A Parameterization of Convex Projective Groups

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1 Introduction

In the hyperbolic plane, fix a triangle with angles $\pi/\alpha, \pi/\beta, \pi/\gamma$ (so, $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}<1$). The set of reflections over the edges of this triangle generate a triangle group with relations

$$a^2 = b^2 = c^2 = 1, (ab)^\gamma = (bc)^\alpha = (ac)^\beta = 1$$

If we consider the Klein model of the hyperbolic plane in $\mathbb{RP}^2$, the reflection over edge e can be represented by a projective transformation, or a 3×3 matrix. The group applied to a triangle will tile the hyperbolic plane, which in the Klein model is the unit circle in $\mathbb{RP}^2$. So, the group fixes the unit circle (and so, under any projective transformation, fixes some conic section). Now, we can perturb each matrix in such a way that the same relations continue to hold and the group remains discrete. However, now, there is no fixed conic section. We call these new groups convex projective groups. These groups are studied extensively in greater generality in [Gol90]. They fix some non-conic convex surface in $\mathbb{RP}^2$. In the first part of this paper, we will develop a parameterization of these modified groups in terms of one parameter. This parameterization limits to the case of the standard reflection transformations. In the second part of the paper, we will examine these convex shapes and make a conjecture about their geometry as well as provide some computational evidence.

2 Background

2.1 Projective Geometry

$\mathbb{RP}^2$, the real projective plane, is the set of all lines through the origin in $\mathbb{R}^3$. We can view the $\mathbb{RP}^2$ as the set of vectors in $\mathbb{R}^3 \setminus \{0\}$ up to scalar multiplication, i.e. v_1 and v_2 are equivalent if and only if $v_1 = cv_2$ for some $c \in \mathbb{R}$. Now, we can represent this plane as the set of vectors $(a, b, 1) \in \mathbb{R}^3$ (which is just $\mathbb{R}^2$) together with the set of vectors $(x, 1, 0)$ and $(1, 0, 0)$, the set of slopes. These are known as points at infinity and the set of all such points is the line at infinity.
Defined in this way, any linear transformation $A \in \mathbf{GL}_3$ induces a transformation on $\mathbb{RP}^2$ because linear transformations preserve lines through the origin. These transformations are also equivalent under scaling, so we can assume they have determinant 1 by using the proper scaling factor. A general matrix in $\mathbf{GL}_3$ can bring any three points to any three points. For projective transformations, we are able to take any four non-collinear chosen points to any four non-collinear points because we have an added degree of freedom.

2.2 Traces

Later in the paper, we will want to check if the product of two reflections $AB = C$ is of a certain order n. Because raising C to the nth power and checking if we get the identity quickly becomes infeasible, we will instead consider the trace of the matrix. We will need the following lemma:

Lemma 2.1. *Let A and B be projective reflections over non-parallel lines. Then, for $n > 1$, $C = AB$ is of order n if and only if it has trace $2\cos(2\pi/n) + 1$.*

Proof. Let C be any projective transformation of order n. Then, C is a rotation by $2\pi/n$ degrees around some line with some scaling factor. However, it is equivalent to the rotation without the scaling factor, so we can assume there is no scaling factor without loss of generality. Now, there is a change of basis matrix Q which will transform it to a rotation around the origin. So, we can write $C = Q^{-1}NQ$ where N is the standard rotation around the z-axis:

$$N = \begin{pmatrix} \cos 2\pi/n & \sin 2\pi/n & 0 \\ -\sin 2\pi/n & \cos 2\pi/n & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which has trace $2\cos(2\pi/n) + 1$. Now, the trace is preserved by conjugation so C has the same trace up to scaling.

Now, suppose that $C = AB$ is the product of two reflections and has trace $2\cos 2\pi/n+1$. Let $(\lambda_0, \lambda_1, \lambda_2)$ be the eigenvalues of C. Now, we claim that one eigenvalue is 1. A and B are both reflections in $\mathbb{RP}^2$ so they each fix a line. These lines in $\mathbb{RP}^2$ can be described by planes through the origin in $\mathbb{R}^3$. Because the lines are not parallel, the planes intersect at a line through the origin. Each transformation A and B will fix this line, so $AB = C$ also fixes this line. So, one of the eigenvalues of C is 1. Without loss of generality, let $\lambda_0 = 1$. Now, we want to show that $C^n = I$. But C^n has eigenvalues $(\lambda_0^n, \lambda_1^n, \lambda_2^n)$ so $C^n = I$ if and only if each λ_i is an nth root of unity. We have already shown that $\lambda_0 = 1$ so it remains to be shown that each other eigenvalue is an nth root of unity.

Now, we know that the sum of the eigenvalues is the trace, so $\lambda_1 + \lambda_2 = 2\cos 2\pi/n$. Also, the product of the eigenvalues is the determinant (which is 1 for a projective transformation), so $\lambda_1\lambda_2 = 1$. Suppose first that each is real. We know that $\lambda_2 = \frac{1}{\lambda_1}$. So, we have that

$$\lambda_1 + \frac{1}{\lambda_1} = 2\cos 2\pi/n \implies \left|\lambda_1 + \frac{1}{\lambda_1}\right| \leq 2$$

However, $\lambda_1 + \frac{1}{\lambda_1}$ has a local min (max) of 2 (-2) at $\lambda_1 = 1$ ($\lambda_1 = -1$). So, the only possibilities are that $\lambda_1 = \lambda_2 = \pm 1$. Now, if $\lambda_1 = 1$, we need the cosine term to be 1, which only happens if $n = 1$ which we have discounted. Now, if $\lambda = -1$, the cosine term is -1 meaning $n = 2$. But -1 is a 2nd root of unity so $C^2 = I$ in this case.

Now, suppose that one of the eigenvalues is non-real. Then, because C is real, the other must be its conjugate. So, $\lambda_2 = \overline{\lambda_1}$. Now, we have that

$$\lambda_1 + \lambda_2 = \lambda_1 + \overline{\lambda_1} = 2\operatorname{Re}\lambda_1 = 2\cos 2\pi/n$$

So, the real part of each eigenvalue is $\cos 2\pi/n$. Now, we also have that

$$\lambda_1\lambda_2 = \lambda_1\overline{\lambda_1} = |\lambda_1| = 1$$

So, the imaginary part of λ_1 must be $\pm \sin 2\pi/n$ to have a norm of 1. So, our two eigenvalues are

$$\lambda_1 = \cos 2\pi/n + i \sin 2\pi/n = e^{2\pi i/n}, \lambda_2 = \cos 2\pi/n - i \sin 2\pi/n = e^{-2\pi i/n}$$

each of which is an nth root of unity as desired. $\square$

So, when we have a group of reflection matrices M_0, M_1, M_2 and want to check if $M_i M_j$ has order n, we can just consider the trace.

2.3 Hyperbolic Geometry

In Euclidean geometry, the Parallel Postulate states that given any line and any point not on that line, there is exactly one line parallel to the given line through the given point. In elliptical geometry, this is modified to read that there is no such parallel line, and in hyperbolic geometry, it is modified to read that there are infinitely many such lines. Another characterization important to this paper is that the angles of triangles in the hyperbolic plane always add to less than π instead of exactly π in Euclidean geometry (and $> \pi$ in elliptical geometry). See Chapter IV, §35 in [HC90] for further discussion of this characterization.

In this paper, I will make use of the Klein disk model of the hyperbolic plane which is best described as a projection of the hyperboloid model. In $\mathbb{R}^3$, we define a Lorentz form L as

$$L(x, y) = x_0 y_0 + x_1 y_2 - x_2 y_2$$

which gives rise to the quadratic form $Q(a, b, c) = a^2 + b^2 - c^2$. Now, we can define the hyperbolic plane to be

$$\mathbb{H}^2 = \{v = (a, b, c) \mid L(v, v) = -1, c > 0\}$$

In this model, geodesics are simply the intersection of the surface with planes through the origin. Because of this, we can describe lines by the vector orthogonal to the intersecting plane. We will use the notation $l^\perp$ as the orthogonal vector describing line l. We can normalize these vectors $l^\perp$ such that $L(l^\perp, l^\perp) = 1$. This also allows us to measure angles in the hyperbolic plane. Suppose lines v and w meet at angle θ; then, we have that

$$\cos \theta = L(v^\perp, w^\perp)$$

See [Rey93], especially section 7, for justification for this formula.

Note that $v^\perp$ and $w^\perp$ are only determined up to sign even with $L(v^\perp, v^\perp) = 1$. So, $L(v^\perp, w^\perp)$ is only determined up to sign as well, θ and $-\theta$ will both be solutions to the above equation. We can assume we are looking for the acute angle at which they meet and instead use

$$\cos^2 \theta = L(v^\perp, w^\perp)^2$$

Now, we also need a formula for where two lines intersect. A point where v and w intersect will be orthogonal to both $v^\perp$ and $w^\perp$. To find a vector orthogonal to two given vectors, we can define a box product

$$v \boxtimes w := (v_2 w_1 - v_1 w_2, v_0 w_2 - v_2 w_0, v_0 w_1 - v_1 w_0)$$

Lemma 2.2. $L(v \boxtimes w, v) = L(v \boxtimes w, w) = 0$

Proof.

$$
\begin{aligned}
L(v \boxtimes w, v) &= (v_2w_1 - v_1w_2)v_0 + (v_0w_2 - v_2w_0)v_1 - (v_0w_1 - v_1w_0)v_2 \\
&= v_0v_2w_1 - v_0v_1w_2 + v_1v_0w_2 - v_1v_2w_0 - v_0v_2w_1 + v_1v_2w_0 \\
&= 0 \\
L(v \boxtimes w, w) &= (v_2w_1 - v_1w_2)w_0 + (v_0w_2 - v_2w_0)w_1 - (v_0w_1 - v_1w_0)w_2 \\
&= w_0v_2w_1 - w_0v_1w_2 + w_1v_0w_2 - w_1v_2w_0 - w_1v_0w_2 + w_0v_1w_2 \\
&= 0
\end{aligned}
$$

□

From this lemma it follows that the box product works as desired.
We will also often need to consider the reflection across a given line. It is important to note that not only will such a reflection fix the line, but it will also fix the vector orthogonal to the line. In fact, any projective transformation of order 2 will fix one line and one point (see §1.2 in [Gol90]). We can use this fact when we generalize reflections by changing the point which they fix.
This model is very useful, but somewhat difficult to represent visually. So, for most of the following, we will consider the Klein disk model, which is the hyperboloid model projected down so that the third coordinate is 1. Now, the hyperbolic plane is the unit circle and lines are simply chords. Equivalently, we can consider the points in $\mathbb{R}^3$ as points in $\mathbb{RP}^2$ projected down onto the plane $z = 1$. Now, reflections can be thought of as projective transformations which will be useful later.

2.4 Triangle Groups

Definition 2.1. A triangle group, denoted by three parameters (α, β, γ) such that $2 \leq \alpha, \beta, \gamma$, is generated by the reflections over the edges of a triangle with angles $\pi/\alpha, \pi/\beta$, and π/γ. The group rule is composition of reflections.

Note that if $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 1$, the triangle sits in Euclidean space. If the sum is greater than 1, it sits in elliptic space and if it is less than 1, it sits in hyperbolic space. We will focus on this third case because the first two are fairly limited (there are only three examples in Euclidean space). If we let the generators of our group be a, b, c where a is the reflection over the edge opposite angle π/α etc., we have the following relations:

$$a^2 = b^2 = c^2 = 1, (ab)^\gamma = (bc)^\alpha = (ac)^\beta = 1$$

The first relation holds because applying any reflection twice is the identity, and the second relation holds because by reflecting over each edge incident to an angle repeatedly, we can rotate around that angle fully, again yielding the identity.
Now, in the Klein Model of the hyperbolic plane, each reflection can be represented by a projective transformation, so we also have a corresponding group of matrices $A \in M_{3,3}$

which follow the same relations. Also, each group defines a tiling of the hyperbolic plane with the corresponding triangle.

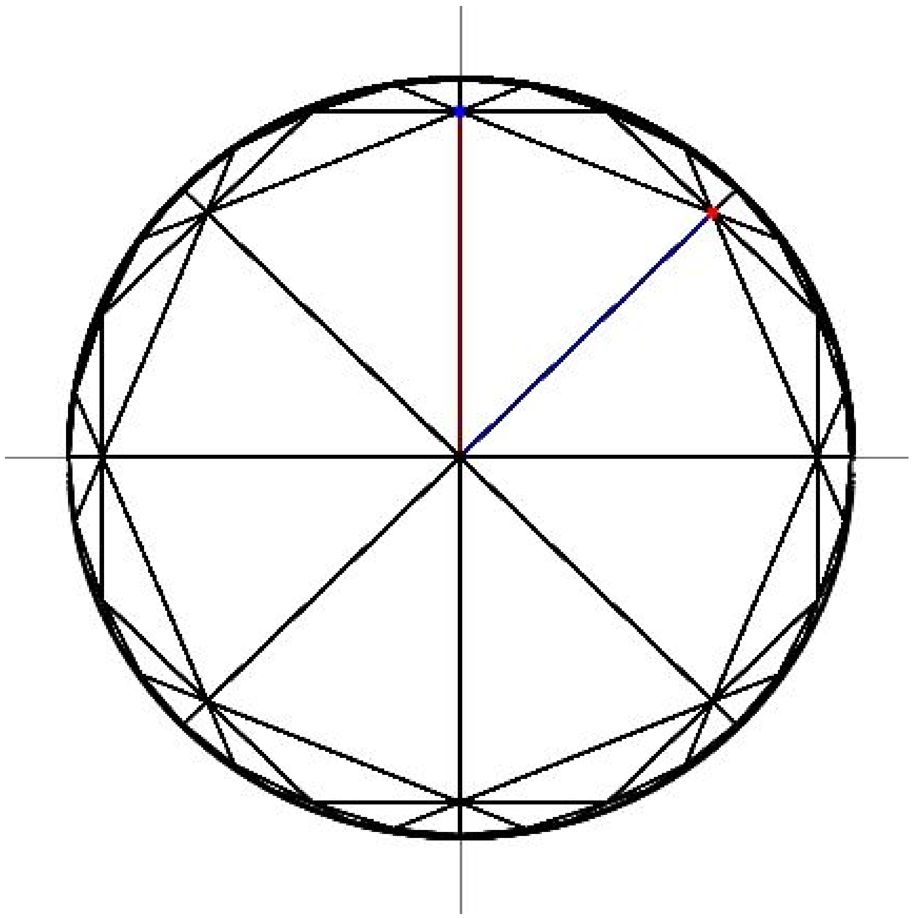


Figure 2.4.1: Tiling of the hyperbolic plane with the (4,4,4) triangle group

Now, since any projective transformation of order 2 fixes a line and a point, we can generalize the concept of triangle groups by changing the point. Given a triangle in the hyperbolic plane, we can find the matrix representation of the triangle group by finding the transformations which correspond to reflecting over each side. These will fix each edge and the orthogonal vector to that edge. By changing the vector fixed by each transformation in the correct way, we are able to find three new generators which still obey the same relations. Now, it turns out that for sufficiently small perturbations, there is some convex domain fixed by this new group instead of just the unit circle. For a more general treatment of these convex projective structures, see [Gol90].
The object of this paper is to describe the one parameter family of these more general groups and to examine some of their properties.

3 Parameterization

3.1 Setting

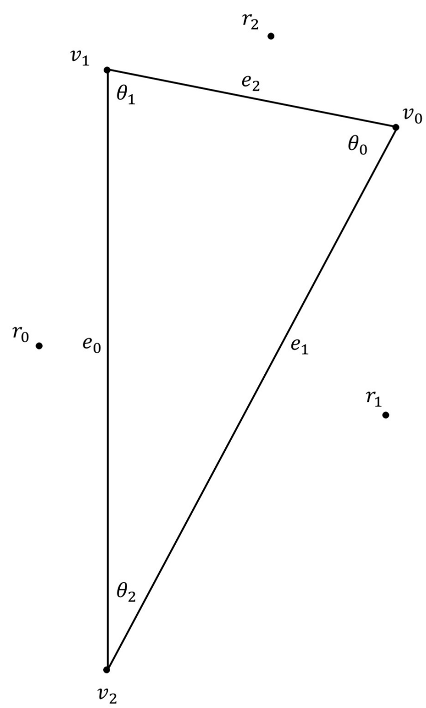


Figure 3.1.1: Labeled Triangle

For a family of convex projective groups based on the triangle group (α, β, γ) we first find the triangle with angles $\theta_0 = \pi/\alpha, \theta_1 = \pi/\beta$, and $\theta_2 = \pi/\gamma$ in the Klein model of the hyperbolic plane. To find the triangle, we set $e_0^\perp = (-1, 0, 0), e_1^\perp = (-\cos\theta_2, \sin\theta_2, 0)$, and $e_2^\perp = (s, t, u)$ where

$$\begin{aligned} s &= \cos\theta_1 \\ t &= \frac{\cos\theta_0 + \cos\theta_2 s}{\sin\theta_2} \\ u &= \sqrt{s^2 + t^2 - 1} \end{aligned}$$

These choices guarantee the correct angles. Recall that the angle θ between two edges v and w obeys the formula

$$\cos^2\theta = L(v^\perp, w^\perp)^2$$

So, we just need to check

$$\begin{aligned}
L(e_0^\perp, e_1^\perp)^2 &= L((-1,0,0),(-\cos\theta_2, \sin\theta_2, 0))^2 \\
&= \cos^2\theta_2 \\
L(e_0^\perp, e_2^\perp)^2 &= L((-1,0,0),(s,t,u))^2 \\
&= (-s)^2 = \cos^2\theta_1 \\
L(e_1^\perp, e_2^\perp)^2 &= L((-\cos\theta_2, \sin\theta_2, 0),(s,t,u))^2 \\
&= (-s\cos\theta_2 + t\sin\theta_2)^2 \\
&= (-\cos\theta_1\cos\theta_2 + \cos\theta_0 + \cos\theta_2\cos\theta_1)^2 \\
&= \cos^2\theta_0
\end{aligned}$$

as desired. From here, we can find the vertices (where the edges intersect) as follows:

$$\begin{aligned}
v_2 &= e_0^\perp \boxtimes e_1^\perp = (0,0,-\sin\theta_2) = (0,0,1) \\
v_1 &= e_0^\perp \boxtimes e_2^\perp = (0,-u,-t) = (0, \frac{u}{t}, 1) \\
v_0 &= e_1^\perp \boxtimes e_2^\perp = (-\sin\theta_2 u, -\cos\theta_2 u, -\cos\theta_2 t - \sin\theta_2 s) \\
&= \left(\frac{\sin\theta_2 u}{\cos\theta_2 t + \sin\theta_2 s}, \frac{\cos\theta_2 u}{\cos\theta_2 t + \sin\theta_2 s}, 1\right) \\
&:= (a,b,1)
\end{aligned}$$

Now, in the standard triangle group, the reflection over each edge fixes that edge and the vector orthogonal to the plane of the edge. When we look for more general groups, we can change each reflection by changing the point which is fixed. However, these groups are only unique up to projective transformations, and a projective transformation can bring any 4 points to any 4 points. So, we can assume that one of these "reflection points" is the original, as we can always bring one of the reflection points back to the original while still fixing the three vertices of the triangle using a projective transformation. So, we can let r_2 be this fixed point:

$$r_2 = e^\perp = (s,t,u)$$

Now, the other two points are variable so we let $r_0 = (x,y,1)$ and $r_1 = (q,w,1)$.

3.2 Formulas

Now we are in a position to state the formulas for the three matrices of our group in terms of the one free parameter q. First, the four variables defined above obey the linear equation

$$\begin{pmatrix} x \\ y \\ q \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ b_{02} \\ 0 \\ b_{12} \end{pmatrix} + q \begin{pmatrix} m_{01} \\ 0 \\ 1 \\ m_{12} \end{pmatrix}$$

where

$$
\begin{aligned}
b_{02} &= \frac{u}{t} \\
m_{01} &= \frac{u \tan\theta_2}{at} \\
b_{12} &= \frac{a}{\sin\theta_2 \cos\theta_2} \\
m_{12} &= -\tan\theta_2
\end{aligned}
$$

For the matrices, we have

$$
M_0 = \begin{pmatrix} 1 & 0 & 0 \\ \dfrac{2b_{02}}{m_{01}q} & -1 & 0 \\ \dfrac{2}{m_{01}q} & 0 & -1 \end{pmatrix}
$$

$$
M_1 = \begin{pmatrix} \dfrac{q(b+m_{12}a)+b_{12}a}{q(b-m_{12}a)-b_{12}a} & \dfrac{-2aq}{q(b-m_{12}a)-b_{12}a} & 0 \\ \dfrac{2bm_{12}q+2bb_{12}}{q(b-m_{12}a)-b_{12}a} & \dfrac{-q(b+m_{12}a)-b_{12}a}{q(b-m_{12}a)-b_{12}a} & 0 \\ \dfrac{2b}{q(b-m_{12}a)-b_{12}a} & \dfrac{-2a}{q(b-m_{12}a)-b_{12}a} & -1 \end{pmatrix}
$$

$$
M_2 = \begin{pmatrix} \cos(2\theta_1) & 2t\cos\theta_1 & -2u\cos\theta_1 \\ 2t\cos\theta_1 & 2t^2-1 & -2ut \\ 2u\cos\theta_1 & 2ut & -\cos(2\theta_1)-2t^2 \end{pmatrix}
$$

where matrix M_i fixes edge e_i and reflection point r_i.
There are two asymptotes for this solution. First, when $q = 0$, terms in M_0 blow up. Also, when $q = \dfrac{b_{12}a}{b - m_{12}a} = a$, terms in M_1 blow up. Solutions on either side of this boundary seem to be equivalent under projective transformations. For a given triangle group, a is the second asymptote. Now, $G(q)$ is equivalent to $G(a-q)$ where $G(x)$ denotes the group of matrices with parameter x. However, vertices 0 and 1 are switched. So, if we swap matrices M_0 and M_1, the trace of any word is equal at q and $a - q$.

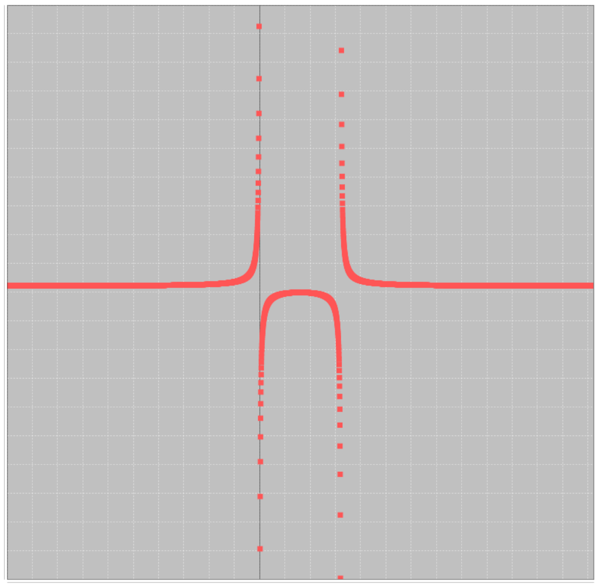
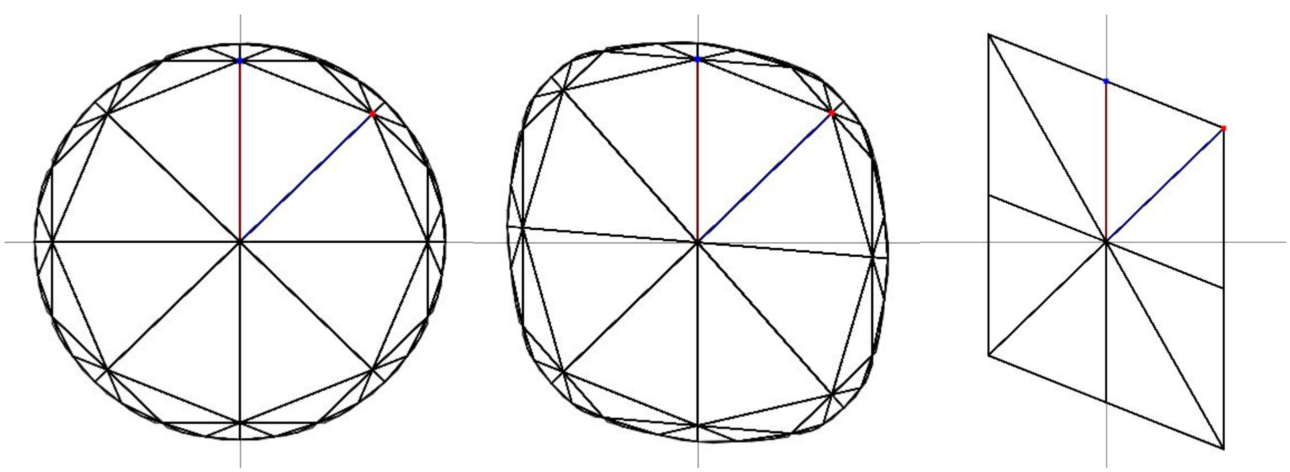


Figure 3.2.1: Trace of $M_0M_1M_2M_1$ with M_0 and M_1 swapped for $q < 0$

With this parameterization, the limit as $q \to 0$ from below or $q \to a$ from above results in the most deformed convex shape, whereas the limit as $q \to \pm\infty$ results in the convex shape formed by a standard triangle group. In this parameterization it is an ellipse, although by using a normalizing projective transformation, we can return it to a circle.



(a) $q = -1000000$ (b) $q = -1$ (c) $q = -0.001$

Figure 3.2.2: Convex shapes for various q values

Solutions between the two asymptotes 0 and a do not fix convex domains. Below is one such example:

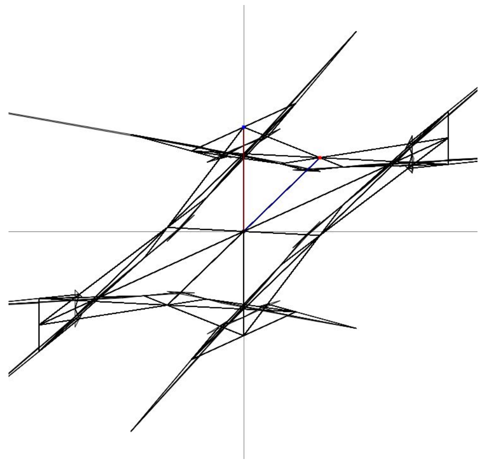


Figure 3.2.3: Shape when q value is between asymptotes (For the (4,4,4) group at $q = 0.5$)

3.3 Normalization

A normalization which creates symmetric shapes limiting to a circle (which is useful for finding evidence for the conjecture below) is as follows: Find the images of vertices 0 and 1 rotated halfway around vertex 2 by applying the matrix $(M_0M_1)^{\gamma/2}$ to the original vertices (note that this only works for even γ). Call these new points $f(v_0)$ and $f(v_1)$. Now, use the projective transformation which brings the four points $(v_0, v_1, f(v_0), f(v_1))$ to $(v_0, v_1, -v_0, -v_1)$. This will normalize the convex shape to be more symmetrical and to limit to a circle as $q \to \infty$. All images in this paper use this normalization, unless otherwise noted.

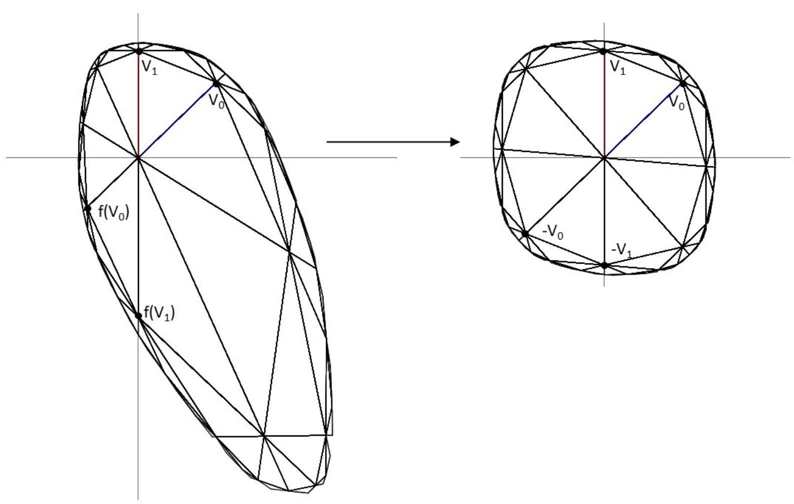


Figure 3.3.1: Normalization Transformation

3.4 Derivations

In this section, we will derive the formulas for the matrices given above. We will do this in three steps:

1. First, we will give a formula for a general perturbed reflection matrix fixing a given line and a given point.

2. Next, we will calculate the perturbed reflections M_0, M_1, M_2 over each edge. M_2 will be deterministic because we assume that this matrix is the standard reflection, but M_0 and M_1 will be in terms of the two variable reflection points $(x, y, 1)$ and $(q, w, 1)$.

3. Finally, we will use trace equations to force the rotation relations to hold. This will give a linear equation for all of our variables in terms of the parameter q.

Step 1 The reflection matrix fixing the edge through vertices $(m_0, m_1, 1)$ and $(n_0, n_1, 1)$ which also fixes the point (p_0, p_1, p_2) has the form $A^{-1}KA$ where

$$K = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$A^{-1} = \begin{pmatrix} m_0 & n_0 & p_0 \\ m_1 & n_1 & p_1 \\ 1 & 1 & p_2 \end{pmatrix}$$

The matrix K fixes the line at infinity (any vector with a third coordinate of 0 remains the same up to scaling) and also fixes the point $(0, 0, 1)$ (the origin). The matrix A^{-1} brings the line at infinity to the desired line and the origin to the desired point. So, by composing $A^{-1}KA$, we bring the desired point and line to the origin and the line at infinity, then perform the desired reflection, and bring everything back.

Step 2 To calculate M_0, we first need to find A^{-1}. From the formulas listed,

$$A^{-1} = \begin{pmatrix} 0 & 0 & x \\ u/t & 0 & y \\ 1 & 1 & 1 \end{pmatrix}$$

From here, we can calculate that

$$A = \frac{t}{xu} \begin{pmatrix} -y & x & 0 \\ y - u/t & -x & xu/t \\ u/t & 0 & 0 \end{pmatrix}$$

Now,

$$\begin{aligned}
M_0 &= A^{-1}KA \\
&= \frac{t}{xu}\begin{pmatrix} 0 & 0 & x \\ u/t & 0 & y \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -y & x & 0 \\ y-u/t & -x & xu/t \\ u/t & 0 & 0 \end{pmatrix} \\
&= \frac{t}{xu}\begin{pmatrix} xu/t & 0 & 0 \\ 2yu/t & -xu/t & 0 \\ 2u/t & 0 & -xu/t \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 2y/x & -1 & 0 \\ 2/x & 0 & -1 \end{pmatrix}
\end{aligned}$$

For M_1, we have

$$A^{-1} = \begin{pmatrix} a & 0 & q \\ b & 0 & w \\ 1 & 1 & 1 \end{pmatrix}$$

From here, we can calculate that

$$A = \frac{1}{qb-wa}\begin{pmatrix} -w & q & 0 \\ w-b & a-q & qb-wa \\ b & -a & 0 \end{pmatrix}$$

Now,

$$\begin{aligned}
M_1 &= A^{-1}KA \\
&= \frac{1}{qb-wa}\begin{pmatrix} a & 0 & q \\ b & 0 & w \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -w & q & 0 \\ w-b & a-q & qb-wa \\ b & -a & 0 \end{pmatrix} \\
&= \frac{1}{qb-wa}\begin{pmatrix} aw+qb & -2aq & 0 \\ 2bw & -aw-qb & 0 \\ 2b & -2a & aw-qb \end{pmatrix}
\end{aligned}$$

For M_2, we have

$$A^{-1} = \begin{pmatrix} 0 & a & s \\ u/t & b & t \\ 1 & 1 & u \end{pmatrix}$$

The determinant of this matrix simplifies to a/t so we have

$$A = \frac{t}{a}\begin{pmatrix} ba-t & s-au & at-bt \\ t-u^2/t & -s & -us/t \\ u/t-b & a & -au/t \end{pmatrix}$$

Now,

$$\begin{aligned} M_2 &= A^{-1}KA \\ &= \frac{t}{a}\begin{pmatrix} 0 & a & s \\ u/t & b & t \\ 1 & 1 & u \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} ba-t & s-au & at-bt \\ t-u^2/t & -s & -us/t \\ u/t-b & a & -au/t \end{pmatrix} \\ &= \begin{pmatrix} -1+2s^2 & 2st & -2su \\ 2st & -1+2t^2 & -2tu \\ 2su & 2tu & 1-2s^2-2t^2 \end{pmatrix} \end{aligned}$$

after some messy calculations. We can write this as

$$M_2 = \begin{pmatrix} \cos(2\theta_1) & 2t\cos\theta_1 & -2u\cos\theta_1 \\ 2t\cos\theta_1 & 2t^2-1 & -2ut \\ 2u\cos\theta_1 & 2ut & -\cos(2\theta_1)-2t^2 \end{pmatrix}$$

after a few further simplifications.

Step 3 Now, we want to find quantities for x, y, q, z such that $(M_0M_1)^\gamma = (M_0M_2)^\beta = (M_1M_2)^\alpha = I$. To simplify these equations, we can instead set the trace of the products equal to the trace of the rotation matrix of the same order (see Lemma 2.1). So, we set the trace of M_0M_2 to

$$2\cos 2\pi/\beta + 1 = 2\cos 2\theta_1 + 1$$

and then solve

$$\begin{aligned} \mathrm{Tr}(M_0M_2) &= 2\cos(2\theta_1)+1 \\ \cos 2\theta_1 + \frac{2y}{x}(2t\cos\theta_1) + 1 - 2t^2 - 4\frac{u}{x}\cos\theta_1 + \cos 2\theta_1 + 2t^2 &= 2\cos(2\theta_1)+1 \\ 2\cos 2\theta_1 + 1 + \frac{4yt\cos\theta_1}{x} - \frac{4u\cos\theta_1}{x} &= 2\cos(2\theta_1)+1 \\ \frac{4yt\cos\theta_1}{x} - \frac{4u\cos\theta_1}{x} &= 0 \\ 4yt\cos\theta_1 &= 4u\cos\theta_1 \\ y &= \frac{u}{t} \end{aligned}$$

Now, we set

$$\mathrm{Tr}(M_1M_2) = 2\cos 2\pi/\alpha + 1 = 2\cos(2\theta_0) + 1$$

This gives

$$\begin{aligned} 2\cos(2\theta_0) + 1 = \frac{1}{qb-wa}\Big(&(aw+qb)\cos 2\theta_1 - 4aqt\cos\theta_1 + 4bwt\cos\theta_1 \\ &- (aw+qb)(2t^2-1) - 4bu\cos\theta_1 \\ &+ 4aut + (aw-qb)(-\cos 2\theta_1 - 2t^2)\Big) \end{aligned}$$

Some simplifying calculations yield $w = m_{12}q + b_{12}$ where $m_{12} = -\tan\theta_2$ and $b_{12} = \dfrac{a}{\sin\theta_2 \cos\theta_2}$.
Now, we set

$$\begin{aligned} \mathrm{Tr}(M_0 M_1) &= 2\cos 2\pi/\gamma + 1 \\ &= 2\cos 2\theta_2 + 1 \\ \frac{1}{qb - wa}\left(aw + q - \frac{4yaq}{x} + aw + qb + qb - aw\right) &= 2\cos 2\theta_2 + 1 \\ aw + 3qb - \frac{4yaq}{x} &= (qb - wa)(2\cos 2\theta_2 + 1) \end{aligned}$$

Simplifying and substituting in for y and w using what we know from above, we get that

$$x = m_{01} q$$

where $m_{01} = \dfrac{u \tan\theta_2}{at}$. Now, we arrive at the linear equation stated above and are able to write out each matrix in terms of a single variable q.

4 Conjecture

4.1 Motivation

The following discussion was motivated by a discussion with Nicolas Tholozan while he was visiting Brown to give a talk in a geometry seminar. The entropy of the Hilbert metric, a related quantity to what we discuss here, and its asymptotic behavior over one parameter families of a more general class of projective groups is studied in [Nie15]. Using the machinery developed above, we have a way of describing the one parameter family of groups for a given triangle group (α, β, γ). We get a valid convex shape for any parameter value in the set $(-\infty, 0) \cup (a, \infty)$ where a is the constant described above. Since the two parts can be seen to be equivalent under projective transformations, we will just focus on the first interval. With the parameterization above, we can consider the family of groups $G(q)$ for $-\infty < q < 0$ for a given triangle group (α, β, γ). Each gives rise to a convex domain in $\mathbb{RP}^2$. At the limit $q \to -\infty$, our convex shape is the same as using the standard reflections in the hyperbolic plane. In the normalization defined above, the convex shape induced by the standard triangle group is just the hyperbolic plane, or the unit disk in $\mathbb{RP}^2$. Now, in each shape, we can consider the size of the triangle. We can use the Hilbert metric to measure distance in each convex shape.

Definition 4.1. For a convex domain Ω which doesn't contain a line, consider two points u and v. Let x and y be the points at which the line through a and b intersects the boundary of Ω and suppose that the order of the four points is x, u, v, y. Then, the Hilbert distance $d_\Omega(u, v)$ is defined by

$$d_\Omega(u, v) = \log\left(\frac{|yu||xv|}{|yv||xu|}\right)$$

and $d_\Omega(u, u) = 0$.

This corresponds with twice the hyperbolic metric on the disk in the Klein model (see page 243 of [HC90]). Now, these convex shapes are equivalent under projective transformation and so is the Hilbert metric. One hypothesis we considered is that the Hilbert metric was monotonically increasing as the parameter q varied from $-\infty$ to 0. One way to prove this would be to show that for $q_1 > q_2$, the convex shape induced by $G(q_1)$ was contained in the convex shape induced by $G(q_2)$. Then, the following lemma would suffice to show the hypothesis:

Lemma 4.1. *Let A and B be convex shapes. Suppose that $A \subsetneq B$. Then for any two points $u, v \in A$, $d_A(u, v) > d_B(u, v)$ where d_A is the Hilbert metric for A and d_B is the Hilbert metric for B*

Proof. Let a_1 and a_2 be the points on the boundary of A intersecting the line through u and v and let b_1 and b_2 be the points on the boundary of B intersecting the same line. Suppose that the order of points is b_1, a_1, u, v, a_2, b_2 (see Figure 4.1.1). Let $|uv| = l$. Note that because $A \subsetneq B$, we have that $|a_1u| < |b_1u|$ and $|a_2v| < |b_2v|$. Now, we can consider $e^{d_A(u,v)}$ for simplicity because the logarithm is monotonic.

$$\begin{aligned}
e^{d_A(u,v)} &= \frac{|a_2u||a_1v|}{|a_2v||a_1u|} \\
&= \frac{(|a_2v| + l)(|a_1u| + l)}{|a_2v||a_1u|} \\
&= \frac{|a_2v||a_1u| + l(|a_1u| + a|_2v|) + l^2}{|a_2v||a_1u|} \\
&= 1 + l\left(\frac{1}{|a_2v|} + \frac{1}{|a_1u|}\right) + \frac{l^2}{|a_2v||a_1u|} \\
&> 1 + l\left(\frac{1}{|b_2v|} + \frac{1}{|b_1u|}\right) + \frac{l^2}{|b_2v||b_1u|} \\
&= e^{d_B(u,v)}
\end{aligned}$$

so $d_A(u, v) > d_B(u, v)$. ∎

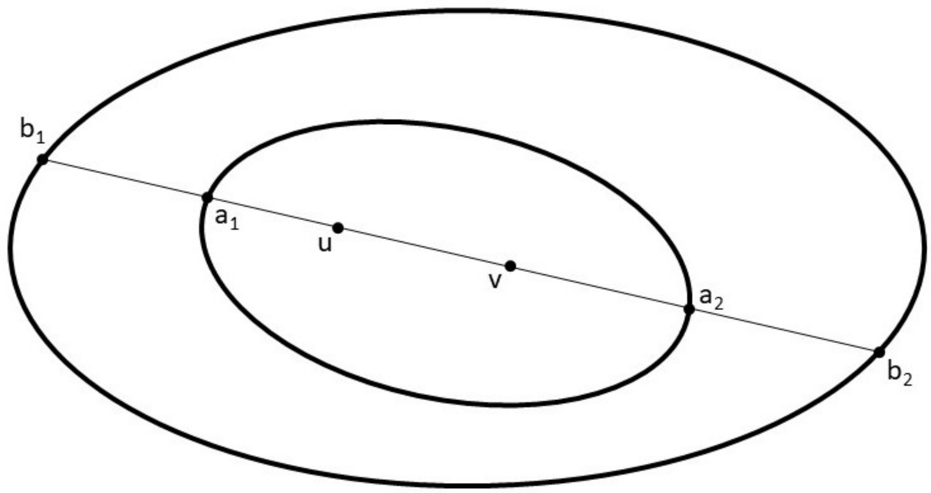


Figure 4.1.1: Diagram for proof of Lemma 4.1

4.2 Statement

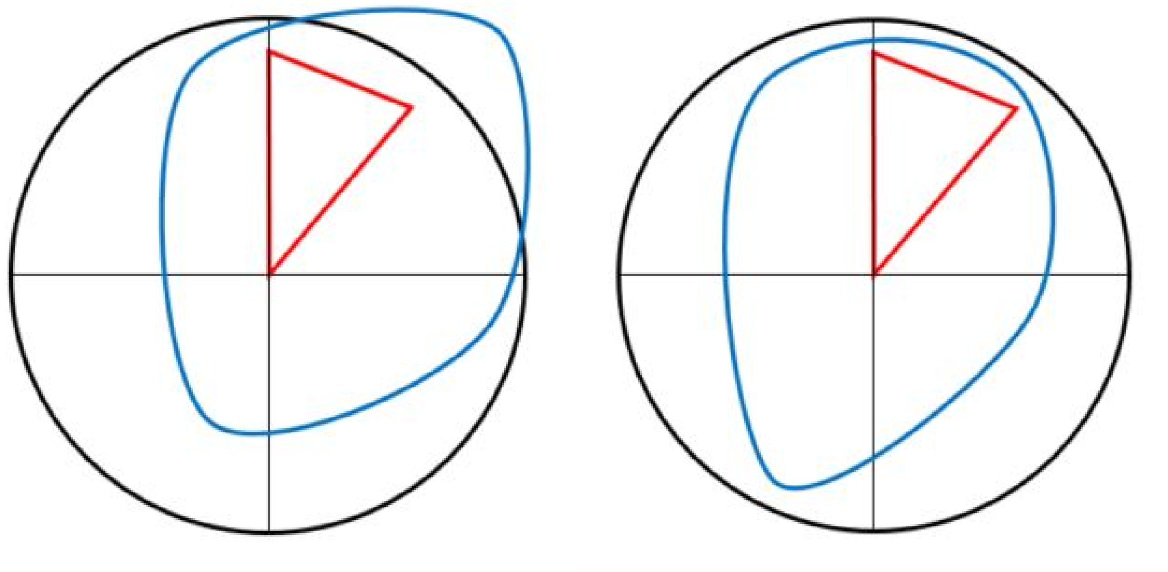
Some computational experimentation suggested that we could not in fact fit shapes with large q values into those with smaller values, leading to the following conjecture:

Conjecture 4.1. *For any triangle group (α, β, γ) and any parameter $q < 0$, let $G(q)$ be the convex projective group formed using the parameter q. Then, if $x < y$, there is no projective transformation of the convex shape of $G(y)$ which fixes the vertices of one triangle and is entirely contained in the convex shape of $G(x)$.*

In order to better understand this, we considered the simpler case of fitting any given shape entirely into the unit circle (which is induced by the group $G(q)$ in the limit as $q \to \infty$). After some further experimentation, we conjectured that for no value of q could the convex shape fit entirely into the unit circle.

Conjecture 4.2. *For any triangle group (α, β, γ) and any parameter $q < 0$, let $G(q)$ be the convex projective group formed using the parameter q. Then, there is no projective transformation of the corresponding convex shape which fixes the vertices of one triangle and is entirely contained in the unit circle.*

If Conjecture 4.2 held, for every $q \in (-\infty, 0)$, we would never be able to find a projective transformation which fixed the original triangle and fit the invariant convex boundary of $G(q)$ into the unit circle. In the below diagram, there would be no projective transformation fixing the red triangle which brings the blue convex boundary in Figure 4.2.1(a) to the blue boundary in Figure 4.2.1(b).



(a) (b)

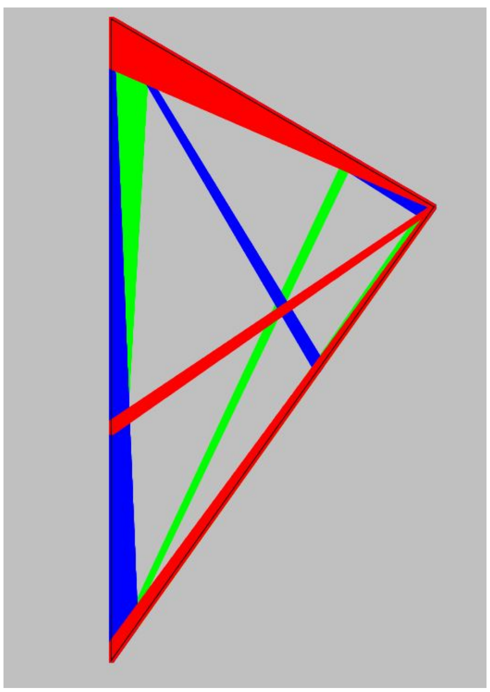
Figure 4.2.1: A projective transformation fixing the triangle which fits the invariant convex boundary of $G(q)$ into the unit circle

If Conjecture 4.1 held, the same would be true except that the unit circle could be replaced by the invariant convex boundary of some other group $G(r)$ for every $r < q$. We were able to computationally verify Conjecture 4.2 for one case and saw evidence that it would be true in general.

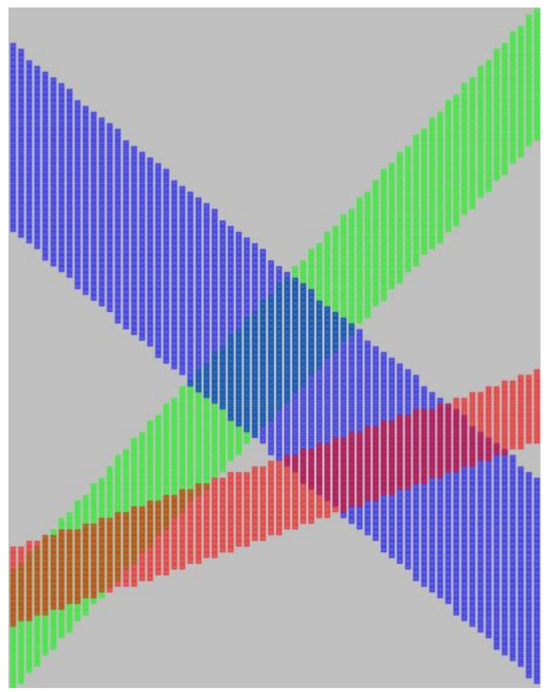
4.3 Evidence

To investigate this conjecture, we studied the $(4, 4, 4)$ triangle group with parameter $q = -1$. To prove Conjecture 4.2 for this case, it is sufficient to find a finite collection of points on the boundary such that no projective transformation which fixed the central triangle would bring them all into the unit circle. We looked at the six points where each edge intersected the boundary. We then considered the set of projective transformations which brought each pair of points into the unit circle (in suitable coordinates) and computationally demonstrated that the intersection of these three sets was empty.
In order to find exact values for the six boundary points, we numerically calculated the eigenvectors of matrices which fixed each edge of the original triangle. To fix e_0, we used $(M_0M_2)^2(M_0M_1)^2$, for e_1 we used $(M_1M_2)^2(M_1M_0)^2$, and for e_2 we used $(M_2M_1)^2(M_2M_0)^2$. Because these matrices fixed each edge, the eigenvectors represented points on each edge which only scaled. However, in projective geometry, points which only scale are fixed. So, we were able to find the fixed points of transformations which fixed each edge, which were the endpoints of each edge. These points are on the boundary of the convex shape. Because we were attempting to fit these points into a circle, we truncated rather than rounded to preserve accuracy with this numerical

method.
Now, we checked computationally if there was any projective transformation which would put the endpoints of all three edges into the unit circle while fixing the central triangle. Each transformation which fixes the central triangle can be described by the way it moves a fixed internal point. A point p inside the triangle was arbitrarily chosen and each transformation T was described by the point $Tp = r$. For simplicity, let $T(r)$ denote the transformation found in this way. For each point r in the triangle, we considered $T(r)$ applied to the eigenvectors calculated above. We had three sets of two endpoints, one set for each edge of the triangle. We plotted the points r for which $T(r)$ fit each pair into the unit circle in different colors and were able to experimentally determine that there was no point r with a corresponding transformation $T(r)$ which fit all three sets of points into the unit circle, verifying the conjecture for this case.



(a) Every point in the triangle defines a transformation...



(b) ...and none can fit all six points into the circle

Figure 4.3.1: Trying to fit the $(4, 4, 4)$ triangle into the circle for $q = -1$

Note that while there were small areas around the edges of the triangle where all three colors overlapped, these transformations brought the fixed point too close to the edge and therefore didn't correctly maintain the convex structure. As an example, below is the convex shape under the projective transformation described by a point in the center of the innermost triangle formed by the three strips:

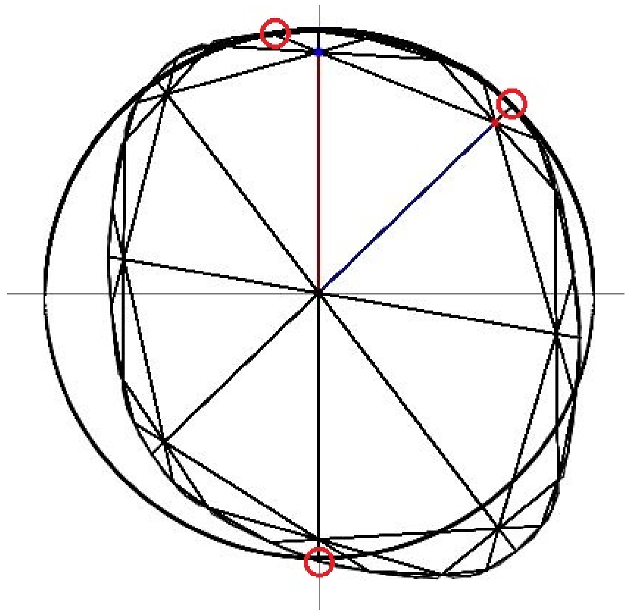


Figure 4.3.2: We see that each of the three pairs of points almost, but doesn't quite fit (and the entire convex shape certainly doesn't fit)

This method relied on picking an arbitrary point p inside the triangle. To eliminate the dependence on such a choice, we instead plotted the eigenvalues of each transformation matrix which did not depend on the fixed point. Because we were only considering projective transformations, we only had a two parameter family of eigenvectors. Without normalizing our transformation matrices, one eigenvalue was always 1 so below, we have plotted the logs of the other two eigenvalues. It is evident that the three sets do not intersect.

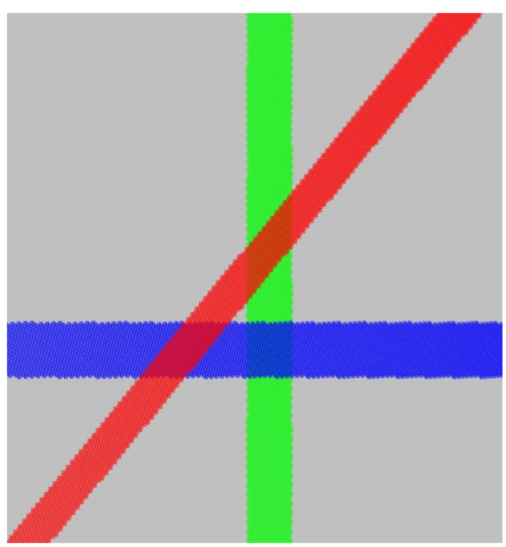


Figure 4.3.3: When we graph the logs of the eigenvalues to remove dependence on the fixed point, there is still no intersection

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