1}{2}(d_1 + d_2 - d_3 - d_4), \\ x_{24} &= -\frac{1}{2}(d_1 - d_2 + d_3 - d_4), \\ x_{14} &= \frac{1}{2}(d_1 + d_2 - d_3 - d_4) + \frac{1}{2}(d_1 - d_2 + d_3 - d_4) + d_4 = d_1, \\ x_{23} &= \frac{1}{2}(d_1 + d_2 - d_3 - d_4) + \frac{1}{2}(d_1 - d_2 + d_3 - d_4) = \frac{1}{2}(d_1 + d_2 + d_3 - d_4), \end{aligned}$$

which is indeed an invariant as $d_1 + d_2 + d_3 - d_4 \geqslant 0$. □

In the most general case of invariants on $(\mathbb{P}^1)^n_{d_1,\ldots,d_n}$, one will need to solve a system of n equations in $\binom{n}{2}$ variables:

$$\begin{aligned} x_{12} + x_{13} + \ldots + x_{1n-1} =& d_1, \\ &\vdots \\ x_{n1} + x_{n2} + \ldots + x_{nn-1} =& d_n, \end{aligned}$$

An analogous computation shows that the following:

Prop 3. *[16] The invariant ring $k[z_1, w_1, \ldots, z_n, w_n]^{\mathrm{SL}_2}_{d_1,\ldots,d_n}$ is non-empty if and only if $2d_i \leqslant \sum_{j=1}^{n} d_j$ for all $1 \leqslant i \leqslant n$. Equivalently, this condition states that $d_{\sigma(1)} + \ldots + d_{\sigma(n-1)} - d_{\sigma(n)} \geqslant 0$ for all permutations $\sigma \in \mathfrak{S}_n$.*

3 Cohomology of Grassmannians

3.1 Schubert Cells and Varieties

From this section onwards, we work solely over $\mathbb{C}$, and the variable k no longer represents a field. Let $\mathrm{Gr}(k, n)$ be the Grassmannian of k-dimensional subspaces of an n-dimesnional complex vector space V. We now consider the collection of subvarieties of $\mathrm{Gr}(k, n)$ called *Schubert varieties*. Our main reference is [11].

Definition 5. *Schubert varieties are defined in terms of a* ***complete flag*** $\mathcal{V}$ *in* V*, which is a sequence of subspaces of* V

$$0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V,$$

with $\dim V_i = i$.

The Grassmannian parameterizes the k-dimensional subspaces of an n-dimensional vector space, so we can write the Grassmannian as the quotient

$$\mathrm{Gr}(k,n) = \{\text{full rank } k \times n \text{ matrices}\} \setminus \mathrm{GL}_k\,.$$

Each GL_k orbit has a unique matrix representation in RREF, so a point in $\mathrm{Gr}(k,n)$ corresponds to a full rank $k \times n$ matrix in RREF. The *Schubert cells* index the subset of the Grassmannian whose points have this particular form. To each matrix representing an element of $\mathrm{Gr}(k,n)$, we assign a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $n - k \geqslant \lambda_1 \geqslant \lambda_2 \cdots \geqslant \lambda_k \geqslant 0$, where λ_i is the number of entries between the $n - i + 1$ entry to the 1 in the i^{th} row, including the endpoints. For example, if we consider the RREF matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

representing an element of $\mathrm{Gr}(2,4)$, the associated partition is $\lambda = (2,2)$. We note that the Young tableaux associated to the partition λ always fits into the $k(n-k)$ ambient rectangle.

A subspace $\Lambda \in \mathrm{Gr}(k,n)$ has the partition λ if and only if

$$\Lambda \cap V_{n-k+i-\lambda_i} = i$$

for $n - k + i = \lambda_i \leqslant r \leqslant n - k + i - \lambda_{i+1}$ for all $1 \leqslant i \leqslant k$. This leads to the following definitions:

Definition 6. *The Schubert cell associated to a partition* λ *is defined as*

$$\Sigma_\lambda^\circ = \{\Lambda \in \mathrm{Gr}(k,n) : \dim(\Lambda \cap V_{n-k+i-\lambda_i}\rangle = i \text{ for all } i\}.$$

and

Definition 7. *The Schubert variety (also called a Schubert cycle) associated to the partition* λ*, denoted by* Σ_λ *is defined as*

$$\Sigma_\lambda(\mathcal{V}) = \{\Lambda \in \mathrm{Gr}(k,n) : \dim(\Lambda \cap V_{n-k+i-\lambda_i}) \geqslant i \text{ for all } i\}.$$

We note that the Schubert variety Σ_λ is the Zariski closure of Σ_λ°, which can be seen through examining the determinants of the minors of the RREF matrices representing the Schubert cells [25].

We consider two special cases of λ:

1. If $\lambda = (0, \ldots, 0)$, then $\dim(\Lambda \cap F_{n-k+i}) \geqslant i$ is trivially satisfied since $\dim(\Lambda) = k$ and $k + (n - k + i) = n + i$, and for any subspace Ψ, we have that $\dim(\Lambda \cap \Psi) = \dim \Lambda + \dim \Psi - \dim(\Lambda + \Psi)$. We have that $\dim(\Lambda + \Psi) \leqslant n$ since they are subspaces of an n-dimensional vector space, and hence the dimension of the intersection is at least i. Therefore, $\Sigma_0 = \mathrm{Gr}(n,k)$.

2. Suppose $\lambda = (n-k, \dots, n-k)$. If $\dim(\Lambda \cap F_i) \geqslant i$ for all $1 \leqslant i \leqslant k$, then we have that
$$\dim(\Lambda) + i - \dim(\Lambda + F_i) \geqslant i.$$
Since $\dim(\Lambda + F_i) \geqslant \dim(\Lambda)$, we have that $\dim(\Lambda) + i - \dim(\Lambda + F_i) < i$ unless $F_i \subseteq \Lambda$ for all $1 \leqslant i \leqslant k$, which occurs if and only if $\Lambda = F_i$. Therefore $\Sigma_{(n-k)^r}$ is a single point in $\mathrm{Gr}(k, n)$.

3.2 Cell Complex Structure

Definition 8. *The size of the partition λ is given by $|\lambda| = \sum_{i=1}^{k} \lambda_i$. We also define a partial ordering on the set of partitions length k by saying $\mu \leqslant \lambda$ if $\mu_i \leqslant \lambda_i$ for $1 \leqslant i \leqslant k$.*

We can see by the structure of the reduced row echelon form matrices that each Schubert cell Σ_λ° is isomorphic to $\mathbb{A}^{k(n-k)-|\lambda|}$, and that the closure of a Schubert cell
$$\Sigma_\lambda = \overline{\Sigma_\lambda^\circ} = \bigcup_{\mu \subseteq \lambda} \Sigma_\mu^\circ.$$
This means that if the closure of a cell intersects another cell, it in fact contains that cell. Hence, the Schubert cells provide an affine stratification
$$\mathrm{Gr}(k, n) = \bigsqcup \mathbb{A}^{k(n-k)-|\lambda|}.$$
Furthermore, the Schubert cells give a cell complex structure on the Grassmannian with only even dimensional cells. We define the 0-skeleton $X^0 = \Sigma_B^\circ$, where B is the $k \times (n-k)$ ambient rectangle. We then define $X^2 = X^0 \cup \Sigma_{\lambda_1}^\circ$, where $\lambda_1 = (n-k, \dots, n-k-1)$. In general, the $2m$-skeleton is formed by attaching the cells with $|\lambda| = k(n-k) - m$ to the previous cell. We see that $\mathrm{Gr}(k, n)$ is a CW complex with even dimensional skeleta
$$X^0 \subseteq X^2 \subseteq \cdots \subseteq X^{2k(n-k)}.$$
Example 1. *We compute the cell decomposition of* $\mathrm{Gr}(2, 4)$. *We have that there are $\binom{4}{2}$ Schubert cells corresponding to matrices in RREF form:*
$$\Sigma_{2,2}^\circ = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$
$$\Sigma_{1,1}^\circ = \begin{pmatrix} 0 & 0 & 1 & * \\ 1 & 0 & * & * \end{pmatrix},$$
$$\Sigma_{0,0}^\circ = \begin{pmatrix} 0 & 1 & * & * \\ 1 & 0 & * & * \end{pmatrix},$$
$$\Sigma_{2,1}^\circ = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & * & 0 \end{pmatrix},$$
$$\Sigma_{2,0}^\circ = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & * & * & 0 \end{pmatrix},$$
$$\Sigma_{1,0}^\circ = \begin{pmatrix} 0 & 0 & 1 & * \\ 1 & * & 0 & * \end{pmatrix}.$$

Then, we have that $X^0 = \Sigma^\circ_{2,2}$, $X^2 = \Sigma^\circ_{2,1} \cup \Sigma^\circ_{2,2}$ (note the boundary of X^2 is X^0), X^4 is formed by attaching $\Sigma^\circ_{1,1}$ and $\Sigma^\circ_{2,0}$, X^6 is formed by attaching $\Sigma^\circ_{1,0}$, and finally X^8 is formed by attaching $\Sigma^\circ_{0,0}$.

We now note a few convenient properties of the Schubert variety indexing convention used:

1. It follows from the definition that $\Sigma_\mu \subseteq \Sigma_\lambda$ if and only if $\mu \leqslant \lambda$.
2. The codimension of a Schubert cycle $\Sigma_\lambda \subseteq \mathrm{Gr}(k,n)$ is given by $|\lambda|$.

Because the Schubert cells give a cell decomposition of $\mathrm{Gr}(k,n)$ with only even dimensional cells, it follows that they generate the homology of $\mathrm{Gr}(k,n)$ [12]. Therefore, the Schubert varieties define a class $[\sigma_\lambda]$ in homology, and Poincare duality gives a correspondence to classes in cohomology. We define

$$\sigma_\lambda := [\Sigma_\lambda] \in H^{2|\lambda|}(\mathrm{Gr}(k,n)).$$

Theorem 3. *The classes σ_λ form a $\mathbb{Z}$-basis for the cohomology ring $H^*(\mathrm{Gr}(k,n),\mathbb{Z})$, and under sufficient conditions, such as Schubert varieties for generic flags (see definition 10), the cup product of two cohomology classes is equivalent to the intersection of the Schubert varieties defining these classes:*

$$\sigma_\lambda \sigma_\mu = [\Sigma_\lambda \cap \Sigma_\mu] \in H^{2(|\lambda|+|\mu|)}\mathrm{Gr}(k,n).$$

We note that GL_n acts transitively on the set of complete flags, so the cohomology class determined by Σ_λ is independent of the choice of flag.

Example 2. *With respect to the flag $p \in L \subseteq H \subseteq \mathbb{P}^3$, the Schubert cycles of $\mathrm{Gr}(2,4) = \mathbb{G}(1,3)$ are given by*

$$\begin{aligned}
\sigma_{0,0} &= \{\textit{all of } \mathrm{Gr}(2,4)\},\\
\sigma_{1,0} &= \{\textit{lines intersecting } L\},\\
\sigma_{1,1} &= \{\textit{lines contained in } H\},\\
\sigma_{2,0} &= \{\textit{lines containing } p\},\\
\sigma_{2,1} &= \{\textit{lines lying in } H \textit{ that contain } p\},\\
\sigma_{2,2} &= \{L\}.
\end{aligned}$$

3.3 Transverse Intersections

Definition 9. *The vector spaces $L, M \subseteq \mathbb{C}^n$ are called transverse if*

$$\dim(L \cap M) = \max(0, \dim(L) + \dim(M) - n).$$

We can extend the notion of transversality to flags $\mathcal{F}$ and $\mathcal{G}$ of an n-dimensional vector space V.

Theorem 4. *Two flags $\mathcal{F}$ and $\mathcal{G}$ on V are transverse if $F_i \cap G_{n-i} = \{0\}$ for all i.*

We now state some equivalent conditions for two flags to intersect transversely.

Theorem 5. *Let $\mathcal{F}$ and $\mathcal{G}$ be flags on V. The following conditions are equivalent:*

1. *$F_i \cap G_{n-i} = \{0\}$ for all i.*

2. $\dim(F_i \cap G_j) = \max(0, i+j-n)$ *for all* i, j.

3. There exists a basis $e_1, \ldots, e_n$ *for* V *such that*

$$F_i = \langle e_1, \ldots, e_i \rangle \text{ and } G_j = \langle e_{n+1-j}, \ldots, e_n \rangle.$$

Because transverse flags form a dense open subset in the space of all pairs of flags, any statement proved for a transverse pair of flags holds for a generic pair of flags. We now make precise the notion of a "generic" flag. The complete flag variety $\mathrm{Fl}(\mathbb{C}^n)$ is the collection of all complete flags in $\mathbb{C}^n$. We can represent a flag via a matrix by defining the span of the first i columns to be the i^{th} flag. Additionally, suppose we fix an ordered basis for $\mathbb{C}^n$ and consider the standard flag associated to this basis (meaning that the i^{th} subspace is spanned by the first i basis vectors). Then, the matrix representing this flag is equivalent up to the action of B_n the subgroup of upper triangular matrices in GL_n. Therefore, the complete flag variety $\mathrm{Fl}(\mathbb{C}^n)$ has the structure of the homogeneous space GL_n / B_n.

Definition 10. *We say that property holds for a "generic" collection of flags if it holds for all tuples of flags in some Zariski open subset of the product variety* $\mathrm{Fl}(\mathbb{C}^n) \times \cdots \times \mathrm{Fl}(\mathbb{C}^n)$.

We also define the notion of a generically transverse intersection, which will be used in the next section.

Definition 11. *Let* A*,* B *be two subvarieties of a variety* X*. We say that* A *and* B *intersect trasversally at a point* $p \in A \cap B$ *if* $X, A,$ *and* B *are smooth at* p *and* $T_pA + T_pB = T_pX$*. We say that* A *and* B *are generically transverse if they intersect transversally at a generic point of each component of their intersection.*

3.4 Intersection Formulas

The product of Schubert classes has a simple formula when one of the classes has the form $\sigma_b = \sigma_{b,0,\ldots,0}$. Such classes are called *special Schubert classes*. Before proving a formula for such intersections, we introduce the concept of *specialization* which will be used in the proof.

Example 3. *Suppose we want to compute* σ_1^2 *in* $\mathrm{Gr}(2,4)$*. Recall that* σ_1 *is the set of lines in* $\mathbb{P}^3$ *passing through a point. Instead of intersecting two general lines* L *and* L'*, the idea of specialization is to choose* L *and* L' *special enough such that the intersection* $\Sigma_1(L) \cap \Sigma_1(L')$ *is easily identifiable but the intersection is still generically transverse. This is accomplished by choosing* L *and* L' *to be distinct lines that intersect. Then the intersection consists of lines passing through* $p = L \cap L'$ *or lines lying on the plane spanned by* L *and* L'*. The first class of lines corresponds to* σ_2 *and the second class of lines corresponds to* $\sigma_{1,1}$*. We can then conclude that*

$$\sigma_1^2 = \sigma_2 + \sigma_{1,1}.$$

We now prove a formula for computing products of Schubert cycles in $\mathrm{Gr}(2, n)$.

Lemma 4. *For any special Schubert class* σ_a *and* σ_b

$$\sigma_a \cdot \sigma_b = \sum_{c \geqslant a,b}^{a+b} \sigma_{c,a+b-c}. \tag{3.1}$$

Proof. We give a proof by induction on n, following [30]. For the $n = 2$ case, the only choices for a and b are 0, and

$$\sigma_0 \cdot \sigma_0 = \sigma_0,$$

so (3.1) is trivially satisfied. Now, suppose $n \geqslant 3$ and the formula (3.1) holds for all $\mathrm{Gr}(2, m)$ with $m < n$. Because the Schubert classes are independent of the flag chosen, we let $\mathcal{W}$ be the standard flag, meaning $W_j = \mathbb{C}^j$, and $\mathcal{V}$ be any other transverse flag with respect to $\mathcal{W}$. We now need to determine the intersection of

$$\Sigma_a(\mathcal{W}) = \{\Lambda \in \mathrm{Gr}(2, n) : \Lambda \cap \mathbb{C}^{n-1-a} \neq \{0\}\}$$

and

$$\Sigma_b(\mathcal{V}) = \{\Lambda \in \mathrm{Gr}(2, n) : \Lambda \cap V_{n-1-b} \neq \{0\}\}.$$

We split the argument into two cases.

Case 1: Suppose that $a + b > n - 2$. Then,

$$\dim(\mathbb{C}^{n-1-a} + V_{n-1-b}) \leqslant \dim \mathbb{C}^{n-1-a} + \dim V_{n-1-b} = (n-1-a)+(n-1-b) = n+(n-2)-(a+b) < n.$$

Also,

$$\dim(\mathbb{C}^{n-1-a} \cap V_{n-1-b}) = \max\{0, n-2-a-b\} = \{0\},$$

since $\mathcal{V}$ and $\mathcal{W}$ are transverse flags. Additionally, we can assume without loss of generality that $\mathbb{C}^{n-1-a} + V_{n-1-b} \subseteq \mathbb{C}^{n-1}$. Thus,

$$\begin{aligned}\sigma_a \cdot \sigma_b =& \Sigma_a(\mathcal{W}) \cap \Sigma_b(\mathcal{V}) = \{\Lambda \in \mathrm{Gr}(2, n) : \Lambda \cap \mathbb{C}^{n-1-a} \neq \{0\}, \Lambda \cap V_{n-1-b} \neq \{0\}\} = \\ &\{\Lambda \in \mathrm{Gr}(2, n-1) : \Lambda \cap \mathbb{C}^{(n-1)-1-(a-1)} \neq \{0\}, \Lambda \cap V_{(n-1)-1-(b-1)} \neq \{0\}\},\end{aligned}$$

which is the product of $\sigma_{a-1}\sigma_{b-1}$ in $\mathrm{Gr}(2, n-1)$. By the inductive hypothesis, we have that

$$\sigma_{a-1} \cdot \sigma_{b-1} = \sum_{c \geqslant (a-1,b-1)}^{a+b-1} \sigma_{c-1,a+b-c-1} = \sum_{c=\max(a,b)}^{\min(n-2,a+b-1)} \sigma_{c-1,a+b-c-1}$$

in $\mathrm{Gr}(2, n-1)$. As a cycles in $\mathrm{Gr}(2, n-1) \subseteq \mathrm{Gr}(2, n)$,

$$\begin{aligned}&\sum_{c=\max(a,b)}^{\min(n-2,a+b-1)} \sigma_{c-1,a+b-c-1} = \sum_{c=\max(a,b)}^{\min(n-2,a+b-1)} \{\Lambda \in \mathrm{Gr}(2, n-1) : \Lambda \subseteq \mathbb{C}^{n-1-(a+b-c-1)}, \Lambda \cap \mathbb{C}^{(n-1)-1-(c-1)} \neq \{0\}\} \\ &= \sum_{c=\max(a,b)}^{\min(n-2,a+b-1)} \{\Lambda \in \mathrm{Gr}(2, n) : \Lambda \subseteq \mathbb{C}^{n-(a+b-c)}, \Lambda \cap \mathbb{C}^{n-1-c} \neq \{0\}\} = \sum_{c \geqslant a,b} \sigma_{c,a+b-c}, \qquad (3.2)\end{aligned}$$

where the bounds on the last sum come from the fact that in this case $\min(n-2, a+b-1) = n-2$ which is the maximum allowed value of c.

Case 2: Suppose $a + b \leqslant n - 2$. If $\mathcal{V}$ is transverse with respect to the standard flag, we have that

$$\dim(\mathbb{C}^{n-1-a} \cap V_{n-1-b}) = \max\{0, (n-1-a) + (n-1-b) - n\} = n - 2 - (a + b).$$

Without loss of generality, we can specialize $\mathcal{V}$ such that $\mathbb{C}^{n-1-a} \cap V_{n-1-b} = \mathbb{C}^{n-1-(a+b)}$ and the intersection is still generically transverse. Then, we have that

$$\dim(\mathbb{C}^{n-1-a} + V_{n-1-b}) = \dim \mathbb{C}^{n-1-a} + \dim V_{n-1-b} - \dim(\mathbb{C}^{n-1-a} \cap V_{n-1-b}) = \\ (n-1-a) + (n-1-b) - (n-1-(a+b)) = n-1.$$

Therefore, we can assume that $\mathbb{C}^{n-1-a} + V_{n-1-b} = \mathbb{C}^{n-1}$. Then,

$$\Sigma_a \cap \Sigma_b = \{\Lambda \in \mathrm{Gr}(2,n) : \Lambda \cap \mathbb{C}^{n-1-a} \neq \{0\}, \Lambda \cap V_{n-1-b} \neq \{0\}\},$$

so it consists of elements $\Lambda \in \mathrm{Gr}(2,n)$ such that either

1. Λ has a 1-dimensional subspace in common with $\mathbb{C}^{n-1-a} \cap V_{n-1-b} = \mathbb{C}^{n-1-(a+b)}$,
2. Λ intersects $\mathbb{C}^{n-1-a}$ and V_{n-1-b} along some one dimensional subspaces $L_1 \subseteq \mathbb{C}^{n-1-a}$ and $L_2 \subseteq V_{n-1-b}$. In this case, Λ must lie in the sum $\mathbb{C}^{n-1-a} + V_{n-1-b} = \mathbb{C}^{n-1}$ since it is a two dimensional subspace.

We can now characterize the intersection $\Sigma_a \cap \Sigma_b$ as

$$\Sigma_a \cap \Sigma_b = \{\Lambda \in \mathrm{Gr}(2,n) : \Lambda \in \mathbb{C}^{n-1-(a+b)} \neq \{0\}\} \cup \\ \{\Lambda \in \mathrm{Gr}(2,n) : \Lambda \subseteq \mathbb{C}^{n-1}, \Lambda \cap \mathbb{C}^{n-1-a} \neq \{0\}, \Lambda \cap V_{n-1-b} \neq \{0\}\}.$$

The first type of subspaces are exactly the elements of Σ_{a+b} with respect to the standard flag in $\mathrm{Gr}(2,n)$. The second case is the intersection of $\Sigma_{a-1}(\mathcal{V})$ and $\Sigma_{b-1}(\mathcal{W})$, and the inductive hypothesis implies

$$\sigma_{a-1} \cdot \sigma_{b-1} = \sum_{c=\max(a,b)}^{\min(n-2,a+b-1)} \sigma_{c-1,a+b-c-1}$$

in $\mathrm{Gr}(2,n-1)$. Then, by the same argument as in the first case, we have that in $\mathrm{Gr}(2,n)$,

$$\sigma_{a-1} \cdot \sigma_{b-1} = \sum_{c=\max(a,b)}^{\min(n-2,a+b-1)} \sigma_{c,a+b-c} = \sum_{c=\max(a,b)}^{a+b-1} \sigma_{c,a+b-c}. \tag{3.3}$$

Combining both cases, we see that

$$\sigma_a \cdot \sigma_b = \sum_{c \geqslant a,b} \sigma_{c,a+b-c},$$

which completes the proof of the lemma. □

We need to compute one more type of intersection before being able to compute arbitrary products of Schubert cycles in $\mathrm{Gr}(2,n)$.

Lemma 5. *In* $\mathrm{Gr}(2,n)$*, the following formula holds:*

$$\sigma_{a_1,a_2} \cdot \sigma_{b,b} = \sigma_{a_1+b,a_2+b}.$$

Proof. With respect to the flags $\mathcal{V}$ and $\mathcal{W}$ the Schubert cycles are of the form

$$\Sigma_{a_1,a_2} = \{\Lambda \in \mathrm{Gr}(2,n) : \Lambda \cap V_{n-1-a_1} \neq \{0\}, \Lambda \subseteq V_{n-a_2}\},$$

$$\Sigma_{b,b} = \{\Lambda \in \mathrm{Gr}(2,n) : \Lambda \subseteq W_{n-b}\}$$

Therefore,

$$\Sigma_{a_1,a_2} \cap \Sigma_{b,b} = \{\Lambda \in \mathrm{Gr}(2,n) : \Lambda \subseteq W_{n-b} \cap V_{n-a_2}, \Lambda \cap (V_{n-1-a_1} \cap W_{n-b}) \neq \{0\}\}.$$

We note that

$$\mathrm{codim}\,(W_{n-b} \cap V_{n-a_2}) = b + a_2$$

and

$$\mathrm{codim}\,(V_{n-1-a_1} \cap W_{n-b}) = a_1 + b + 1.$$

Therefore, letting $\mathcal{U}$ be a new flag containing $V_{n-1-a_1} \cap W_{n-b}$ and $W_{n-b} \cap V_{n-a_2}$, we have that

$$\Sigma_{a_1,a_2} \cap \Sigma_{b,b} = \{\Lambda : \Lambda \subseteq U_{n-(a_2+b)}, \Lambda \cap U_{n-1-(a_1+b)}\} = \sigma_{a_1+b,a_2+b},$$

thereby completing the proof. □

Indeed, Lemmas (4) and (5) are sufficient to compute the intersection of arbitrary Schubert cycles in $\mathrm{Gr}(2,n)$, which are given by the following formula.

Theorem 6. *Assuming that $a_1 - a_2 \geqslant b_1 - b_2$, then*

$$\sigma_{a_1,a_2}\sigma_{b_1,b_2} = \sigma_{a_1+b_1,a_2+b_2} + \sigma_{a_1+b_1-1,a_2+b_2+1} + \ldots + \sigma_{a_1+b_2,b_1+a_2} = \sum_{\substack{|c|=|a|+|b| \\ a_1+b_2 \geqslant c_1 \geqslant a_1+b_2}} \sigma_{c_1,c_2}.$$

Proof. We first consider the case where $b_1 = b_2 = b$. Then,

$$\Sigma_{b,b}(\mathcal{W}) = \{\Lambda \in \mathrm{Gr}(2,n) : \Lambda \subseteq W_{n-b}\},$$

so for any a_1, a_2 we have that

$$\begin{aligned}&\Sigma_{a_1,a_2}(\mathcal{V}) \cap \Sigma_{b,b}(\mathcal{W}) = \{\Lambda \in \mathrm{Gr}(2,n) : \Lambda \cap V_{n-1-a_1} \neq \{0\}, \Lambda \subseteq V_{n-a_2}, \Lambda \subseteq W_{n-b}\} = \\ &\{\Lambda \in \mathrm{Gr}(2,n) : \Lambda \cap (V_{n-1-a_1} \cap W_{n-b}) \neq \{0\}, \Lambda \subseteq (V_{n-a_2} \cap W_{n-b}\} = \\ &\Sigma_{a_1+b,a_2+b}(V_{n-1-a_1} \cap W_{n-b}, V_{n-a_2} \cap W_{n-b}).\end{aligned}$$

By Lemma 5, we have that $\sigma_{a_1,a_2}\sigma_{b,b} = \sigma_{a_1+b,a_2+b}$.

Now we consider the general case. Using Lemma (5), we can write

$$\sigma_{a_1,a_2}\sigma_{b_1,b_2} = (\sigma_{a_1-a_2,0}\sigma_{a_2,a_2})(\sigma_{b_1-b_2,0}\sigma_{b_2,b_2}) = \sigma_{a_1-a_2,0}\sigma_{b_1-b_2,0}\sigma_{a_2+b_2,a_2+b_2}.$$

From Lemma (4), we know that

$$\sigma_{a_1-a_2,0}\sigma_{b_1-b_2,0} = \sigma_{(a_1-a_2)+(b_1-b_2),0} + \sigma_{(a_1-a_2)+(b_1-b_2)-1,1} + \ldots + \sigma_{a_1-a_2,b_1-b_2},$$

so

$$\begin{aligned}\sigma_{a_1,a_2}\sigma_{b_1,b_2} =&(\sigma_{(a_1-a_2)+(b_1-b_2),0}+\sigma_{(a_1-a_2)+(b_1-b_2)-1,1}+\ldots+\sigma_{a_1-a_2,b_1-b_2})\sigma_{a_2+b_2,a_2+b_2} = \\ &\sigma_{(a_1-a_2)+(b_1-b_2),0}\sigma_{a_2+b_2,a_2+b_2}+\ldots+\sigma_{a_1-a_2,b_1-b_2}\sigma_{a_2+b_2,a_2+b_2}.\end{aligned}$$

Then, using the first case, we can expand

$$\begin{aligned}\sigma_{a_1,a_2}\sigma_{b_1,b_2} =&\sigma_{(a_1-a_2)+(b_1-b_2),0}\sigma_{a_2+b_2,a_2+b_2}+\ldots+\sigma_{a_1-a_2,b_1-b_2}\sigma_{a_2+b_2,a_2+b_2} = \\ &\sigma_{a_1+b_1,a_2+b_2}+\ldots+\sigma_{a_1+b_2,b_1+a_2},\end{aligned} \tag{3.4}$$

which completes the proof. □

In general, there exist combinatorial formulas for the intersection of Schubert varieties Grassmannians $\mathrm{Gr}(k,n)$ where k is not necessarily 2 (see, for example, [11]).

Theorem 7. *(Pieri's Formula) Let σ_λ be a special Schubert cycle in $\mathrm{Gr}(k,n)$, meaning that $\lambda = (\lambda_1, 0, \ldots, 0)$, suppose that σ_μ is any Schubert cycle. Then*

$$\sigma_\lambda \cdot \sigma_\mu = \sum_{\substack{\mu_i \leqslant \nu_i \leqslant \mu_{i-1} \\ |\nu|=|\lambda|+|\mu|}} \sigma_\nu.$$

Proof. We can prove Pieri's formula in $\mathrm{Gr}(2,n)$ as a corollary of (6). Assuming without loss of generality that $\lambda \geqslant \mu_1-\mu_2$, Theorem (6) implies that

$$\sigma_{\lambda,0}\cdot\sigma_{\mu_1,\mu_2} = \sigma_{\lambda+\mu_1,\mu_2}+\sigma_{\lambda+\mu_1-1,\mu_2+1}+\ldots+\sigma_{\lambda+\mu_2,\mu_1} = \sum_{\substack{\mu_1\leqslant\nu_1 \\ \mu_1\leqslant\nu_2\leqslant\mu_2 \\ |\nu|=|\lambda|+|\mu|}} \sigma_\nu,$$

which exactly matches Pieri's formula. For a general proof see [11, Theorem 4.9]. □

In order to describe the formula for the intersection of arbitrary Schubert cycles, it is necessary to introduce some more combinatorial machinery on which more details can be found in [12], [28].

Definition 12. *A Semi-Standard Young Tableaux (SSYT) is a filling of boxes of the Young diagram of shape ν/λ with positive integers such that within each row, the integers weakly increase from left to right and within each column they strictly increase from top to bottom. The content of a SSYT is $\mu = (\mu_1, \ldots, \mu_n)$ if there are μ_i boxes labeled i. The reading word is the word formed by concatenating rows from bottom to top. A reading word is called a lattice if when read backwards from the n^{th} to $(n-m)^{\text{th}}$ term, the sequence contains at least as many i as $i+1$.*

Theorem 8. *(Littlewood-Richardson Rule) If σ_λ and σ_μ are arbitrary Schubert cycles, then*

$$\sigma_\lambda\cdot\sigma_\mu = \sum c^\nu_{\lambda\mu}\sigma_\nu,$$

where the sum ranges over all ν in the $(n-k)\cdot k$ ambient rectangle, and $c^\nu_{\lambda\mu}$ is the number if semi-standard Young tableaux having content μ whose reading word is a lattice.

[28, Theorem 5.1] It is important to note that there is an isomorphism

$$H^*(\mathrm{Gr}(k,n)) \cong \frac{\Lambda}{\langle s_\lambda | \lambda \notin B\rangle},$$

where Λ is the ring of symmetric functions, and the ideal $\langle s_\lambda | \lambda \notin B\rangle$ is the ideal generated by Schur polynomials corresponding to partitions which do not fit in the ambient $(n-k)\times k$ rectangle B. Because the Schur polynomials form a basis for the ring of symmetric functions, it follows that the Schur polynomials corresponding to partitions fitting inside the ambient rectangle form a basis for $\frac{\Lambda}{\langle s_\lambda | \lambda \notin B\rangle}$. Due to this isomorphism, computations with Schubert classes can be turned into computations with Schur functions.

3.5 Torus Orbit Formula

We have seen that the n-dimensional torus acts on $\mathrm{Gr}(k,n)$ from the right. For $L \in \mathrm{Gr}(k,n)$ we now consider the orbit closure $\overline{T\cdot L}$ and want to decompose the cohomology class $[\overline{T\cdot L}]$ with respect to the Schubert cycles. For an arbitrary subspace $L \in \mathrm{Gr}(k,n)$, the torus orbit closures can be quite unwieldy, so we restrict ourselves to "generic L", which we now define. Our main reference is [20].

Definition 13. *Let $I \subseteq \{1,\ldots,n\}$, $|I| = k$ and let L_I be the subspace in $\mathbb{C}^n$ defined by the equations $\{x_i = 0$ for $i \in I\}$. Additionally, we define $\mathbb{C}^I$ as $\mathrm{Span}_{\mathbb{C}}\{e_i\}$ for $i \in I$. A subspace $L \in \mathrm{Gr}(k,n)$ is generic if for any $I \in \{1,\ldots,n\}$, $L \cap L_I = 0$.*

The space of all generic subspaces, denoted $\mathrm{Gr}^0(k,n)$ is a T-invariant subset in $\mathrm{Gr}(k,n)$ called the generic stratum.

Prop 4. *A subspace $L \in \mathrm{Gr}(k,n)$ is generic if and only if none of its $k\times k$ minors in the matrix representing L vanish.*

Proof. Suppose that $L_I = \mathrm{span}(e_{i_1},\ldots,e_{i_{n-k}})$ and that L is represented by the $k\times n$ matrix by taking the row span of the matrix

$$M(L) = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{k1} & \ldots & a_{kn} \end{pmatrix}.$$

If $L \cap L_I \neq 0$, $L \subseteq L_I$, so then there exist $c_1,\ldots,c_k$ and $d_1,\ldots,d_{n-k}$ such that if $r_i = a_{i1} + \ldots + a_{in}$ are the rows of the matrix of L, then

$$c_1 r_1 + \ldots + c_k r_k = d_1 e_{i_1} + \ldots + d_{n-k} e_{n-k} \tag{3.5}$$

We now form a new matrix $\tilde{M}$ by attaching the matrix of L_I below $M(L)$. For example, in $\mathrm{Gr}(2,4)$ if $L_I = \mathrm{span}(e_1,e_2)$, then

$$\tilde{M} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

If equation (3.5) holds, we see that the rows of $\tilde{M}$ are not linearly independent, so $\det \tilde{M} = 0$. However, this also means that no two columns of M are linearly independent, so the $k\times k$ minor spanned by the orthogonal complement of L_I must vanish. Hence, we have shown that if L is generic, then none of its $k \times k$ minors vanish.

Conversely, if none of the $k \times k$ minors of $M(L)$ vanish, then none of the $k \times k$ minors of $\tilde{M}$ vanish so equation (3.5) is satisfied if and only if $c_i = d_i = 0$ for all i, showing that $L \cap L_I = 0$. □

Definition 14. *A Lie complex is the torus orbit closure* $\overline{T \cdot L}$ *for a generic* $L \in \mathrm{Gr}^0(k, n)$.

Theorem 9. *[20, Proposition 1.1.5] Each Lie complex is an* $n-1$ *dimensional variety with fixed points under the* T*-action given by* $\mathbb{C}^I$. *In fact, these* $\binom{n}{k}$ *points are the only singular points of a Lie complex.*

Example 4. *We compute the torus orbit closure in* $\mathrm{Gr}(2, 4)$. *If* L *is generic,* L *is represented by a matrix*

$$M(L) \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$$

such that none of the two by two minors vanish. Then, under the torus action, L *is mapped to a matrix*

$$T \cdot L = \begin{pmatrix} t_1 a_{11} & t_2 a_{12} & t_3 a_{13} & t_4 a_{14} \\ t_1 a_{21} & t_2 a_{22} & t_3 a_{23} & t_4 a_{24} \end{pmatrix}$$

for $t_i \neq 0$. *We note that none of the minors of* $T \cdot L$ *vanish as the minor consisting of the* i, j *columns is simply scaled by a factor of* $t_i t_j \neq 0$. *However, if we let up to two of the* t_i *in* $T \cdot L$ *we see that there is still one non-vanishing minor, so the resulting matrix which we denote* $\tilde{T} \cdot L$ *still represents an element of* $\mathrm{Gr}(2, 4)$ *and is therefore contained in the orbit closure* $\overline{T \cdot L}$.

Example 5. *The torus orbit orbit closure in* $\mathrm{Gr}(2, 4)$ *has a nice geometric interpretation. We view* $\mathrm{Gr}(2, 4)$ *as the set of lines in* $\mathbb{P}^3$. *Letting* x_1, x_2, x_3, x_4 *be homogeneous coordinates in* $\mathbb{P}^3$, *we define* L_i *to be the plane defined by* $x_i = 0$. *The configuration of the planes* L_i *form a tetrahedron, and* l *is general if and only* l *doesn't intersect any of the* 6 *edges of the tetrahedron. Then,* l *intersects each* L_i *at pairwise distinct points, and we can define the cross ratio*

$$r(l \cap L_1, \ldots, l \cap L_4) = \lambda.$$

The tetrahedral complex K_λ *is then defined to be the closure of the set of lines* $l \in \mathrm{Gr}^0(2, 4)$ *with cross ratio* λ. *In terms of the Plücker coordinates, this is the zero locus of*

$$p_{12} p_{34} + \lambda p_{13} p_{24} = 0.$$

This argument shows that the Lie complexes of $\mathrm{Gr}(2, 4)$ *are exactly the tetrahedral complexes* K_λ.

Because K_λ is cut out by a quadratic irreducible polynomial, we see that it has codimension 1 in $\mathrm{Gr}(2, 4)$. It follows that $[K_\lambda]$ is a $\mathbb{Z}$-linear combination of codimension 1 Schubert cycles, so $[K_\lambda] = 2\sigma_{1,0}$, where the coefficient comes from the fact that $\sigma_{1,0}^4 = 2\sigma_{2,2}$. In general, we know that all Lie complexes represent the same class in $H^* \mathrm{Gr}(k, n)$. In [23, Theorem 5], Kylachko gives a formula for the class of a Lie complex in terms of the Schubert cycle basis for $H^* \mathrm{Gr}(k, n)$. which we now explain.

Theorem 10. *Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\lambda_1 \geqslant \lambda_2 \cdots \geqslant \lambda_k$ be a Young tableaux with no more than k-rows and $(n-k)$ columns consisting of $(n-1)$ squares. The coefficient at σ_λ in the decomposition of a Lie complex $[T \cdot L]$ for with respect to Schubert cycles is*

$$\sum_{i=0}^{k} (-1)^i \binom{n}{i} \dim \Sigma^\lambda(\mathbb{C}^{k-i}) \tag{3.6}$$

where $\Sigma^\lambda(\mathbb{C}^{k-i})$ is the irreducible representation of $\mathrm{GL}(k-i)$ with highest weight λ.

We note that the dimension of the irreducible representation of $\mathrm{GL}(k-i)$ with highest weight λ is combinatorially determined by the number $\mathcal{N}$ of semi-standard Young tableaux of shape λ. The number $\mathcal{N}$ is given by the *hook-length formula*,

$$\mathcal{N} = \frac{|\lambda|!}{\prod_{s \in \lambda} \mathrm{hook}(s)},$$

where hook(s) for s a square in a Young diagram is the

$$\{\text{number of squares strictly below } s\} + \{\text{number of squares strictly to the right of } s\} + 1.$$

We note that this formula obscures the fact that the coefficients of the Schubert classes in this decomposition are always positive. In fact, in [6, Theorem 5.1], Berget and Fink give an equivalent formula

$$[\overline{T \cdot L}] = \sum_{\lambda} \sigma_\lambda \sigma_{\tilde{\lambda}},$$

where the sum is over the partitions λ that fit inside the $(k-1) \times (n-k-1)$ rectangle, and $\tilde{\lambda}$ is the complementary partition to λ with respect to this rectangle. Explicitly, if $\lambda = (\lambda_1, \ldots, \lambda_{k-1})$, then $\tilde{\lambda} = (n-k-1-\lambda_1, \ldots, n-k-1-\lambda_{k-1})$. A comparison of the formulas in ([6]) and ([23]) for the class of the torus orbit closure is given in [24].

In $\mathrm{Gr}(2,n)$, Klyachko's formula (3.6) reduces to

$$[\overline{T \cdot L}] = (n-2)\sigma_{n-3,0} + (n-4)\sigma_{n-4,1} + (n-6)\sigma_{n-5,2} + \ldots$$

We see that $[\overline{T \cdot L}] = 2\sigma_{1,0}$ which aligns with the fact that Lie complexes in $\mathrm{Gr}(2,4)$ have codimension 1, and $\sigma_{1,0}$ spans $H^2(\mathrm{Gr}(2,4))$. In $\mathrm{Gr}(2,5)$ for example, a Lie complex has codimension 2, and the formula tells us that

$$[\overline{T \cdot L}] = 3\sigma_{2,0} + \sigma_{1,1}.$$

In general, a Lie complex has codimension $k(n-k) - (n-1)$, and must therefore be a $\mathbb{Z}$-linear combination of Schubert classes with this codimension.

It seems that a full proof of the torus orbit formula (3.6) has not been published, and is a fruitful avenue for future work. There are multiple properties of the torus orbit closures that may be useful in the proof of this formula which we now state.

Let $e_1, \ldots, e_n$ be the standard basis vectors in $\mathbb{R}^n$. We define the convex polytope $\Delta(k,n)$ as the convex hull of $\binom{n}{k}$ points of the form $e_I \coloneqq e_{i_1} + \ldots + e_{i_k}$ for $1 \leqslant i_1 \leqslant \cdots \leqslant i_n \leqslant n$. These points are the vertices of $\Delta(k,n)$. For any subspace $L \in \mathrm{Gr}(k,n)$ we define its *matroid polytope* $M(L)$ as the convex hull of e_I where I runs over all bases for L. For a generic L, we we see that $M(L) = \Delta(k,n)$. There is a very concrete description of $\Delta(k,n)$ given in [1],[20].

1. Each face of $\Delta(k,n)$ is a hypersimplex.
2. Edges of $\Delta(k,n)$ are segments $[e_I, e_J]$ where J differs by I by swapping one element $i \in I$ with an element $j \notin I$.

Theorem 11. *[20, Proposition 1.2.4], [3] Let $L \in \mathrm{Gr}(k,n)$ be a generic subspace. Then, the torus orbit closure $\overline{T \cdot L}$ is a projective normal toric variety corresponding to the matroid polytope $\Delta(k,n)$, meaning the fan associated to $\overline{T \cdot L}$ is the normal fan of $\Delta(k,n)$, which means that the cones are the cones over the proper faces of $\Delta(k,n)$.*

We now consider the example of $\mathrm{Gr}(2,4)$.

Example 6. *Let L be a generic subspace in $\mathrm{Gr}(2,4)$. The matroid polytope $M(L) = \Delta(2,4)$ is the convex hull*

$$\Delta(2,4) = \mathrm{Conv}\{e_1 + e_2, e_1 + e_3, e_1 + e_4, e_2 + e_3, e_2 + e_4, e_3 + e_4\},$$

which is an octahedron.

To decompose the cohomology class of a generic torus orbit closure $X = \overline{T \cdot L}$ into the Schubert classes, one wants to consider the pullback of the inclusion map $\iota : X \to \mathrm{Gr}(k,n)$. Specifically, the coefficient of each Schubert class in the decomposition will be given by the image of the Schubert cycles under this pullback map. It is important to note that for $n = k - 1$, the fan associated to $\overline{T \cdot L}$ is complete, simplicial, and rational, which means that the cohomology ring of X has a simple combinatorial structure as described in [5], which may be of use in carrying out this computation.

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