THE WEIL-ÉTALE TOPOLOGY FOR NUMBER RINGS

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§0. Introduction

The purpose of this paper is to serve as the first step in the construction of a new Grothendieck topology (the Weil-étale topology) for arithmetic schemes X (schemes of finite type over Spec $\mathbb{Z}$), which should be in many ways better suited than the étale topology for the study of arithmetical invariants and of zeta-functions. The Weil-étale cohomology groups of "motivic sheaves' or "motivic complexes of sheaves" shold be finitely generated abelian groups, and the special values of zeta-functions should be very closely related to Euler characteristics of such cohomology groups.

As an example of the above philosophy, let $\bar{X}$ be a compactification of X. This involves first completing X to obtain a scheme X_1 such that X is dense in X_1 and $f : X_1 \to \text{Spec } \mathbb{Z}$ is proper over its image, and then, if f is dominant, adding fibers over the missing points of Spec $\mathbb{Z}$ and the archimedean place of $\mathbb{Q}$ to obtain $\bar{X}$.

Let ϕ be the natural inclusion of X into $\bar{X}$. The following should be true:

a) The Weil-étale hypercohomology groups with compact support $H^q(\bar{X}, \phi_!\mathbb{Z})$ are finitely generated abelian groups which are equal to 0 for all but finitely many q, and independent of the choice of compactification $\bar{X}$. We will denote them by $H^q_c(X, \mathbb{Z})$.

b) If $\tilde{\mathbb{R}}$ denotes the "sheaf of real-valued functions" on X, then the cohomology groups $H^q(\bar{X}, \phi_!\tilde{\mathbb{R}})$ are independent of the compactification, and we denote them by $H^q_c(X, \tilde{\mathbb{R}})$. The

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natural map from $H^q_c(X,\mathbb{Z})\otimes\mathbb{R}$ to $H^q_c(X,\tilde{\mathbb{R}})$ is an isomorphism. (Note that this is not at all a formality, and would for instance be false if we considered cohomology on all of $\bar{X}$).

c) There exists an element ψ in $H^1(\bar{X},\tilde{\mathbb{R}})$ such that the complex $(H^*_c(X,\tilde{\mathbb{R}}),\cup\psi)$ (this is a complex under Yoneda product with ψ) is exact.

Then the Euler characteristic $\chi_c(X)$ of the complex $H^q_c(X,\mathbb{Z})$ is well-defined (See Section 7), and we can describe the behavior of the zeta-function $\zeta_X(s)$ at $s=0$ ($\zeta^*(0) = lim_{s\to 0}\zeta(s)s^{-a}$ where a is the order of the zero of $\zeta(s)$ at $s=0$) by the formula $\zeta^*_X(0) = \pm\chi_c(X)$.

Defining $\zeta^*(X,-n)$ in the analogous fashion, and taking advantage of the formula $\zeta_X(s) = \zeta_{X\times\mathbb{A}^n}(s+n)$, we can conjecturally describe the behavior of the zeta-function of any arithmetic scheme at any non-positive integer $-n$ by the formula $\zeta^*(X,-n) = \pm\chi_c(X\times\mathbb{A}^n)$,

There should exist motivic complexes $\mathbb{Z}(-n)$ whose Euler characteristics give the values of $\zeta^*(X,-n)$ directly, and the above conjectural formula should give a guide to a possible definition.

In this paper we only define the Weil-étale topology in the case when F is a global number field and $X = \text{Spec } O_F$. We then compute the cohomology groups $H^q_c(X,\mathbb{Z})$ for $q = 0.1.2, 3$, and verify that our conjectured formula holds true under the assumption that the groups $H^q_c(X,\mathbb{Z})$ are zero for $q>3$.

It is not hard to guess possible extensions of the definition given here to arbitary X, once we have defined Weil groups and Weil maps for higher-dimensional fields, both local and global. Kato has made a very plausible suggestion of such a definition, and we hope to return to this question in subsequent papers.

We close the introduction with two remarks:

1) The definition given here would work for any open subscheme of a smooth projective curve over a finite field. Do the cohomology groups thus obtained agree with the ones defined in our earlier paper [L]? This seems highly likely, but we haven't checked it.

2) What is the relation of these conjectures to the celebrated Bloch-Kato conjectures? In general, they are not even about the same objects. The Bloch-Kato conjectures concern

the Hasse-Weil zeta-function of a variety over a number field, and our conjectures concern the scheme zeta-function of a scheme over Spec $\mathbb{Z}$. If the scheme is smooth and proper over Spec $\mathbb{Z}$, then the zeta-function of the scheme is the same as the Hasse-Weil zeta-function of the generic fiber, so then we can ask if the conjectures are compatible. Even this seems far from obvious, although presumably true.

§1. Cohomology of topological groups

Let G be a topological group. We define a Grothendieck topology T_G as follows:

Let the category $Cat(T_G)$ be the category of G-spaces and G-morphisms. A collection of maps $\{\pi_i : X_i \to X\}$ will be called a covering (so an element of $Cov(T_G)$ if it admits local sections: for every $x \in X$ there exists an open neighborhood V of x, an index i and a continuous map $s_i : V \to X_i$ such that $\pi_i s_i = 1$.

We verify easily that $Cat(T_G)$ has fibered products. It is immediate that T_G satisfies the axioms for a Grothendieck topology, and we call T_G the "local-section topology".

Let A be a topological G-module. We define a presheaf of abelian groups $\tilde{A}$ on T_G by putting $\tilde{A}(X) = Map_G(X, A)$ (the set of continuous G-equivariant maps from X to A).

Proposition 1.1. *$\tilde{A}$ is a sheaf.*

Proof. We have to show $\tilde{A}$ verifies the sheaf axiom: Let $\{\pi_i : X_i \to X\}$ be a cover. Let θ_1 and θ_2 be the maps from $\prod Map_G(X_i, A)$ to $\prod Map_G(X_i \times_X X_j, A)$ induced by the two projections, and let ψ be the natural map from $Map_G(X, A)$ to $\prod Map_G(X_i, A)$. We have to check that if f is in $\prod Map_G(X_i, A)$ and $\theta_1(f) = \theta_2(f)$, there is a unique g in $Map_G(X, A)$ such that $f = \psi(g)$.

It is clear that g exists and is unique as a map of sets; we need only show that g is continuous. This follows immediately from the existence of local sections.

Define $C^p(G, A)$ to be $Map_G(G^{p+1}, A)$, where G acts diagonally on G^{p+1}. Let δ_p map $C^p(G, A)$ to $C^{p+1}(G, A)$ by the standard formula

$$\delta_p f(g_0, \dots g_{p+1}) = \sum_0^{p+1} (-1)^i f(g_0, \dots \hat{g}_i, \dots g_{p+1})$$

Then the cohomology $H_c^p(G, A)$ of this complex is the continuous (homogeneous) cochain cohomology of G with values in A.

Remark. By the usual computation, this cohomology is the same as the inhomogeneous continuous cohain complex of G with values in A.

Let $*$ denote a point, with trivial G-action.

Definition 1.2. . *We define the cohomology groups* $H^i(G, A)$ *to be* $H^i(T_G, *, \tilde{A})$.

Proposition 1.3. *Let* $0 \to A \to B \to C \to 0$ *be an exact (as abelian groups) sequence of* G*-maps of topological* G*-modules . Assume that the topology of* A *is induced from that of* B *and that the map from* B *to* C *admits local sections. Then the sequence of sheaves on* T_G *:* $0 \to \tilde{A} \to \tilde{B} \to \tilde{C} \to 0$ *is also exact, and consequently there is a long exact sequence of cohomology*

$$0 \to H^0(G, A) \to H^0(G, B) \to H^0(G, C) \to H^1(G, A) \dots$$

Proof. It is immediate that the sequence of sheaves is left exact. Let X be a G-space and let $f : X \to C$. Then the projection on the first factor makes the fibered product $X \times_C B$ a local section cover of X. Let p_1 and p_2 be the projections from $X \times_C B$ to X and B respectively, and λ the map from B to C. Then $p_1^* f = \lambda_* p_2$, so the map from $\tilde{B}$ to $\tilde{C}$ is surjective.

Proposition 1.4. . *The Čech cohomology groups* $\check{H}^p(*.\tilde{A}) = \check{H}^p(T_G, *, \tilde{A})$ *are functorially isomorphic to* $H_c^p(G, A)$.

Proof. By definition $\check{H}^p(*, \tilde{A})$ is the direct limit of the groups $\check{H}^p(\mathcal{U}, \tilde{A})$, where $\mathcal{U}$ runs through the set of coverings of $*$. It is immediate that the map from G to $*$ is an initial object in the category of covers, so $\check{H}^p(*, \tilde{A}) = \check{H}^p(\{G\}, \tilde{A})$. But, by definition, this is the cohomology of the complex

$$Map_G(G, A) \to Map_G(G \times G, A) \to Map_G(G \times G \times G, A) \ldots$$

which is just the definition of the homogeneous continuous cochain complex.

Corollary 1.5. *Let A be a G-module with trivial G-action. Then the cohomology group $H^1(G, A)$ is naturally isomorphic to $Hom_{cont}(G, A)$.*

Proof. In any Grothendieck topology, $H^1(F) = \check{H}^1(F)$ for any sheaf F. On the other hand the continuous cochain cohomology group $H^1_c(G, A)$ (G acting trivially) is well-known to be the group of continuous homomorphisms from G to A.

Our next goal is to relate the cohomology of G to the Čech cohomology of the underlying topological spaces of G and its products.

Lemma 1.6. *Let G be a topological group and X a topological space. Let $W = G \times X$, and let G act on W by $g(h, w) = (gh, w)$. In the local section topology on W every cover $\{\pi_i : U_i \to W\}$ has a refinement by a cover of the form $\{G \times V_x\}$, where V_x is a topological neighborhood of the point x in X.*

Proof. Let $x \in X$. There exists an open neighborhood V_x of x, an open neighborhood T_x of the identity e of G, an index i, and a section $\lambda_x : T_x \times V_x \to U_i$. Let i_x be the inclusion of V_x in X. Define a map $\rho_x : G \times V_x \to U_i$ by $\rho_x(g, v) = g(\lambda_x(e, v))$. Clearly $\{G \times V_x\}$ is a local section cover of $G \times X$.

We have

$$\pi_i \rho_x(g, v) = \pi_i g \lambda_x(e, v) = g \pi_i \lambda_x(e, v) = g(e, v) = (g, v)$$

This shows that $\pi_i \rho_x = id \times i_x$, and hence that $\{G \times V_x\}$ refines $\{U_i\}$.

Corollary 1.7. *a) Let E be a local-section sheaf on $G \times X$. Define a local-section presheaf $\alpha_* E$ on X by $\alpha_* E(Y) = E(G \times Y)$. Then $\alpha_* E$ is a sheaf for the local-section topology on X and α_* is exact.*

b) $H^q_G(G \times X, E)$ is isomorphic to $H^q(X, \alpha_* E)$.

We observe that we may restrict $\alpha_* F$ to the usual topology on X and obtain the same cohomology, since usual topological covers are cofinal in local-section covers.

Proof. a) Let $\{U_i \to Y\}$ be a local-section cover of Y. Then $\{G \times U_i \to G \times Y\}$ is a local-section G-cover of $G \times Y$ and $G \times (U_i \times_Y U_j)$ is naturally isomorphic to $(G \times U_i) \times_{G \times Y} (G \times U_j)$, so $\alpha_* F$ is a sheaf.

We note that α_* is clearly left exact. Let $E \to F$ be a surjective map of sheaves on $G \times X$. Let $x \in \alpha_* F(Y) = F(G \times Y)$. There exists a local-section cover $\{\pi_i : U_i \to G \times Y\}$ such that $\pi_i^*(x) \in F(U_i)$ lifts to $E(U_i)$. By Lemma 1.6, we may assume that U_i is $G \times V_i$, where the V_i's are an open cover of Y, and $\pi_i = (id, \lambda_i)$, where λ_i is the inclusion of V_i in Y. Clearly $\lambda_i^* x$ comes from $\alpha_* E(V_i)$, so $\alpha_* E \to \alpha_* F$ is surjective, and α_* is exact.

It is immediate that α^{-1}, defined by $\alpha^{-1}(Y) = Y \times G$ is a map of topologies, so we have a Leray spectral sequence for α_*. This spectral seguence degenerates because α_* is exact, yielding the desired isomorphism.

In any Grothendieck topology, we have the presheaf $\mathbb{Z}'$, defined by assigning the group $\mathbb{Z}$ to any object amd the identity to any map. We define $\mathbb{Z}$ to be the sheaf associated with the presheaf $\mathbb{Z}'$. If the topology is T_G, we also have the sheaf $\tilde{\mathbb{Z}}$, which corresponds to the trivial G-module $\mathbb{Z}$ and is characterized by $\tilde{\mathbb{Z}}(X) = Map_G(X, \mathbb{Z})$. We can define a map from the presheaf $\mathbb{Z}$ to $\tilde{\mathbb{Z}}$ by sending n to the map with the constant value n. This is clearly injective, and induces an injective map from $\mathbb{Z}$ to $\tilde{\mathbb{Z}}$.

This map is also surjective. Let $f : X \to \mathbb{Z}$, and let $X_n = f^{-1}(n)$. The X_n's form a disjoint open cover of X, and the Cech cohomology H^0 of the presheaf $\mathbb{Z}'$ with respect to this cover contains an element g which is n on X_n. Then g determines an element of $\mathbb{Z}$ which maps onto f.

§2. An alternative definition

Let G be a topological group. We construct a simplicial G-space S_n as follows:

Let $S_n = G^{n+1}$, and let G act on S_n by $g(g_0, g_1 \dots g_n) = (gg_0, g_1 \dots, g_n)$.

Now define face maps $\rho_i : S_n \to S_{n-1}$ by: $\rho_i(g_0, \dots g_n) = (g_0, \dots, g_i g_{i+1}, \dots g_n)$ for $0 \le i < n$, and $\rho_n(g_0, \dots g_n) = (g_0, \dots g_{n-1})$.

The maps ρ_i are maps of G-spaces, and a straightforward verification shows that $\rho_i \rho_j = \rho_{j-1}\rho_i$ if $i < j$.

We will not use the degeneracy maps, so we omit the definition.

Now let $\tilde{S}_n = G^{n+1}$, but with G acting diagonally on $\tilde{S}_n : g(g_0 \dots g_n) = (gg_0 \dots gg_n)$. Let $\pi_i : \tilde{S}_n \to \tilde{S}_{n-1}$ by $\pi_i(g_0 \dots g_n) = (g_0 \dots \hat{g}_i \dots g_n)$. Computation shows a) π_i is a G-map, and b) $\pi_i \pi_j = \pi_{j-1}\pi_i$ if $i < j$.

Let $\phi : \tilde{S}_n \to S_n$ by $\phi(g_0 \dots g_n) = (g_0, g_0^{-1} g_1, g_1^{-1} g_2 \dots g_{n-1}^{-1} g_n)$.

We verify that ϕ is a G-map and that $\rho_i \phi = \phi \pi_i$.

Let F be a local-section sheaf on the site T_G. Let F_n be the sheaf on G^n (as topological space) defined by $F_n(U) = F(G \times U)$, where G acts on $G \times U$ by acting by left translation on G and trivially on U.

We define maps $\tilde{\rho}_i : G^n \to G^{n-1}$ by

$$\tilde{\rho}_i(g_1 \dots g_n) = (g_1 \dots g_i g_{i+1} \dots g_n)(1 \le i \le n)$$

$$\tilde{\rho}_0(g_1 \dots g_n) = (g_2 \dots g_n)$$

$$\tilde{\rho}_n(g_1 \dots g_n) = (g_1 \dots g_{n-1})$$

Let p_n be the natural projection from $S_n = (G \times G^n)$ to G^n. We check that $p_{n-1}\rho_i = \bar{\rho}_i p_n$, and so automatically $\bar{\rho}_i \bar{\rho}_j = \bar{\rho}_{j-1}\bar{\rho}_i$.

Now take the (second) canonical flabby resolution $T_{j,n}$ of F_n on G^n. [Some words are in order. If X is a topological space and F is a sheaf on X the usual canonical flabby resolution is obtained by defining $C^0(F)$ to be $\prod_{x \in X} (i_x)_* F_x$, embedding F in $C^0(F)$, taking the quotient G, embedding G in $C^0(G)$, and continuing this process to obtain a flabby resolution $0 \to F \to C^0(F) \to C^0(G) \to \dots$. On the other hand, the second canonical flabby resolution looks like (see [Go] §6.4 for details) $0 \to F \to C^0(F) \to C^0(C^0(F)) \to \dots,$ after defining suitable coboundary maps. We have to use this construction to compare our definition with that of Wigner, who when he says "canonical flabby resolution" means this one.] By construction we have for each i a map from F_{n-1} to $(\bar{\rho}_i)_* F_n$, which is easily seen to induce inductively a map from $T_{j,n}$ to $(\bar{\rho}_i)_* T_{j,n}$, and hence a map from $\Gamma(G^{n-1}, T_{j,n-1})$ to $\Gamma(G^n, T_{j,n})$. By taking the alternating sum of these maps we get a map $\delta_{j,n} : \Gamma(G^{n-1}, T_{j,n-1}) \to \Gamma(G^n, T_{j,n})$, and thus a double complex. We define $\tilde{H}^*(G, F)$ to be the hypercohomology of this double complex.

Proposition 2.1. . *The cohomology groups $H^i(T_G, *, F)$ are functorially isomorphic to the cohomology groups $\tilde{H}^i(G, F)$.*

Proof. We need only check that $H^0 = \tilde{H}^0$, that the $\tilde{H}^i$'s form a cohomological functor, and that the $\tilde{H}^i$'s vanish on injectives for $i > 0$.

Since the canonical flabby resolution takes short exact sequences of sheaves into short exact sequences of complexes, the $\tilde{H}^i$'s form a cohomological functor. (Recall that corollary 1.7 implies that an exact sequence of sheaves on T_G gives rise to an exact sequence of sheaves on G^n for every n).

If $F = I$ is injective, I restricts to an injective sheaf J_n on $G \times G_n$ (If f is the map from $G \times G_n$ to a point, f^* takes injectives to injectives, since it has the exact left adjoint $f_!$). We know that $I_n = \alpha_* J_n$ is flabby, hence acyclic, and so the homology of the flabby resolution of I_n reduces to $H^0(G^n, I_n)$ and the spectral sequence of a double complex shows that our hypercohomology is the cohomology of the complex $H^0(G_n, I_n) = I_n(G_n) = I(G \times G_n)$. The equality $\rho_i \phi = \phi \pi_i$ shows that the homology of $I(G \times G_n)$ is the same as the homology of $I(G^{n+1})$ with diagonal action, which is the Čech cohomology $\check{H}^i(G, I)$. Since Čech cohomology vanishes for injectives for $i > 0$ so does $\tilde{H}^i(G, F)$.

Finally, it follows again from the formula $\rho_i \phi = \phi \pi_i$ that if F is any presheaf on the category of G -spaces. the cohomology of the complex $F(S_n)$ is naturally isomorphic to the cohomology of the complex $\alpha_* F(G^n) = F(\tilde{S}_n)$, where the coboundary maps are the alternating sums of the maps induced by p_i and π_i respectively. It follows that the cohomology $\tilde{H}^0(G,F)$ is naturally isomorphic to the Čech cohomology $\check{H}^0(G,F)$ which in turn is $H^0(T_G,*,F)$.

Remark 2.2 We observe that if F is a sheaf of the form $\tilde{A}$, then our cohomology groups are exactly the cohomology groups denoted by $\hat{H}^*(G,A)$ by David Wigner ([W], p.91). We then obtain as a corollary of Wigner's Theorem 2 ([W], p.91) that if G is locally compact, σ-compact, finite dimensional, and A is separable and has Wigner's "property F", then our $H^*(G,A)$ are naturally isomorphic to the groups $H^*(G,A)$ defined by Wigner in [W], (which we will call $H^*_{Wig}(G,A)$). We further point out that under the same conditions Wigner's groups are naturally isomorphic to the groups (which we wil call $H^*_M(G,A)$) defined by Calvin Moore in [M] and used by C.S. Rajan in [R]. (Wigner's Theorem 2 does not explicitly require separability, but his proof that certain categories of modules are quasi-abelian is not valid without it.) In order to apply this result, we recall that Proposition 3 of [W] tells us that any locally connected complete metric topological group (for instance, $\mathbb{Z}$, S^1, or $\mathbb{R}$) has property F.

Theorem 2.3. *There is a spectral sequence* $E_1^{p,q} = H^q_{top}(G^p, \alpha_* F) \Rightarrow H^{p+q}(T_G,*,F)$.

Proof. This is just the spectral sequence of the double complex defining $\tilde{H}^*(G,F)$.

Corollary 2.4. *Let G be a) a profinite group, or b) the Weil group of a global function field , and A a topological G-module. Then the cohomology groups $H^i(G,A)$ are canonically isomorphic to the usual groups $H^i(G,A)$ given by complexes of continuous cochains.*

Proof. We show first that the cohomological dimension of a profinite space X is zero. To do this it suffices to show (by using alternating cochains)) that every open cover has a refinement by a disjoint cover. It is immediate that X has a base for its topology consisting of sets U_i which are both open and closed. By compactness, any cover has a refinement $\{U_1, \dots U_n\}$ consisting of finitely many such U_i . Let $C(U) = X - U$. Then $\{U_1, U_2 \cap C(U_1), U_3 \cap C(U_2) \cap C(U_3) \dots\}$ is a further refinement which is disjoint.

In case a) each G^q is profinite, so has cohomological dimension zero, and in case b) the Weil group G is the topological product of a profinite group and a discrete group, so G^q is the disjoint union of open profinite spaces, so again has cohomological dimension zero. So in each case the spectral sequence degenerates, to yield that $H^*(G, F)$ is the cohomology of the complex $F(G \times G^p)$. We have seen that this is the same as the cohomology of the complex $F(G^{p+1})$, with G acting diagonally, which is just the homogeneous continuous cochain complex of the G-module A if $F = \tilde{A}$.

Lemma 2.5. *Let X be the product of a compact space and a metrizable space, and let E be a sheaf of modules over the sheaf of continuous real-valued functions on X. Then $H^q(X^p, E) = 0$ for all $p, q > 0$.*

Proof. The hypothesis implies that for any p, X^p is again the product of a compact space and a metrizable space, and so paracompact. We recall from ([Go], p.157) that any sheaf of modules over the sheaf of continuous real-valued functions on a paracompact space is fine, so "mou", so acyclic.

Corollary 2.6. *Let G be a topological group which is, as topological space, the product of a compact space and a metrizable space (e.g. the Weil group of a global or local field) and let $\mathbb{R}$ denote the real numbers with trivial G-action. Then the cohomology groups $H^p(G, \tilde{\mathbb{R}})$ are given by the cohomology of the complex of homogeneous continuous cochains from G to $\mathbb{R}$.*

Proof. Let $F = \mathbb{R}$. We first observe that $\alpha_*(F)(U) = F(G \times U) = Map_G(G \times U, \mathbb{R})$, which is naturally isomorphic to $Map(U, \mathbb{R}) = \tilde{\mathbb{R}}(U)$. Then Lemma 2.5 implies that the spectral sequence of Theorem 2.3 degenerates, so that the cohomology $H^p(G, \tilde{\mathbb{R}})$ is given by the cohomology of the complex $H^0(G^p, \alpha_*\mathbb{R}) = Map_G(G \times G^p, \mathbb{R}) = Map_G(S_p, \mathbb{R})$. As above, this is the same as the cohomology of the homogeneous cochain complex $Map_G(\tilde{S}_p, \mathbb{R})$.

§3. Cohomology of the Weil group.

Let F be a number field (resp. a local field), $\bar{F}$ an algebraic closure of F, and G_F the galois group of $\bar{F}$ over F. Let K be a finite Galois extension of F. and let C_K denote the idèle class group of K (resp. K^*).

Now fix a Weil group W_F associated with the topological class formation $Lim(C_K)$, where the limit is taken over fields K finite and Galois over F. We recall that W_F is equipped with a continuous homomorphism $g : W_F \to G_F$. If K is such a field, let $W_K = g^{-1}(G_K)$, and let W_K^c be the closure of the commutator subgroup of W_K in W_F. Then it is shown in [Artin-Tate] that W_F/W_K^c is a Weil group for the pair $(G(K/F), C_K)$ (resp. $(G(K/F), K^*)$). So having fixed a Weil group W_F, we have canonical maps from it to $W_{K/F} = W_F/W_K^c$. The standard construction of the Weil group W_F (See [A-T]) shows that W_F is the projective limit of the groups $W_{K/F}$.

Now let F be global and S be a finite set of valuations of F including the archimedean valuations and all valuations which ramify in K, but not including the trivial valuation. Let $U_{K,S}$ be the subgroup of the idèle group I_K consisting of those idèles which are 1 at valuations lying over S, and units at valuations not lying over S. It is well known (see [N]. p. 393) that $U_{K,S}$ is a cohomologically trivial $G(K/F)$-module. The natural map from $U_{K,S}$ to the idèle class group C_K is obviously injective and we identify $U_{K,S}$ with its image. Let the S-idèle class group $C_{K,S}$ be defined by $C_{K,S} = C_K/U_{K,S}$. Then the natural maps from the Tate cohomology groups $\hat{H}^i(G(K/F), C_K)$ to $\hat{H}^i(G(K/F), C_{K,S})$ are isomorphisms for all i.

Let α be the fundamental classs in $\hat{H}^2(G(K/F), C_K)$ and β its image in $\hat{H}^2(G(K/F), C_{K,S})$. It follows immediately from the fact that C_K is a class formation that. for all i. cup-product with β induces an isomorphism between $\hat{H}^i(G(K/F), \mathbb{Z})$ and $\hat{H}^{i+2}(G(K/F), C_{K,S})$. We then define the S-Weil group $W_{K/F,S}$ to be the extension of $G(K/F)$ by $C_{K,S}$ determined by β, There is clearly a natural surjection p_S from $W_{K/F}$ to $W_{K/F,S}$, and it follows from the arguments in [A-T] (p.238) that there is a natural isomorphism from $W^{ab}_{K/F,S}$ to $C_{F,S}$.

Let $N_{K,S}$ be the kernel of the natural map from W_F to $W_{K/F,S}$. Let A be a topological W_F-module, and let $A_{K,S}$ be the topological $W_{K/F,S}$ module consisting of the invariant elements $A^{N_{K,S}} \subseteq A$. Assume that $A = \bigcup A_{K,S}$.

Lemma 3.1. *The Weil group W_F is the projective limit over K and S of the groups $W_{K/F,S}$.*

Proof. It suffices to show that the relative Weil group $W_{K/F}$ is the projective limit over S of the groups $W_{K/F,S}$. The maps p_S induce a map p from $W_{K/F}$ to the projective limit.

Let $W^1_{K/F}$ (resp.$W^1_{K/F,S}$) be the kernel of the absolute value map of $W_{K/F}$ (resp.$W_{K/F,S}$) to $\mathbb{R}^*$. Since $W^1_{K/F}$ is compact and the maps p_S are surjective, p is surjective as a map from $W^1_{K/F}$ to the projective limit of $W^1_{K/F,S}$, and hence p is surjective. The proof that p is injective immediately reduces to showing that the map from C_K to the projective limit of the $C_{K/F,S}$ is injective, which in turn follows from the corresponding fact for the idèle groups.

Definition 3.2. . *We define the cohomology group $H^q(W_F, A)$ to be the direct limit of the cohomology groups $H^q(W_{K/F,S}, A_{K,S})$.*

The cohomology groups of $W_{K/F,S}$-modules are the ones defined in Section 1. We observe that $W_{K/F,S}$ is locally compact, σ-compact and finite-dimensional, so Wigner's comparison theorem applies and the cohomology of these groups with coefficients in $\mathbb{Z}$, $\mathbb{R}$ or S^1 are the same as Wigner's cohomology groups and therefore also Moore's cohomology groups.

We now compute the cohomology groups $H^q(W_F, \mathbb{Z})$: Evidently $H^0(W_F, \mathbb{Z}) = \mathbb{Z}$. Since $H^1(W_{K/F,S}, \mathbb{Z}) = Hom_{cont}(W_{K/F,S}, \mathbb{Z}) = 0$, (because $W_{K/F,S}$ is an extension of a compact group by a connected group), we have $H^1(W_F, \mathbb{Z}) = 0$.

We have the following result of Moore ([M2], Theorem 9, p.29), as quoted by Rajan ([R], Proposition 5):

Lemma 3.3. *Let G be a locally compact group. Let N be a closed normal subgroup of G and let A be a locally compact, complete metrizable topological G-module. Then there is a spectral sequence*

$$E_2^{p,q} \Rightarrow H_M^{p+q}(G, A)$$

where $E_2^{p.q} = H^p_M(G/N, H^q_M(N, A))$ if $q = 0, q = 1$, or $H^q_M(N, A) = 0$

Lemma 3.4. *The cohomology groups $H^q(W_{K/F,S}, \mathbb{R})$ are: $\mathbb{R}$ if $q = 0$, $Hom_{cont}(\mathbb{R}, \mathbb{R}) \simeq \mathbb{R}$ if $q = 1$, and 0 if $q > 1$.*

Proof. We know by Corollary 2.5 that these cohomology groups are given by the continuous cochain cohomology. It is well-known (See [B-W]) that if G is compact then

$H^q(G,\mathbb{R})=0$ for $q>0$, and $H^0(\mathbb{R},\mathbb{R})=\mathbb{R}$, $H^1(\mathbb{R},\mathbb{R})=Hom_{cont}(\mathbb{R},\mathbb{R})$, and $H^q(\mathbb{R}.\mathbb{R})=0$ for $q>1$. The result then follows from the fact that we have the exact sequence:

$$1\to W^1_{K/F,S}\to W_{K/F,S}\to\mathbb{R}\to 1$$

with $W^1_{K/F,S}$ compact. and applying Lemma 3.3.

Lemma 3.5. $H^q_M(\mathbb{R},\mathbb{Z})=0$ *for* $q>0$.

Proof. $H^q_M(\mathbb{R},\mathbb{Z})=H^q_{Wig}(\mathbb{R},\mathbb{Z})=$ (by Theorem 4 of [W]) $H^q(B_{\mathbb{R}},\mathbb{Z})=0$ because $\mathbb{R}$ is contractible.

We see from Lemma 3.4 and the exact sequence $0\to\mathbb{Z}\to\mathbb{R}\to S^1\to 0$ that we have $0\to Hom_{cont}(\mathbb{R},\mathbb{R})\to H^1(W_{K/F,S},S^1)\to H^2(W_{K/F,S},\mathbb{Z})\to 0$. Since the abelianized Weil group $(W_{K/F,S})^{ab}$ is naturally isomorphic to $C_{F,S}$, $H^1(W_{K/F,S},S^1)$ is the Pontriagin dual $C^D_{F,S}$ of $C_{F,S}$, which yields that $H^2(W_{K/F,S},\mathbb{Z})$ is the Pontriagin dual $(C^1_{F,S})^D$ of the idéle class group of norm one. By taking limits over K and S we obtain that $H^2(W_F,\mathbb{Z})=(C^1_F)^D=0$.

We next wish to show that $H^3(W_F,\mathbb{Z})=0$, and to do this it is, by Lemma 3.3, enough to show that $H^2(W_F,S^1)=0$. We first observe that Rajan's proof in [R] that the Moore cohomology groups $H^2_M(W_F,S^1)=0$ works equally well to show that $H^2_M(W^1_F,S^1)=0$. Since for Moore cohomology, the cohomology of the projective limit of compact groups is the direct limit of the cohomologies, ([M] or [R]) we have that $0=H^2_M(W^1_F,S^1)=\underrightarrow{Lim}H^2_M(W^1_{K/F,S},S^1)=$ (by Remark 2.2) $\underrightarrow{Lim}H^2(W^1_{K/F,S},S^1)=$ (by Lemma 3.4) $\underrightarrow{Lim}H^3(W^1_{K/F,S},\mathbb{Z})$.

It is easy to see that the Weil group $W_{K/F,S}$ is the direct product (in both the algebraic and topological senses) of $W^1_{K/F,S}$ and $\mathbb{R}$. Applying the Hochschild-Serre spectral sequence (Lemma 3.3) coming from the exact sequence $1\to\mathbb{R}\to W_{K/F,S}\to W^1_{K/F,S}\to 1$, and using Lemma 3.5, we conclude that $H^q(W_{K/F,S},\mathbb{Z})=H^q(W^1_{K/F,S},\mathbb{Z})$. So $H^3(W_F,\mathbb{Z})=$ (by definition) $\underrightarrow{Lim}H^3(W_{K/F,S},\mathbb{Z})=\underrightarrow{Lim}H^3(W^1_{K/F,S},\mathbb{Z})=0$. We sum up what we have shown in the following theorem:

Theorem 3.6. *The cohomology groups $H^q(W_F,\mathbb{Z})$ are given by: $H^0(W_F,\mathbb{Z}) = \mathbb{Z}$, $H^1(W_F,\mathbb{Z}) = 0$, $H^2(W_F,\mathbb{Z}) = (C^1_F)^D$ (the Pontriagin dual of C^1_F) and $H^3(W_F,\mathbb{Z}) = 0$.*

Unfortunately so far we have not succeeded in computing the cohomology groups $H^q(W_F,\mathbb{Z})$ for $q > 3$.

Lemma 3.7. *If v is not in S, the natural map induced by θ_v from W_v to $W_{K/F,S}$ annihilates the kernel I_v of the natural map from W_{F_v} to $\mathbb{Z}$.*

Proof. Let w be the valuation lying over v determined by θ_v. The following diagram is commutative:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K_w^* & \longrightarrow & W_{K_w/F_v} & \longrightarrow & G(K_w/F_v) & \longrightarrow & 1 \\ & & \Big\downarrow{\scriptstyle w} & & \Big\downarrow{\scriptstyle \pi_v} & & \Big\downarrow{\scriptstyle i} & & \\ 1 & \longrightarrow & \mathbb{Z} & \xrightarrow{f} & \mathbb{Z} & \longrightarrow & G(\kappa(w)/\kappa(v)) & \longrightarrow & 1 \end{array}$$

because fundamental classes of the two extensions correspond. (Here i is the natural isomorphism and f is the residue field degree). It follows that the image of I_v in W_{K_w/F_v} is isomorphic to the unit group $Ker(w)$, which goes to zero in $C_{K/F,S}$ and so a fortiori in $W_{K/F,S}$.

§4. The global Weil-étale topology

Let F be a global field choose an algebraic closure $\bar{F}$ of F. Let $G_F = G(\bar{F}/F)$ be the Galois group of $\bar{F}/F$.

Let v be a valuation of F, and F_v the completion of F at v. Choose an algebraic closure $\bar{F}_v$ of F_v, and an embedding of $\bar{F}$ in $\bar{F}_v$. Choose a global Weil group W_F and a local Weil group W_{F_v}. For each finite extension E of F in $\bar{F}$, let $E_v = EF_v$ be the induced completion of E. Let w be a valuation of $\bar{F}$ lying over v, and let i^*_w be the natural inclusion of G_{F_v} in G_F whose image is the decomposition group of w.

Definition 4.1. *A Weil map θ_v is a continuous homomorphism from W_{F_v} to W_F such that there exists a valuation w of $\bar{F}$ such that the following diagrams are commutative for all finite extension fields E of F:*

$$\begin{array}{ccc} W_{F_v} & \longrightarrow & G_{F_v} \\ {\scriptstyle \theta_v}\downarrow & & \downarrow{\scriptstyle i_w^*} \\ W_F & \longrightarrow & G_F \end{array} \qquad \begin{array}{ccc} E_v^* & \longrightarrow & W_{E_v}^{ab} \\ {\scriptstyle n_v}\downarrow & & \downarrow \\ C_E & \longrightarrow & W_E^{ab} \end{array}$$

where n_v maps $a \in E^*$ to the class of the idéle whose v-component is a and whose other components are 1, and the map from $W_{E_v}^{ab}$ to W_E^{ab} is induced by θ_v.

It is an easy consequence of [T] that Weil maps always exist, and are unique up to an inner automorphism of W_F.

The local Weil group $W_v = W_{F_v}$ maps to ${W_v}^{ab} = F_v^*$, which in turn maps to $\mathbb{Z}$ by the valuation map v. Let I_v be the kernel of the composite map from W_v to $\mathbb{Z}$.

We choose once and for all a set of Weil maps $\theta_v : W_v \to W_{\mathbb{Q}}$ for all valuations v of $\mathbb{Q}$. If w is any valuation of a number field F, the inclusion of W_w in W_v and θ_v induce a Weil map $\theta_w : W_w \to W_F$.

Let $\bar{Y} = \bar{Y}_F$ be the set of all valuations of F. We require the trivial valuation v_0 to be in $\bar{Y}$, corresponding to the generic point of Spec O_F, where O_F is the ring of integers of F¿. Let $W_{\kappa(v)}$ be $\mathbb{Z}$ if v is non-archimedean, $\mathbb{R}$ if v is archimedean. and W_F if $v = v_0$. We say that v is a specialization of w if w is v_0 and v is not. In each case there is a natural map π_v from the local Weil group W_v to $W_{\kappa(v)}$, and we let I_v be its kernel. It is an easy exercise to verify that if K_w is a finite Galois extension of F_v, then the map π_v factors through W_{K_w/F_v}.

Let K be a finite Galois extension of F. Let S be a finite set of non-trivial valuations of F, containing all the valuations of F which ramify in K. We now define a Grothendieck topology $T_{K,S,\bar{Y}}$:

We first define a category Cat $T_{K,S,\bar{Y}}$ The objects of Cat $T_{K,S,\bar{Y}}$ are collections $((X_v),(f_v))$, where v runs through all points of $\bar{Y}$, X_v is a $W_{\kappa(v)}$-space, and if v is a specialization of w, $f_v : X_v \to X_w$ is a map of W_v-spaces. (We regard X_v as a W_v-space via π_v, and X_w as a W_v-space via the Weil map θ_v). If $v = v_0$, we require that the action of W_F on X_v factor through $W_{K/F,S}$.

A morphism g from $\mathcal{X} = ((X_v), (f_v))$ to $\mathcal{X}' = ((X'_v), (f'_v))$ is a collection of W_v-maps $g_v : X_v \to X'_v$ such that $g_{v_0} f_v = f'_v g_v$.

We say that g is a local section morphism if the maps g_v from X_v to $g_v(X_v)$ admit local sections.

The fibered product of $((X_{1,v}), (g_{1,v}))$ and $((X_{2,v}), (g_{2,v}))$ over $((X_{3,v}), (g_{3,v}))$ is given by $((X_{1,v} \times_{X_{3,v}} X_{2,v}), ((g_{1,v} \times g_{2,v}))$.

We define the coverings Cov $(T_{K,S,\bar{Y}})$ by:

A family of morphisms in our category $\{((X_{i,v}), (f_{i,v})) \to ((X_v), (f_v))\}$ is a cover if $\{X_{i,v} \to X_v\}$ is a local section cover for all v.

Our category clearly has a final object $*_{(K,S)}$ whose components are the one-point space for each v in $\bar{Y}$.

If E is a sheaf for our topology, we define $H^i(\bar{Y}_{K,S}, E)$ to be $H^i(T_{K,S,\bar{Y}}, *_{(K,S)}, E)$, and $H^i(\bar{Y}, E)$ to be the direct limit over K and S of the $H^i(\bar{Y}_{K,S}, E)$.

We define a morphism of topologies i_v^{-1} from $T_{K,S}$ to $T_{W_{\kappa(v)}}$ by $i_v^{-1}((X_v), (f_v)) = X_v$. We have the corresponding direct image maps $(i_v)_*$ from sheaves on $T_{W_{\kappa(v)}}$ to sheaves on $T_{K,S}$ by $(i_w)_*(E)((X_v), (f_v)) = E(X_w)$. For psychological reasons we define j to be i_{v_0}. It is clear that i_v^{-1} preserves covers and fibered products, and so is a morphism of topologies.

Definition 4.2. *Let $\theta : H \to G$ be a morphism of topological groups, and X an H-space. Define $X \times^H G$ to be the quotient (with the quotient topology) of $X \times G$ by the equivalence relation $(x, g) \sim (x', g')$ iff there exists a $\tau \in H$ such that $x' = \tau x$ and $g' = g\tau^{-1}$.*

Remark. The functor which takes an H-space X to the G-space $X \times^H G$ is easily seen to be left adjoint to the forgetful functor from G-spaces to H-spaces, regarding a G-space as an H-space via θ.

Lemma 4.3. *Let G be a topological group and let I be a closed subgroup such that the projection ρ from G to G/I admits local sections. Then the category of G-spaces with maps to G/I is equivalent to the category of I-spaces and the covers in the respective categories correspond.*

Proof. If X is an I-space let $\alpha(X) = (X \times^I G, \lambda)$, where $\lambda : X \times^I G \to G/I$ is given by $\lambda(x, \sigma) =$ the coset σI.

If Z is a G-space with a map $\pi : Z \to G/I$, let $\beta(Z, \pi) = \pi^{-1}(I)$. It is straightforward to verify that α and β are inverse functors.

We now claim that the covers correspond.

Lemma 4.4. *If $\rho : G \to G/I$ admits local sections and the cover $\{X_i \to X\}$ admits local sections, then the cover $\{X_i \times^I G \to X \times^I G\}$ admits local sections.*

Proof. Let $y = [x, \sigma]$ be the class of (x, σ) in $X \times^I G$. Let U be a neighborhood of $\rho(\sigma)$ such that there exists a continuous section $s : U \to G$ of ρ. Let $V = \rho^{-1}(U)$. Let $U^* = s(U)$. We claim that $X \times^I V$ is functorially isomorphic to $X \times U^*$. It is immediate that given $[x, v]$ in $X \times^I V$, there exists a unique pair (x', v') in $X \times U^*$ such that $[x, v] = [x', v']$. In fact $(x', v') = ((s\rho(v))^{-1}vx, s\rho(v))$.

So if $\{X_i \to X\}$ admits local sections so does $\{X_i \times U^* \to X \times U^*\}$, and then so does $\{X_i \times^I V \to X \times^I V\}$, and therefore also $\{X_i \times^I G \to X \times^I G\}$.

Lemma 4.5. . *If I is a locally compact subgroup of a Hausdorff topological group G, the natural projection from G to G/I is a fibration, and hence admits local sections.*

Proof. This is [W], Proposition 2, p.88.

So we have proved

Theorem 4.6. . *Let G be a Hausdorff topological group, I a locally compact subgroup, and A a continuous G-module. Then $H^i(T_G, G/I, \tilde{A})$ is naturally isomorphic to $H^i(I, A)$.*

Theorem 4.7. *Let $j = j_{\bar{Y}}$, and let A be a topological W_F-module. There exists a spectral sequence*

$$E_2^{p \cdot q} = H^p(\bar{Y}, R^q j_* \tilde{A}) \Rightarrow H^{p+q}(W_F, A)$$

Proof. This follows from [A](p. 44) by applying his Theorem 4.11 to $j = j_{K,S}$ and taking direct limits over K and S.

The rest of this section will be devoted to computing the sheaves $R^q j_* \tilde{A}$. Let v be in $\bar{Y}$ Our goal is to prove:

Theorem 4.8. . *Let $q > 0$, and let $B = B_q$ be $R^q(j_{K,S})_*\tilde{A}$. Then the natural map from B to $\coprod_{v\in S} i_{v*}i_v^*B$ given by adjointness is an isomorphism of sheaves.*

We begin with:

Lemma 4.9. . *Let E be a Weil-étale sheaf on $T_{K.S.\bar{Y}}$. Then $i_v^*E = 0$ for all $v \in \bar{Y}$ implies that $E = 0$.*

Proof. We know that i_v^*E is the sheafification of the presheaf inverse image i_v^pE. If X_v is a $W_{\kappa(v)}$-space, $i_v^pE(X)$ is the direct limit of $E(U)$, where $U = ((X_v'), (f_v))$ is an object of Cat (T_K) such that there is a map from X_v to $i_v^{-1}(U) = X_v'$. Since there exists a U (for example the object which has X_v at v, $X_v \times^{W_v} W_{K/F,S}$ at the generic point, and the empty set elsewhere) with $i_v^{-1}(U) = X_v$ we may always assume that $i_v^{-1}(U) = X_v$, i.e. that $X_v' = X_v$.

More generally, if $h_v : Z_v \to X_v$ is a map of $W_{\kappa(v)}$-spaces, and $f_v : X_v \to X_{v_0}$ is a map of W_v-spaces, then the map $h_v' : X_{v_0}' = Z_v \times^{W_v} W_{K/F,S} \to X_{v_0}$ given by $h_v'(z, w) = wf_vh_v(z)$ is well-defined and so we get a map of $(X_{v_0}', Z_v, \phi, ...\phi)$ to $(X_{v_0}, X_v, (X_w))$ which induces the original map h_v.

If $U = (X_v)$ ia an object of Cat $T_{K,S,\bar{Y}}$, and $\alpha \in E(U)$, there is a covering $\{X_{v_i}\}$ of X_v such that α goes to zero in each $i^pE(X_{v_i}) = E(U_i)$, where the v-component of $U_i = X_{v,i}$. By the argument in the preceding paragraph, we can induce these coverings from families of maps to U, and the collection of all these families will be a covering of U in which α goes to zero, thus making $\alpha = 0$.

Lemma 4.10. *a)i_v^* is exact. b) $i_v^*i_{v*}i_v^*E$ is canonically isomorphic to i_v^*E. c) $i_w^*i_{v*} = 0$ if $v \neq w$. d) i_{v*} is exact.*

Proof. a) Since i_v^* is a left adjoint, it is right exact. Suppose that the sheaf E injects into the sheaf E', and that $\alpha \in i_v^*E(X_v)$ goes to zero in $i_v^*E'(X_v)$. There exists a Weil-étale cover $(X_{v,i})$ of X_v and objects $\mathcal{X}_i$ of Cat $T_{K,S,\bar{Y}}$ such that for each i, α restricted to $X_{v,i}$ comes from an element β_i in $E(\mathcal{X}_i)$, $(\mathcal{X}_i)_v = X_{v,i}$, and the image of β_i in $E'(\mathcal{X}_i)$ is equal to zero. Hence $\beta_i = 0$ and since α goes to zero in a Weil-étale cover, we have $\alpha = 0$.

b) This is a formal consequence of the fact that i_v^* is left adjoint to i_{v*}.

c) $(i_w^p F)(X_w)$ is the direct limit of $F(U = (X_{w_0}, X_w, \phi, ...\phi))$ where $X_w \to X_{w_0}$. If $F = (i_v)_* E$ and $v \neq w$ then $F(U) = E(\phi) = 0$.

d) This follows immediately from the fact that v is a specialization of v_0, and if $\mathcal{X} = (X_{v_0}, X_v, (X_w(w \neq v, v_0))$, then any covering $X_{v,i}$ of $X_v = i_v^{-1}(\mathcal{X})$ comes from the covering $\mathcal{X}_i = (X_{v_0}, X_{v_i}, (X_w))$ of $\mathcal{X}$.

Lemma 4.11. *Let E now be the sheaf $R^q(j_{K,S})_*\tilde{A}$, with $q > 0$. If v is not in S, then $i_v^* E = 0$.*

Proof. Given a $W_{\kappa(v)}$ - space X_v and an element α in $i_v^p(E)(X_v)$, we wil produce a cover $\{X_{v,i}\}$ of X_v such that the restriction of α vanishes on each $X_{v,i}$. By Lemma 3.7, if v is not in S, the Weil map θ_v from W_v to $W_{K/F,S}$ factors through $W_{\kappa(v)}$. So let us define X_{v_0} to be $X_v \times^{W_{\kappa(v)}} W_{K/F,S}$. By using the definition of i_v^p, , $i_v^p(E)(X_v)$ is easily seen to be $E(X_{v_0}, X_v, \ldots, \phi, \ldots)$, where all the spaces X_w for $w \neq v, v_0$ are empty. By passing to a cover, we may assume that α comes from an element β is $H^q(X_{v_0}, \tilde{A})$. Since $q > 0$ and higher cohomology dies in a cover, we may choose a cover $X_{v_0,i}$ of X_{v_0} such that β goes to zero in $H^q(X_{v_0,i}, \tilde{A})$. Letting $X_{v,i} = X_{v_0} \times_{X_{v_0}} X_v$, we see that α goes to zero on each $X_{v,i}$.

Proof of Theorem 4.8: By lemma 4.11, $\prod_{v \in \bar{Y}} (i_v)_* i_v^* B$ is equal to $\prod_{v \in S} (i_v)_* i_v^* B$. By Lemma 4.10, the map from B to $\prod_{v \in S} (i_v)_* i_v^* B$ induces an isomorphism on stalks, and hence is an isomorphism by Lemma 4.9.

Let $j = j_{K/F,S}$. By Theorem 4.8, Lemma 4.10b, and the fact that cohomology commutes with direct products, we obtain

Corollary 4.12. *If $q > 0$, $H^p(\bar{Y}, R^q j_* \tilde{A}) = \coprod H^p(W_{\kappa(v)}, i_v^* R^q j_* \tilde{A})$, where the sum is taken over all $v \in S$.*

The next section will be devoted to computing these cohomology groups for small values of p and q.

Proposition 4.13. *The natural map ϕ from the sheaf $\mathbb{Z}$ on $T_{K,S}$ to the sheaf $j_* j^* \mathbb{Z}$ is an isomorphism.*

Proof. It is clear that ϕ is injective. Let $\mathcal{X} = (X_v)$ be an object of $T_{K,S}$. Let f be in $j_*j^*\mathbb{Z}(\mathcal{X}) = j_*\tilde{\mathbb{Z}}(\mathcal{X}) = Map(X_{v_0}, \mathbb{Z})$ (see the remarks at the end of §1). Let $\{X_n\}$ be the disjoint open cover of X_{v_0} defined by $X_n = f^{-1}(n)$. Let $X_{n,v} = f_v^{-1}(X_n)$. Then the collection $(X_{n,v})$ is a disjoint cover of $\mathcal{X}$, and the element g which takes each $(X_{n,v)}$ to n lives in the Čech cohomology of $\mathbb{Z}$ with respect to this cover of $\mathcal{X}$, and its image in $\mathbb{Z}(\mathcal{X})$ maps to f.

§5. The computation of $H^p(\bar{Y}, R^q j_*\mathbb{Z})$ and $H^p(\bar{Y}, R^q j_*\mathbb{R})$

Lemma 5.1. *. Let G be a discrete group, and let E be a sheaf on T_G. Then the canonical map from $\widetilde{E(G)}$ to E induces an isomorphism of cohomology.*

.

Proof. Since G is discrete, any covering of a discrete G-space X by G-spaces X_i has a refinement consisting of the X_i's with the discrete topology. It then follows by a standard comparison theorem in the theory of Grothendieck topologies that the T_G-cohomology of any discrete G-space X is the same as the cohomology of X in the standard topology of discrete G-sets and families of surjective morphisms. But sheaves in this topology may be identified with G-modules by making a sheaf F correspond to the G-module $F(G)$. (G is a left G-space by left multiplication, and the G-action on $F(G)$ is induced by letting $\sigma \in G$ act on G by right multiplication by σ^{-1}).

Putting this together, we may identify the cohomology groups $H^p_{T_G}(*, E)$ with the groups $H^p(G, E(G))$, where the cohomology is defined by the usual cochain definition.

Lemma 5.2. *Let v be a finite place, let $j = j_{K,S}$, let A be a continuous W_F-module, let $E = i_v^* R^q j_* \tilde{A}$, and let $G = \mathbb{Z} = W_{\kappa(v)}$. Then a) $E(G) = H^q(\theta_v(I_v), A)$, and hence $H^p(W_{\kappa(v)}, i_v^* R^q j_* \tilde{A}) = H^p(W_{\kappa(v)}, H^q(\theta_v(I_v), A))$*

b) $\underrightarrow{Lim}_{K,S} H^p(W_{\kappa(v)}, i_v^* R^q j_* \tilde{A}) = H^p(W_{\kappa(v)}, H^q(I_v, A))$.

Proof. Since G has no non-trivial covers, it is immediate that $i_v^p R^q j_* \mathbb{Z}(G)$ is naturally isomorphic to $E(G)$. (Here i_v^p denotes the presheaf inverse image). Recall that, if C is a presheaf on $T_{K,S}$, $i_v^p(C)(G)$ is given by the direct limit of those $C(\mathcal{X})$ for which there is a map $\phi : G \to i_v^{-1}(\mathcal{X})$. This then is equal to $H^q(T_{W_{K/F,S}}, W_{K/F,S}/\theta_v(I_v), A)$.

By Theorem 4.6, this is just $H^q(\theta(I_v), A)$, and an application of Lemma 5.1 completes the proof of a). Now observe that $\underrightarrow{Lim}_{K.S}(H^q(W_{\kappa(v)}, H^q(\theta_v(I_v), A)) =$ (since $W_{\kappa(v)} = \mathbb{Z}$) $H^p(W_{\kappa(v)}, \underrightarrow{Lim}_{K,S} H^q(\theta_v(I_v), A) =$(since $\theta(I_v)$ is compact and Moore cohomology commutes with limits for compact groups) $H^p(\underrightarrow{Lim}_{K,S}\theta_v(I_v), A) =$ (by Lemma 3.1) $H^p(W_{\kappa(v)}, H^q(I_v, A))$, which shows b).

Lemma 5.3. *Let v be a finite place. Then a) $H^1(I_v, \mathbb{Z}) = H^1(I_v, \mathbb{R}) = 0$,*

b) $H^0(W_{\kappa(v)}, H^2(I_v, \mathbb{Z}))$ is naturally isomorphic to the Pontriagin dual U_v^D of the local units U_v in the completion F_v of the field F at v, c) $H^0(W_{\kappa(v)}, H^2(I_v, \mathbb{R})) = 0$.

Proof. If $A = \mathbb{Z}$ or $\mathbb{R}$, $H^1(I_v, A) = Hom(I_v, A) = 0$. From the exact sequence $1 \to I_v \to G_v \to \hat{\mathbb{Z}} \to 1$, we get the Hochschild-Serre spectral sequence $H^p(\hat{Z}, H^q(I_v, \mathbb{Z})) \Rightarrow H^{p+q}(G_v, \mathbb{Z})$. This spectral sequence yields the short exact sequence $0 \to H^2(\hat{\mathbb{Z}}, \mathbb{Z}) \to H^2(G_v, \mathbb{Z}) \to H^0(\hat{\mathbb{Z}}, H^2(I_v, \mathbb{Z})) \to 0$.

By local class field theory $H^2(G_v, \mathbb{Z})$ is naturally isomorphic to $Hom(F_v^*, \mathbb{Q}/\mathbb{Z})$, so the above exact sequence shows that $H^0(\hat{Z}, H^2(I_v, \mathbb{Z}))$ is naturally isomorphic to $Hom(U_v, \mathbb{Q}/\mathbb{Z})$ which (since U_v is profinite) is the Pontriagin dual of U_v. But since $W_{\kappa(v)} = \mathbb{Z}$ is dense in $\hat{\mathbb{Z}}$, 5.3b) follows immediately. Since I_v is compact, $H^2(I_v, \mathbb{R}) = 0$, which proves c).

Lemma 5.4. *Let $\theta : H \to G$ be a map of topological groups, so we may regard any G-space as an H-space via θ. Let I be a topological subgroup of H. Let Z be any topological space, regarded as a G-space with trivial G-action, and let X be any G-space. Then any H-map $\phi : H/I \times Z$ to X factors through the G-space $G/\theta(I) \times Z$.*

Proof. This follows immediately from the remark after Definition 4.2.

Lemma 5.5. *Let G be a connected topological group, and let X be a topological space on which G acts trivially. Then $\check{H}^q(X, \mathbb{Z})$ is naturally isomorphic to $\check{H}^q_{top}(X, \mathbb{Z})$.*

Proof. We first claim that any local-section G-cover $\rho_i : \{X_i \to X\}$ has a refinement by a cover of the form $\{G \times V_i\}$, where $\{V_i\}$ is an open cover of X, and G acts on $G \times V_i$ by left multiplication on the first factor. Given $x \in X$, let U_x be an open neighborhood of x such that $s_x : U_x \to X_{i(x)}$ is a section of $\rho_{i(x)}$. Define $\phi_x : G \times U_x \to X_{i(x)}$ by $\phi_x(g, u) = gs_x(u)$, and verify first that ϕ_x is a G-map and next that $pr_2 = \rho_i(x)\phi_x$, thus showing that $\{G \times U_x\}$ refines $\{X_i\}$.

We next claim that the Čech complex for the sheaf $\mathbb{Z}$ of the G-cover $\{G \times V_i\}$ is the same as the Čech complex of the cover $\{V_i\}$ of X. This follows immediately because any map from a power G^n of the connected group G to the discrete group $\mathbb{Z}$ is constant.

Lemma 5.6. *Let Z be a contractible topological space. Let v be a fixed archimedean place of $\bar{Y}_{K/F,S}$, so $I_v = S^1$, and let $H = W_{\kappa(v)}$. Let H act on $H \times Z$ by left multiplication on the first factor. We claim that*

a)$(i_v^p R^1 j_* \mathbb{Z})(H \times Z) = 0$.

b) $(i_v^p R^2 j_* \mathbb{Z})(H) = H^2(I_v, \mathbb{Z})$.

c) $(i_v^* R^2 j_* \mathbb{Z})(H) = H^2(I_v, \mathbb{Z})$.

Proof. Let E be any sheaf on $\bar{Y}_{K/F,S}$. Then by definition, $(i_v^p(E))(H \times Z)$ is equal to the direct limit of the $E((X_w, f_w))$, where $H \times Z \to i_v^{-1}((X_w, f_w)) = X_v$. Now let $E = R^q j_* \mathbb{Z}$. It is immediate that we may assume in the direct limit that $X_v = H \times Z$ and that X_w is the empty set if w is neither v nor the generic point v_0. Lemma 5.1 shows that we may assume that X_{v_0} is $W_{K/F}/\theta_v(I_v)$ and hence that $R^q j_* \mathbb{Z}((X_w, f_w)) = H^q_{W_{K/F}}((W_{K/F}/(\theta_v(I_v)) \times Z), \mathbb{Z})$. By Lemma 4.3, this is the same as $H^q_{I_v}(Z, \mathbb{Z})$. If $q = 1$ this is equal to $\check{H}^1_{I_v}(Z, \mathbb{Z})$ which in turn is equal by Lemma 5.5 to $\check{H}^1_{top}(Z, \mathbb{Z})$ which is zero since Z is contractible.

If $q = 2$ and Z is a point we have that $i_v^p(R^2 j_* \mathbb{Z})(H) = H^2_{I_v}(*, \mathbb{Z}) = H^2(I_v, \mathbb{Z})$. c) then follows immediately because H has no non-trivial covers.

Lemma 5.7. *Let G be a topological group, and n a positive integer. Then G^n, regarded as a G-space with G acting diagonally, is isomorphic to $G \times G^{n-1}$ where G acts by left multiplication on the first factor and trivially on G^{n-1}.*

Proof. Let $\phi : G^n \to G \times G^{n-1}$ by $\phi(g_1, \ldots g_n) = (g_1, g_1^{-1} g_2, \ldots g_{n-1}^{-1} g_n)$. It is easy to see that ϕ is a G-isomorphism.

Proposition 5.8. *Let v be an archimedean place.*

a) $H^p(W_{\kappa(v)}, i_v^* R^1 j_* \mathbb{Z}) = 0$ for $p = 0, 1,$ and 2.

a') $H^p(W_{\kappa(v)}, i_v^* R^1 j_* \mathbb{R}) = 0$ for $p = 0, 1,$ and 2.

b) $H^0(W_{\kappa(v)}, i_v^* R^2 j_* \mathbb{Z}) = H^2(I_v, \mathbb{Z})^{W_{\kappa(v)}}$

b') $H^0(W_{\kappa(v)}, i_v^* R^2 j_* \mathbb{R}) = 0$.

c) $H^2(I_v, \mathbb{Z})^{W_{\kappa(v)}} = {U_v}^D$.

Proof. We have the standard spectral sequence from Čech to derived functor cohomology:

$$E_2^{p.q} = \check{H}^p(\kappa(v), \underline{H}^q(i_v^* R^1 j_* \tilde{A})) \Rightarrow H^{p+q}(\kappa(v), i_v^* R^1 j_* \tilde{A})$$

where we know that $E_2^{0,q} = 0$ for $q > 0$.

We begin with the case $p = 2$. The spectral sequence immediately gives the exact sequence;

$$0 \to \check{H}^2(\kappa(v), i_v^* R^1 j_* \tilde{A}) \to H^2(\kappa(v), i_v^* R^1 j_* \tilde{A}) \to \check{H}^1(\kappa(v), \underline{H}^1(i_v^* R^1 j_* \tilde{A}))$$

So it suffices to show that the first and third terms in this exact sequence are zero. We begin with the first:

We first let $A = \mathbb{Z}$ and show that, more generally, $\check{H}^p(\kappa(v), i_v^* R^1 j_* \mathbb{Z}) = 0$. Since the covering $\{H\}$ of $*$ is initial, it is enough to show that $(i_v^* R^1 j_* \mathbb{Z})(H^n) = 0$. By lemma 5.7, this is equivalent to showing that $(i_v^* R^1 j_* \mathbb{Z})(H \times H^{n-1}) = 0$, where H acts trivially on H^{n-1}. But this is an immediate consequence of Lemma 1.6 and Lemma 5.5, since H is contractible and locally contractible.

Now let $A = \mathbb{R}$. If E is any sheaf of $\mathbb{R}$ -vector spaces on $H \times H^{n-1}$, and $q > 0$, Corollary 1.7 shows that $H_H^q(H \times H^{n-1}, E)$ is isomorphic to $H^q(H^{n-1}, \alpha_* E)$ which is equal to zero by Lemma 2.5.

Now we look at the third term. Since $H^1 = \check{H}^1$, we have to show that $\check{H}^1(H \times H^{n-1}, i_v^* R^1 j_* \tilde{A}) = 0$. A typical term in a coinitial cover of $H \times H^{n-1}$ is $H^r \times X$ with

X contractible, locally contractible, and metrizable. But rewriting this as $H \times (H^{r-1} \times X)$ and again using Lemma 5.5 in the case when $A = \mathbb{Z}$ and Lemma 2.5 when $A = \mathbb{R}$ enables us to copy the arguments of the preceding paragraph, since $H^{r-1} \times X$ is also contractible, locally contractible, and metrizable.

The case when $p = 1$ is similar but easier.

Now b) and b') follow immediately from Lemma 5.6c.

If v is complex $H^2(I_v, \mathbb{Z}) = \mathbb{Z}$, and if v is real we have the exact sequence $1 \to S^1 \to I_v \to \mathbb{Z}/2\mathbb{Z} \to 1$. The Hochschild-Serre spectral sequence shows that $H^2(I_v, \mathbb{Z}) = H^2(S^1, \mathbb{Z})^{\mathbb{Z}/2\mathbb{Z}} = \mathbb{Z}/2\mathbb{Z}$. In both cases the Weil group $W_{\kappa(v)}$ acts trivially, and we get $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$, the duals of S^1 and ± 1 respectively.

Theorem 5.9. *Let A be either $\mathbb{Z}$ or $\mathbb{R}$. a) $H^p(\bar{Y}_{K,S}, R^1(j_{K,S})_*\tilde{A}) = 0$ for $p = 0, 1, 2$.*

b) $H^p(\bar{Y}, R^1 j_*\tilde{A}) = 0$ for $p = 0, 1, 2$.

c) $H^0(\bar{Y}_{K,S}, R^2(j_{K,S})_*\mathbb{Z}) = \coprod_{v \in S}(U_v)^D$.

c') $H^0(\bar{Y}, R^2(j_{K,S})_*\mathbb{R}) = 0$.

d) $H^0(\bar{Y}, R^2 j_*\mathbb{Z}) = \coprod_{v \neq v_0}(U_v)^D$.

d') $H^0(\bar{Y}, R^2 j_*\mathbb{R}) = 0$.

Proof. Parts a) and c) follow immediately from Corollary 4.12, Lemma 5.2, Lema 5.3, and Proposition 5.8. Parts b) and d) follow from a) and c) by taking limits.

Let $Pic(\bar{Y})$ be the Arakelov class group of F, i. e., the group obtained by taking the idèle group of F and dividing by the principal idèles and the unit idèles (a unit idèle (u_v) is defined by $|u_v|_v = 1$ for all v). Let $Pic^1(\bar{Y})$ be the kernel of the absolute value map from $Pic(\bar{Y})$ to $\mathbb{R}^*$. Let $\mu(F)$ denote the group of roots of unity in F.

Theorem 5.10. *a) $H^0(\bar{Y}, \mathbb{Z}) = \mathbb{Z}$*

b) $H^1(\bar{Y}, \mathbb{Z}) = 0$

c) $H^2(\bar{Y}, \mathbb{Z}) = (Pic^1(\bar{Y})^D$

d) $H^3(\bar{Y}, \mathbb{Z}) = \mu(F)^D$.

Proof. a) is clear. The Leray spectral sequence for j_* gives first that $H^1(\bar{Y},\mathbb{Z}) = H^1(W_F,\mathbb{Z}) = 0$, which proves b). Next it gives (using Theorem 5.9) the exact sequence

$$0 \to H^2(\bar{Y},\mathbb{Z}) \to H^2(W_F,\mathbb{Z}) \to \coprod_{v\neq v_0} (U_v)^D \to H^3(\bar{Y},\mathbb{Z}) \to H^3(W_F,\mathbb{Z}) = 0$$

.

This is easily seen (using Theorem 3.6) to be the Pontriagin dual of the sequence:

$$0 \to H^3(\bar{Y},\mathbb{Z})^D \to \prod_{v\neq v_0} U_v \to C_F^1 \to H^2(\bar{Y},\mathbb{Z})^D \to 0$$

which completes the proof, since the roots of unity are the kernel of the map from the unit idèles to $C^1(F)$ and $Pic^1(F)$ is defined to be the cokernel.

Theorem 5.11. *a)* $H^0(\bar{Y},\mathbb{R}) = \mathbb{R}$

b) $H^1(\bar{Y},\mathbb{R}) = H^2(\bar{Y},\mathbb{R}) = 0$.

Proof. a) is clear, and b) follows from the Leray spectral sequence, using Theorem 5.9.

§6. Cohomology with compact support

Let ϕ be the natural inclusion of Y in $\bar{Y}$. Let E be any sheaf on Y. We define the sheaf $\phi_! E$ on $\bar{Y}$ to be the sheaf associated with the presheaf P defined by $P(\mathcal{X} = (X_v)) = E((X_v))$ if $X_v = \phi$ for all v not in Y, and $P(\mathcal{X}) = 0$ otherwise.

Proposition 6.1. *Let F be any sheaf on $\bar{Y}$. There exists an exact sequence of sheaves on $\bar{Y}$:*

$$0 \to \phi_! \phi^* F \to F \to i_* i^* F \to 0$$

where $i_* i^* F = \prod_{v\in Y_\infty} (i_v)_* i_v^* F$.

Proof. We first show that for all v in $\bar{Y}$, there exists an exact sequence

$$0 \to i_v^* \phi_! \phi^* F \to i_v^* F \to i_v^* i_* i^* F \to 0$$

We first see easily that if v is non-archimedean that $i_v^*\phi_!\phi^*F = i_v^*F$, and $i_v^*(i_*i^*F) = 0$ by Lemma 4.10c), so we get exactness. If v is archimedean, $i_v^*\phi_!\phi^*F = 0$ and $i_v^*(i_*i^*F) = i_v^*i_*F$ by Lemma 4.10b), so again we get exactness.

The exactness of the above exact sequences implies the Proposition, using Lemma 4.9 and the fact that i_v^* is exact (Lemma 4.10 a)).

Lemma 6.2. *Let v be an archimedean valuation. Then a) $H^i(W_{\kappa(v)}, \mathbb{Z}) = 0$ for $i > 0$.*

b) $H^i(\bar{Y}, i_*\mathbb{Z}) = 0$ for $i > 0$.

Proof. a) is immediate because $W_{\kappa(v)} = \mathbb{R}$, $\mathbb{R}$ is contractible, and $\mathbb{Z}$ is discrete. Then b) follows becaue i_* is exact.

Theorem 6.3. *a) $H^0(\bar{Y}, \phi_!\mathbb{Z}) = 0$*

b) $H^!(\bar{Y}, \phi_!\mathbb{Z}) = (\coprod_{S_\infty} \mathbb{Z})/\mathbb{Z})$

c) $H^2(\bar{Y}, \phi_!\mathbb{Z}) = Pic^1(\bar{Y})^D$

d) $H^3(\bar{Y}, \phi_!\mathbb{Z}) = \mu(F)^D$

Proof. This follows immediately from Theorem 5.10, Proposition 6.1, and Lemma 6.2.

Proposition 6.4. *There is a natural exact sequence*

$$0 \to Pic(Y)^D \to Pic^1(\bar{Y})^D \to Hom(U_F, \mathbb{Z}) \to 0$$

Proof. Let F_v denote the completion of F at the archimedean valuation V. Then we have a natural inclusion i of $\prod_v F_v^*$ into the idèle group J_F. We then obtain an exact sequence;

$$0 \longrightarrow \coprod \mathbb{R}^*_{>0} \xrightarrow{\tilde{i}} Pic(\bar{Y}) \longrightarrow Pic(Y) \longrightarrow 0$$

where $\tilde{i}$ is induced by i.

Then the logarithmic embedding of the units yields the exact sequence

$$0 \to V/L \to Pic^1(\bar{Y}) \to Pic(Y) \to 0$$

where V is the kernel of the sum map from $\coprod_v \mathbb{R}$ to $\mathbb{R}$, and L is the lattice in V obtained by taking the image of the unit group U_F under the map which sends a unit u to the vector $(log|u|_v)$.

We now examine the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (V/L)^D & \longrightarrow & V^D & \longrightarrow & L^D & \longrightarrow & 0 \\
 & & \uparrow & & \alpha\uparrow & & \uparrow & & \\
 & & 0 & \longrightarrow & Hom(V,\mathbb{R}) & \xrightarrow{\beta} & Hom(L.\mathbb{R}) & & \\
 & & & & \uparrow & & \gamma\uparrow & & \\
 & & & & 0 & \longrightarrow & Hom(L,\mathbb{Z}) & &
\end{array}$$

where α and β are isomorphisms, and γ is injective. This defines an isomorphism between $Hom(L,\mathbb{Z})$ and $(V/L)^D$, and the proposition follows, after we observe that the natural map from $Hom(L,\mathbb{Z})$ to $Hom(U_F,\mathbb{Z})$ is an isomorphism.

§7. Euler characteristics

Let $n \geq 1$ and let

$$0 \longrightarrow V_0 \xrightarrow{T_0} V_1 \xrightarrow{T_1} \dots \xrightarrow{T_{n-1}} V_n \longrightarrow 0$$

be an exact sequence of real vector spaces, and let B_i denote an ordered basis for V_i. We recall the definition of the determinant of the above data. If $n = 1$ the data determine an $n \times n$ matrix, and we take the determinant of that matrix.

If $n = 2$, let $B_0 = (a_1, \dots a_r)$, let $B_1 = (b_1, \dots b_{r+s})$, and let $B_2 = (c_{r+1}, \dots c_{r+s})$. For $1 \leq i \leq r$, let $d_i = T_0(a_i)$. Choose $(d_{r+1}, \dots d_{r+s})$ in V_1 such that $T_1(d_i) = c_i$. In the one-dimensional space $\Lambda^{r+s} V_1$ the element $d_1 \wedge d_2 \wedge \dots d_{r+s}$ is clearly independent of the choice of d_i, and we define our determinant δ so that $d_1 \wedge d_2 \cdots \wedge d_{r+s} = \delta(b_1 \wedge b_2 \wedge \cdots \wedge b_{r+s})$.

We finish by giving an inductive definition. Assume we have defined the determinant for $n \leq N$ and we wish to define it for $n = N+1$. We let I be the image of T_{n-1} so that we have the two exact sequences :

$$0 \longrightarrow V_0 \xrightarrow{T_0} V_1 \xrightarrow{T_1} \dots \xrightarrow{T_{n-1}} I \longrightarrow 0$$

$$0 \longrightarrow I \xrightarrow{i} V_n \xrightarrow{T_n} V_{n+1} \longrightarrow 0$$

where i is the inclusion of I in V_n. We choose any basis C for I. We now define the determinant δ of the sequence

$$0 \longrightarrow V_0 \xrightarrow{T_0} \dots \xrightarrow{T_n} V_{n+1} \longrightarrow 0$$

with bases $B_0, \dots, B_{n+1}$, to be $\delta_1(\delta_2)^{(-1)^n}$, where δ_1 is the determinant of the sequence

$$0 \longrightarrow V_0 \xrightarrow{T_0} V_1 \xrightarrow{T_1} \dots \xrightarrow{T_{n-1}} I \longrightarrow 0$$

where V_i has basis B_i and I has basis C, and δ_2 is the determinant of the sequence

$$0 \longrightarrow I \xrightarrow{i} V_n \xrightarrow{T_n} V_{n+1} \longrightarrow 0$$

where I has basis C, and V_n and V_{n+1} have bases B_n and B_{n+1}. It is easy to see that this definition is independent of the choice of C.

Now let $A_0, A_1, \dots, A_n$ be finitely generated abelian groups, and let $V_i = A_i \otimes \mathbb{R}$. Assume that there exist $\mathbb{R}$-linear transformations $T_i : V_i \to V_{i+1}$ such that the sequence

$$0 \longrightarrow V_0 \xrightarrow{T_0} V_1 \xrightarrow{T_1} \dots V_{n-1} \xrightarrow{T_n} V_n \longrightarrow 0$$

is exact.

We define the Euler characteristic $\chi(A_0, A_1, \dots A_n, T_0 \dots T_{n-1})$ to be the alternating product $\prod_{i=0}^{n} |((A_i)_{tor})|^{(-1)^i}$ divided by the determinant of $(V_0, \dots V_n, T_0, \dots T_{n-1}, B_0, \dots, B_n)$, where the B_i are the images of bases of the free abelian groups $A_i/(A_i)_{tor}$.

The B_i are of course not unique, but a change of basis only changes the determinant by the determinant of a matrix in $GL(r, \mathbb{Z})$, i. e. by ± 1.

So our Euler characteristic is well-defined up to sign.

§8. Dedekind zeta-functions at zero

In this section we wish to verify that the conjecture stated in the introduction is true for Dedekind zeta-functions, modulo the assumption that the higher cohomology groups are zero.

We first define our Euler characteristic. Let F be a number field, let O_F be the ring of integers in F, and let $Y = \text{Spec } O_F$. Let $\bar{Y}$ be Y together with the archimedean primes of F, given the Weil -étale topology as above. Let ϕ be the inclusion of Y in $\bar{Y}$.

Let $\mathbb{R}$ denote the sheat on $\bar{Y}$ determined by defining $\mathbb{R}((X_v))$ to be the set of compatible continuous W_v-maps from X_v to $\mathbb{R}$, where W_v acts trivially on $\mathbb{R}$. It is clear both that this is a sheaf and that such a set is determined by giving a W_{v_0}-map from X_{v_0} to $\mathbb{R}$. It is also clear that this is the same sheaf as the sheaf $j_*\tilde{\mathbb{R}}$.

The Leray spectral sequence for the map j_* yields:

$$0 \to H^1(\bar{Y}, \mathbb{R}) \to H^1(W_F, \mathbb{R}) \to H^0(\bar{Y}, R^1 j_* \mathbb{R}) \to H^2(\bar{Y}, \mathbb{R}) \to H^2(W_F, \mathbb{R}) = 0$$

where $H^2(W_F, \mathbb{R}) = 0$ by Lemma 3.4.

But $R^1 j_* \mathbb{R}$ is isomorphic to $\coprod (i_v)_* i_v^* R^1 j_* \mathbb{R}$, and so we conclude easily that $H^1(\bar{Y}, R^1 j_* \mathbb{R})$ is isomorphic to $\coprod H^1(I_v, \mathbb{R})$, where the sums are taken over all non-trivial valustions of F. But whether v is archimedean or non-archimedean, I_v is compact, so $H^1(Iv, \mathbb{R}) = Hom(I_v, \mathbb{R}) = 0$. We conclude that $H^1(\bar{Y}, \mathbb{R}) = H^1(W_F, \mathbb{R}) = Hom(W_F, \mathbb{R})$, and that $H^2(\bar{Y}, \mathbb{R}) = 0$. Let ψ in $H^1(\bar{Y}, \mathbb{R})$ be the homomorphism obtained by mapping W_F to its abelianization C_F and then taking the logarithm of the absolute value.

We next observe that first, by standard arguments the category of sheaves of $\mathbb{R}$-modules has enough injectives, and next, that any injective sheaf of $\mathbb{R}$-modules is injective as a sheaf of abelian groups. These observations imply that taking the Yoneda product with ψ in $H^1(\bar{Y}, \mathbb{R}) = Ext^1_{\bar{Y}}(\mathbb{R}, \mathbb{R})$ induces a map from $H^q(\bar{Y}, F) = Ext^q_{\bar{Y}}(\mathbb{R}, F)$ to $H^{q+1}(\bar{Y}, F) = Ext^{q+1}_{\bar{Y}}(\mathbb{R}, F)$, where F is any sheaf of R-modules.

Theorem 8.1. *Assume that $H^q(\bar{Y}, \phi_! \mathbb{Z}) = 0$ for $q > 3$. Let ζ_F be the Dedekind zeta-function of F. Then the Euler characteristic $\chi(H^*(\bar{Y}, \phi_! \mathbb{Z}))$ is well-defined and is equal to*

$\pm\zeta_F^*(0)$.

Proof. We first observe that the groups $H^i(\bar{Y}, \phi_!\mathbb{Z})$ are finitely-generated by Theorem 6.3 and Proposition 6.4. We must show next that the natural map from $H^i(\bar{Y}, \phi_!\mathbb{Z}) \otimes \mathbb{R}$ to $H^i(\bar{Y}, \phi_!\mathbb{R})$ is an isomorphism. Look at the commutative diagram:

$$
\begin{array}{ccccccccc}
 & & H^2(\bar{Y}, \mathbb{R}) = 0 & & & & & & \\
 & & \uparrow & & & & & & \\
 & & H^2(\bar{Y}, \phi_!\mathbb{R}) & & & & & & \\
 & & \uparrow & & & & & & \\
0 & \longrightarrow & H^1(\bar{Y}, i_*\mathbb{R}) & \xrightarrow{\alpha} & H^1(\bar{Y}, i_*S^1) & \longrightarrow & H^2(\bar{Y}, i_*\mathbb{Z}) = 0 & & \\
 & & \gamma\uparrow & & \uparrow & & \uparrow & & \\
 & & H^1(\bar{Y}, \mathbb{R}) & \longrightarrow & H^1(\bar{Y}, S^1) & \longrightarrow & H^2(\bar{Y}, \mathbb{Z}) & \longrightarrow & H^2(\bar{Y}, \mathbb{R}) = 0 \\
 & & \delta\uparrow & & \uparrow & & \beta\uparrow & & \uparrow \\
 & & H^1(\bar{Y}, \phi_!\mathbb{R}) & \longrightarrow & H^1(\bar{Y}, \phi_!S^1) & \longrightarrow & H^2(\bar{Y}, \phi_!\mathbb{Z}) & \xrightarrow{\epsilon} & H^2(\bar{Y}, \phi_!\mathbb{R})
\end{array}
$$

It is easy to see that γ is injective, so δ is the zero map, so $H^1(\bar{Y}, \phi_!\mathbb{R})$ may be identified with $H^1(\bar{Y}, \phi_!\mathbb{Z}) \otimes \mathbb{R}$, and we take a basis of $H^1(\bar{Y}, \phi_!\mathbb{R})$ obtained by choosing $r_1 + r_2 - 1$ archimedean primes of F.

By a tedious but straightforward calculation with injective resolutions, we see that the map ϵ may be computed by applying β, lifting to $H^1(\bar{Y}, S^1)$, mapping to $H^1(\bar{Y}, i_*S^1)$, applying α^{-1}, and mapping to $H^2(\bar{Y}, \phi_!\mathbb{Z})$.

Now by comparing this diagram with the diagram at the end of Section 6, we see that we may first identify $H^2(\bar{Y}, \phi_!\mathbb{R})$ with $Hom(V_0, \mathbb{R})$, where $V = \coprod_{v \in S_\infty} \mathbb{R}^*_{>0}$ and V_0 is the kernel of the product map to $\mathbb{R}^*_{>0}$. Next, we may take as a basis of this group coming from $H^2(\bar{Y}, \phi_!\mathbb{Z})$ the dual basis of any basis for the units of F modulo torsion, identifying V_0 with $U_F \otimes \mathbb{R}^*_{>0}$ via the map $u \mapsto (|u|_v)$ for the same set of $r_1 + r_2 - 1$ $v's$ we used above. Finally the Yoneda product with ψ clearly takes 1_v to the map f_v where $f_v((x_w)) = \log x_v$.

It is now easy to see that the determinant of the pair consisiting of $H^*(\bar{Y}, \phi_!\mathbb{Z})$ and Yoneda product with ψ is R^{-1} where R is the classical regulator.

Since $H^0(\bar{Y}, \phi_! \mathbb{Z}) = 0$, $(H^1(\bar{Y}, \phi_! \mathbb{Z}))_{tor} = 0$. $|(H^2(\bar{Y}, \phi_! \mathbb{Z}))_{tor}| = h$, and $|H^3(\bar{Y}, \phi_! \mathbb{Z})| = w$, the Euler characteristic of $H^*(\bar{Y}, \phi_!, \mathbb{Z})$ is equal to hR/w which up to sign is $\zeta_F^*(0)$.

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