# THE WEIL- ÉTALE TOPOLOGY

S.Lichtenbaum

## §0. Introduction

In this paper we introduce a new, or at least neglected, Grothendieck topology on the category of schemes of finite type over a finite field. This topology, which we call the Weil-étale topology, bears the same relation to the étale topology as the Weil group bears to the Galois group.

Recall that if $K$ is the function field of a curve over a finite field $k$, $\bar{K}$ is a fixed algebraic closure of $K$, and $G_K$ is the Galois group of $\bar{K}$ over $K$, $G_K$ comes with a natural surjection $\pi$ to $Gal(K\bar{k}/K)$, which is isomorphic to $\hat{\mathbb{Z}}$ and topologically generated by the Frobenius element $\phi$. The Weil group $W_K$ is just $\pi^{-1}(\mathbb{Z})$ where $\mathbb{Z}$ is the subgroup of $Gal(K\bar{k}/K)$ consisting of all integral powers of $\phi$. (Of course, this works equally well if "curve" is replaced by "algebraic variety" but classically only curves were considered).

The Weil-étale topology should have several advantages over the étale topology. Conjecturally, all the motivic cohomology groups in this topology should be finitely generated, whereas in the étale topology these groups can be quite complicated. Also, the Weil-étale cohomology groups should contain more information, in the sense that they should determine the étale cohomology groups, but not vice versa.

Finally, there should be a natural notion of Euler characteristic for these groups, which is closely related to special values of zeta-functions. In short, it is our hope that the Weil-étale

The author has been partially supported by N. S. F. grant DMS-9970333.

motivic cohomology groups of algebraic varieties over finite fields should be better suited than the étale groups to any possible arithmetic application.

In the body of this paper we first prove a global duality theorem for cohomology of Weil-étale sheaves on curves over finite fields, starting from a similar theorem in the étale case, which is essentially a version of global class field theory. Our duality theorem is a Verdier-type duality and is properly stated (Theorem 6.5) in terms of the derived category of abelian groups.

We then go on to state a conjecture (Conjecture 8.1) relating the behavior of zeta-functions of algebraic varieties $V$ over finite fields at the point $s = 0$ to Weil-étale Euler characteristics and prove this conjecture for curves, smooth surfaces, and arbitrary projective smooth varieties. In the case when $V$ is projective and smooth this is a variant of Theorem 0.4a of [M2], but if $V$ is not projective, we do not know even a conjectural statement in terms of étale cohomology.

## §1. The cohomology of $\mathbb{Z}$-modules and $\hat{\mathbb{Z}}$-modules

We begin with a warning to the reader; the phrase "$\mathbb{Z}$-module" does not denote an abelian group but instead a module over the group ring $\mathbb{Z}[\mathbb{Z}]$. Similarly, a "$\hat{\mathbb{Z}}$-module" is a module over $\mathbb{Z}[\hat{\mathbb{Z}}]$.

**Lemma 1.1.** *Let $M$ be a torsion $\hat{\mathbb{Z}}$-module. The natural map from $H^i(\hat{\mathbb{Z}}, M)$ to $H^i(\mathbb{Z}, M)$ is an isomorphism.*

Proof. Both are zero for $i \geq 2$ and it is clear for $i = 0$. So we may assume $i = 1$. Let $\sigma$ be a generator of $\mathbb{Z}$. $H^1(\mathbb{Z}, M)$ may be identified with $M/(\sigma - 1)M$, while $H^1(\hat{\mathbb{Z}}, M)$ may be identified with $\varinjlim(M_{(n)}^{\sigma^n}/(\sigma - 1)M^{\sigma^n})$, where $M_{(n)}$ is the kernel of multiplication by $N_n = 1 + \sigma \cdots + \sigma^{n-1}$ on $M$. Since $M$ is torsion, it is easy to see that $M = \bigcup_n M_{(n)}^{\sigma^n}$. (Let $x$ be in $M$. Since $\hat{\mathbb{Z}}$ acts continuously on $M$, there exists an $r$ such that $\sigma^r x = x$ and there exists an $m$ such that $mx = 0$. Then $N_{mr}(x) = 0$).

**Lemma 1.2.** *Let $M$ be any $\hat{\mathbb{Z}}$-module. Then there are functorial isomorphisms:*

*a)* $H^0(\hat{\mathbb{Z}}, M) \cong H^0(\mathbb{Z}, M)$.

*b)* $H^1(\hat{\mathbb{Z}}, M) \cong H^1(\mathbb{Z}, M)_{tor}$

*c)* $H^2(\hat{\mathbb{Z}}, M) \cong H^1(\mathbb{Z}, M) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$

Proof. a) is clear.

To prove b) and c), look at the commutative diagram

$$\begin{array}{ccccccccc}
H^1(\mathbb{Z}, M) & \xrightarrow{n} & H^1(\mathbb{Z}, M) & \longrightarrow & H^1(\mathbb{Z}, M/nM) & \longrightarrow & 0 & & \\
\uparrow & & \uparrow & & \uparrow\alpha & & \uparrow & & \\
H^1(\hat{\mathbb{Z}}, M) & \xrightarrow{n} & H^1(\hat{\mathbb{Z}}, M) & \longrightarrow & H^1(\hat{\mathbb{Z}}, M/nM) & \longrightarrow & H^2(\hat{\mathbb{Z}}, M)_n & \longrightarrow & 0
\end{array}$$

where $\alpha$ is is an isomorphism by the preceding lemma.

It follows from diagram-chasing that there is a natural isomorphism

$$H^2(\hat{\mathbb{Z}}, M)_n \xrightarrow{\sim} (H^1(\mathbb{Z}, M) \otimes (\mathbb{Z}/n\mathbb{Z}))/(H^1(\tilde{\mathbb{Z}}, M) \otimes \mathbb{Z}/n\mathbb{Z})$$

and hence in the limit an isomorphism

$$H^2(\hat{\mathbb{Z}}, M) \xrightarrow{\sim} H^1(\mathbb{Z}, M) \otimes \mathbb{Q}/\mathbb{Z}$$

since $H^2(\hat{\mathbb{Z}}, M)$ is torsion and $H^1(\hat{\mathbb{Z}}, M) \otimes \mathbb{Q}/\mathbb{Z} = 0$.

Since $H^0(\mathbb{Z}, M) \xrightarrow{\sim} H^0(\hat{\mathbb{Z}}, M)$ and $H^0(\mathbb{Z}, M/nM) \xrightarrow{\sim} H^0(\hat{\mathbb{Z}}, M/nM)$, it is immediate that $H^1(\mathbb{Z}, M)_n \xrightarrow{\sim} H^1(\hat{\mathbb{Z}}, M)_n$ from which b) follows.

## §2. Definition of the Weil -étale topology

Let $k$ be a finite field and $\bar{k}$ a fixed algebraic closure of $k$. Let $X$ be a scheme of finite type over $k$ and let $\bar{X} = X \times_k \bar{k}$. We define the *Weil-étale topology* $\mathcal{W} = \mathcal{W}_X$ on $X$ by letting $Cat(\mathcal{W})$ be the category defined as follows: the objects of $Cat(\mathcal{W})$ are schemes étale and of finite type over $\bar{X}$. Let $\pi_1$ be the projection from $\bar{X}$ to $X$, and let $\pi_2$ be the projection from $\bar{X}$ to $\bar{k}$. If $(W, f : W \to \bar{X})$ and $(Z, g : Z \to \bar{X})$ are objects in $Cat(\mathcal{W})$ with $W$ connected, a morphism $\phi$ from $(W, f)$ to $(Z, g)$ is an $X$ - morphism $\phi$ from $W$ to $Z$ such that $\pi_2 \circ f =$

$\pi_2 \circ g \circ \phi$ up to an integral power of Frobenius on $\bar{k}$. A morphism from an arbitrary $W$ is a collection of morphisms on the connected components of $W$. The coverings are the usual étale coverings.

We recall that the Galois group of $\bar{k}$ over $k$ is naturally isomorphic to $\hat{\mathbb{Z}}$, and we let $\mathbb{Z}$ be the subgroup of $\hat{\mathbb{Z}}$ consisting of the integral powers of Frobenius. If $G$ is a Weil-étale sheaf on $X$ then $G(\bar{X})$ is in a natural way a $\mathbb{Z}$-module. We define $H^0_{\mathcal{W}}(X,G)$ to be $G(\bar{X})^{\mathbb{Z}}$, and $H^i_{\mathcal{W}}(X,G)$ to be the derived functors of $H^0$.

Note that there is no final object in our category, so our functors $H^i$ are not cohomology in the usual sense. The topology could be enlarged so that the $H^i$ do become cohomology, but at the cost of some additional complications and no obviously apparent advantage.

The category of lisse sheaves for this topology is equivalent to the category of "Weil sheaves" introduced by Deligne in [Del]. but Deligne did not actually define a topology, nor consider sheaves of the type we work with here.

**Definition 2.1.** *Let $G$ be a group of automorphisms of a scheme $X$. We say that $G$ acts on a sheaf $F$ on $X$ if we have a compatible system of maps $\psi_\sigma : F \to \sigma_* F$ for all $\sigma$ in $G$.*

**Proposition 2.2.** *The category of Weil-étale sheaves on $X$ is equivalent to the category of étale sheaves $F$ on $\bar{X}$ equipped with a $\mathbb{Z}-$action.*

Proof. A Weil-étale sheaf on $X$ certainly determines an étale sheaf on $\bar{X}$. if $U$ is an étale scheme over $\bar{X}$,and $\sigma$ is in $\mathbb{Z}$, let $U_\sigma = U \times_{\bar{X}} \bar{X}$, where the map from $\bar{X}$ to $\bar{X}$ is given by $\sigma$. Then the projection map from $U_\sigma$ to $U$ is a map in our category, and so determines a functorial map $F(U)$ to $F(U_\sigma)$, which is exactly a map from $F$ to $\sigma_* F$.

To go in the other direction, a map in our category from $V$ to $W$ gives rise to a commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & W \\ \uparrow & & \uparrow \\ \bar{X} & \xrightarrow{\sigma} & \bar{X} \end{array}$$

where $\sigma$ is in $\mathbb{Z}$. This clearly determines an $\bar{X}$-map from $V$ to $W_\sigma$, and thus a map from $F(W)$ to $F(V)$.

**Proposition 2.3.** *let $G$ be a Weil-étale sheaf on $X$. There is a spectral sequence whose $E_2^{p,q}$-term is $H^p(\mathbb{Z}, H^q_{\acute{e}t}(\bar{X}, G))$ and which converges to $H^{p+q}_{\mathcal{W}}(X, G)$.*

Proof. The category of sheaves for the Weil-étale site on $k$ is equivalent to the category of $\mathbb{Z}$-modules and the cohomology groups $H^i_{WD}(k, G)$ are canonically isomorphic to the cohomology groups $H^i(\mathbb{Z}, G(\bar{k}))$. The functor $(\pi_2)_*$ has the exact left adjoint $\pi_2^*$, and so takes injectives to injectives. Our spectral sequence now just becomes a usual composite functor spectral sequence.

We define a pair of functors: $\psi$, which maps Weil-étale sheaves on $X$ to étale sheaves on $X$, and $\phi$, which maps étale sheaves on $X$ to Weil-étale sheaves on $X$. We define $\psi$ as follows: If $G$ is a Weil-étale sheaf, define $\psi(G)(U)$ to be $(G(U \times_X \bar{X}))^{\mathbb{Z}}$. If $F$ is an étale sheaf on $X$ define $\phi(F)$ to be $\pi_1^*(F)$ (which is endowed with a natural $\mathbb{Z}$-action).

It is readily verified that these functors have the following properties:

**Proposition 2.3.**

*(a) $\phi$ is left adjoint to $\psi$.*

*(b) $\psi\ \phi = 1$*

*(c) $\psi$ is left exact and $\phi$ is exact, so $\psi$ takes injectives to injectives.*

*(d) $\psi(\mathbb{Z}) = \mathbb{Z}$, and $\psi(\mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$*

*(e) There is a functorial map $c_i : H^i_{\acute{e}t}(X, \psi(G)) \to H^i_{\mathcal{W}}(X, G)$ which is an isomorphism when $i = 0$, and, by 2), a functorial map $H^i_{\acute{e}t}(X, F) \to H^i_{\mathcal{W}}(X, \phi(F))$.*

*(f) There is a functorial map of spectral sequences from*

$$H^p(\hat{\mathbb{Z}}, H^q_{\acute{e}t}(\bar{X}, F)) \Rightarrow H^{p+q}_{\acute{e}t}(X, \psi(F))$$

*to*

$$H^p(\mathbb{Z}, H^q_{\acute{e}t}(\bar{X}, F)) \Rightarrow H^{p+q}_{\mathcal{W}}(X, F)$$

,

*(g) $c_i$ is an isomorphism if $G$ is torsion.*

Note that in Proposition 2.2(f),the first spectral sequence is the standard Hochschild-Serre spectral sequence and the second is the one from Proposition 2.1. Proposition 2.2(g) then follows from a comparison of these spectral sequences, using Lemma 1.1.

## §3. The finite generation of cohomology groups

Throughout this section let $k$ be a fixed finite field.

**Lemma 3.1.** *Let $U$ be a curve over the finite field $k$. Let $j : U \to X$ embed $U$ as a dense open subset of a projective curve $X$. Then the groups $H^i_{\mathcal{W}}(X, j_!\mathbb{Z})$ are a) independent of $j$ and b) finitely generated.*

Proof. a) Let $j : U \to X$ and $j' : U \to X'$ be two completions of $U$. Replacing $X'$ by the closure of the image of $U$ in $X \times X'$ we may assume that there is a map $\pi : X' \to X$ such that $\pi \circ j' = j$. Since $\pi$ is finite, so is $\bar{\pi}$, and $\bar{\pi}_*$ is exact in the ètale topology. It follows that the natural map from $H^p_{\acute{e}t}(\bar{X}, \bar{\pi}_* F)$ to $H^p_{\acute{e}t}(\bar{X}', \bar{F})$ is an isomorphism, and hence by the basic spectral sequence of Proposition 2.1 ,the natural map from $H^p_{\mathcal{W}}(X, \pi_* F)$ to $H^p_{\mathcal{W}}(X', F)$ is an isomorphism. But $\pi_* j_! \mathbb{Z}$ is naturally isomorphic to $j'_!\mathbb{Z}$, and so the cohomology groups $H^p_{\mathcal{W}}(X', j'_!\mathbb{Z})$ are isomorphic to the groups $H^p_{\mathcal{W}}(X, j_!\mathbb{Z})$

b) Let $U$ be an open dense subscheme of $V$ which in turn is an open dense subscheme of the projective curve $X$. Let $\phi : U \to X$ and $j : V \to X$ be the given open immersions, and $i$ the closed immersion of $V - U$ in $X$. We have the exact sequence

$$0 \to \phi_!\mathbb{Z} \to j_!\mathbb{Z} \to i_*\mathbb{Z} \to 0$$

.

Then part b) follows from the consideration of the related long exact cohomology sequence, showing that first the cohomology groups of smooth projective curves, then smooth curves, and then arbitrary curves are finitely generated.

**Theorem 3.2.** *Let $X$ be a projective smooth variety over $k$. The Weil-étale cohomology groups $H^q_{\mathcal{W}}(X,\mathbb{Z})$ are finitely generated for all $q$, finite for $q \geq 2$ and zero for $q$ large.*

Proof. a)We may assume $X$ connected. Let $\bar{X}$ be $X \times_k \bar{k}$. It follows from [M2] that $H^q_{\acute{e}t}(X,\mathbb{Z})$ is finite for $q \geq 3$ and zero for $q$ large and $q = 1$, that $H^2_{\acute{e}t}(X,\mathbb{Z})$ is the $\mathbb{Q}/\mathbb{Z}$-dual of a finitely generated group of rank one, and of course $H^0_{\acute{e}t}(X,\mathbb{Z}) = \mathbb{Z}$. A comparison of the spectral sequences given at the end of the preceding section then shows that $H^q_{\mathcal{W}}(X,\mathbb{Z})$ is canonically isomorphic to $H^q_{\acute{e}t}(X,\mathbb{Z})$ (and hence finite and zero for $q$ large) for $q \geq 3$, that $H^2_{\mathcal{W}}(X,\mathbb{Z})$ is isomorphic to $H^2_{\acute{e}t}(X,\mathbb{Z})/(\mathbb{Q}/\mathbb{Z})$ and hence finite. It also follows immediately that $H^0_{\mathcal{W}}(X,\mathbb{Z})$ and $H^1_{\mathcal{W}}(X,\mathbb{Z})$ are both isomorphic to $\mathbb{Z}$.

**Theorem 3.3.** *Let $d \leq 2$ and et $U$ be a smooth $d$-dimensional quasi-projective variety over $k$. By resolution of singularities we can find a smooth projective variety $X$ containing $U$ as an open dense subvariety. Let $j : U \to X$ be the corresponding open immersion. Then the cohomology groups $H^q_{\mathcal{W}}(X, j_!\mathbb{Z})$ are finitely generated, zero for $q$ large and independent of the choices of $X$ and $j$.*

Proof. We write $H^q$ for $H^q_{\mathcal{W}}$. We first show finite generation. Let $j : U \to X$ as above,and let $Z = X - U$. Let $i : Z \to X$ be the corresponding closed immersion. We have the exact sequence of sheaves on $X$: $0 \to j_!\mathbb{Z} \to \mathbb{Z} \to i_*\mathbb{Z} \to 0$. Since $i_*$ is exact, Lemma 3.1 implies that the cohomology groups $H^q(X, i_*\mathbb{Z}) \xrightarrow{\sim} H^q(W,\mathbb{Z})$ are finitely generated and zero for $q$ large and Theorem 3.2 implies the cohomology groups $H^q(X,\mathbb{Z})$ are finitely generated and zero for $q$ large. Hence the long exact cohomology sequence implies that the groups $H^q(X, j_!\mathbb{Z})$ are also finitely generated and zero for $q$ large.

Now we show independence. Suppose we have $j : U \to X$ and $j' : U \to W$ with $X$ and $W$ being smooth projective. Let $\rho : U \to V = X \times W$ be the map induced by $j$ and $j'$, and let $Z$ be the closure of $U$ in $V$. By resolution we can find a $\pi : Z' \to Z$ such that $Z'$ is projective and smooth. Replacing $W$ by $Z'$, we may asume that there is a map $\pi : W \to X$ such that $\pi \circ j' = j$. We next observe that $\pi_* j'_!\mathbb{Z} = j_!\mathbb{Z}$, and then the functorial map from $H^q(X, \pi_* F)$ to $H^q(W, F)$ gives us the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^q(X, j_!\mathbb{Z})/n & \longrightarrow & H^q(X, j_!(\mathbb{Z}/n)) & \longrightarrow & (H^{q+1}(X, j_!\mathbb{Z}))_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_q} & & \downarrow{\scriptstyle \beta_q} & & \downarrow{\scriptstyle \gamma_{q+1}} & & \\ 0 & \longrightarrow & H^q(W, j'_!\mathbb{Z})/n & \longrightarrow & H^q(W, j'_!(\mathbb{Z}/n)) & \longrightarrow & (H^{q+1}(W.j'_!\mathbb{Z}))_n & \longrightarrow & 0 \end{array}$$

We know that $\beta$ is an isomorphism because the standard étale cohomology groups with compact support of constructible sheaves are well-defined. We proceed by descending induction. Let $\delta_q : H^q(X, j_!\mathbb{Z}) \to H^q(W, j'_!\mathbb{Z})$. If $\delta_{q+1}$ is an isomorphism then $\gamma_{q+1}$ is an isomorphism, hence $\alpha_q$ is an isomorphism. Now if we have a map $f$ from one finitely generated group to another such that $f$ becomes an isomorphism after tensoring with $\mathbb{Z}/n$ for every $n$ then it is easy to see that $f$ must be an isomorphism. So $\delta_q$ is an isomorphism. We can start the induction because both $H^q$'s are zero for large $q$.

**Proposition 3.4.** *Let $X$ be a geometrically connected smooth curve over a finite field $k$. Then the cohomology groups $H^q_{\mathcal{W}}(X, G_m)$ are finitely generated for all $q$ and zero if $q \geq 3$. If $X$ is projective, $H^0_{\mathcal{W}}(X, G_m) = k^*$ and $H^1_{\mathcal{W}}(X, G_m) = Pic(X)$ just as in the étale case, $H^2_{\mathcal{W}}(X, G_m) = \mathbb{Z}$, and the rest are zero.*

.

Proof. We begin with the spectral sequence of Proposition 2.1:

$$H^p(\mathbb{Z}, H^q_{ét}(\bar{X}, G_m)) \Rightarrow H^{p+q}_{\mathcal{W}}(X, G_m)$$

This spectral sequence degenerates to give the series of short exact sequences

$$0 \to H^1(\mathbb{Z}, H^{q-1}_{ét}(\bar{X}, G_m) \to H^q_{\mathcal{W}}(X, G_m) \to H^0(\mathbb{Z}, H^q_{ét}(\bar{X}, G_m) \to 0$$

. We now plug in the fact [M1, Example 2.22d] that $H^q_{ét}(\bar{X}, G_m) = 0$ for $q \geq 2$. We immediately obtain $H^q_{\mathcal{W}}(X, G_m) = 0$ for $q \geq 3$.

We also see $H^2_{\mathcal{W}}(X, G_m)$ is isomorphic to $H^1(\mathbb{Z}, Pic(\bar{X}))$. Let $U(\bar{X}) = H^0(\bar{X}, G_m)$. We have an exact sequence

$$0 \to H^1(\mathbb{Z}, U(\bar{X})) \to H^1_{\mathcal{W}}(X, G_m) \to (Pic(\bar{X}))^{\mathbb{Z}} \to 0$$

Now write $X$ as $Y - S$ with $Y$ smooth and projective and $S$ finite. let $T$ be the finite set of ppoints of $\bar{Y}$ lying over $S$. First assume that $S$ is empty. Then $(Pic(\bar{Y}))^{\mathbb{Z}} \xrightarrow{\sim} Pic(Y)$ since $H^2(\mathbb{Z}, \bar{k}^*) = 0$ Since $H^1(\mathbb{Z}, Pic_0(\bar{Y}) = 0)$ by Lang's theorem, $H^1(\mathbb{Z}, Pic(\bar{Y}) \xrightarrow{\sim} H^1(\mathbb{Z}, \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}$. The statement about $H^0$ is clear.

Now let $X$ be arbitrary. We have $U(\bar{Y}) = \bar{k}^*$ and the exact sequences:

$$0 \to U(\bar{Y}) \to U(\bar{Y} - T) \to M \to 0$$

$$0 \to N \to Pic(\bar{Y}) \to Pic(\bar{Y} - T) \to 0$$

where $M$ and $N$ are $\mathbb{Z}$-modules which are finitely generated as abelian groups and so whose $\mathbb{Z}$-cohomology is also finitely generated. Since $H^q(\mathbb{Z}, \bar{k}^*)$ and $H^q(\mathbb{Z}, Pic(\bar{Y}))$ are both finitely generated for all $q$ the result follows.

## §4. Verdier duality for abelian groups.

In this section we state the surely well-known derived category version of duality in the category of abelian groups. Let $\mathcal{D}$ be the full subcategory of the bounded derived category $D^b(\mathbb{Z})$ of abelian groups consisting of those complexes with finitely-generated homology groups. If $A$ is in $\mathcal{D}$, let $A^*$ be $RHom(A, \mathbb{Z})$. Since $\mathbb{Z}$ has finite injective dimension, $A^*$ is again in $\mathcal{D}$.

We recall Theorem 10.8.7 of [Weib]: If $R$ is a commutative ring and $B$ is a bounded above complex of $R$-modules, then $\otimes^L_R B : D^-(R) \to D^-(R)$ is left adjoint to the functor $RHom_R(B, -) : D^+(R) \to D^+(R)$. That is, for $A$ in $D^-(R)$ and $C$ in $D^+(R)$ there is a natural isomorphism:

$$Hom_{D(R)}(A, RHom_R(B, C)) \xrightarrow{\sim} Hom_{D(R)}(A \otimes^L_R B, C)$$

Now let $R = \mathbb{Z}$, $B = A^*$,and $C = \mathbb{Z}$. By applying Weibel's Theorem 10.8.7 twice, we obtain

$$Hom_{\mathcal{D}}(A, RHom(A^*, \mathbb{Z}) \xrightarrow{\sim} Hom_{\mathcal{D}}(A \otimes A^*, \mathbb{Z}) \xrightarrow{\sim} Hom_{\mathcal{D}}(A^* \otimes A, \mathbb{Z}) \xrightarrow{\sim} Hom_{\mathcal{D}}(A^*, A^*).$$

so there is a canonical map $\alpha$ in the derived category from $A$ to $A^{**}$, corresponding to the identity in $Hom_{\mathcal{D}}(A^*, A^*)$.

**Proposition 4.1.** *The map $\alpha$ is an isomorphism from $\mathcal{D}$ to $\mathcal{D}$.*

Proof. The proof is by an easy induction on the length $n$ of the complex $A$. If $n = 1$ it follows immediately, because if $F$ is free and finitely generated $Hom(F, \mathbb{Z})$ is dualizing, and if $M$ is finite, $Ext^1(M, \mathbb{Z}) \xrightarrow{\sim} Hom(M, \mathbb{Q}/\mathbb{Z})$ is dualizing.

## §.5 R-categories

Throughout this section, let $R$ be a commutative ring of finite global dimension, so every $R$-module has a finite projective and a finite injective resolution. If $\mathcal{B}$ is any additive category, let $K(\mathcal{B})$ be the category of complexes of objects of $\mathcal{B}$, up to homotopy. We are primarily interested in the case when $R = \mathbb{Z}$. Let $D(\mathbb{Z})$ be the derived category of the category of abelian groups.

We begin with some lemmas about the category of $R$-modules. We refer to [Weib] for basic definitions and results.

**Lemma 5.1.** *Let $P^\bullet = \cdots \to P_{i-1} \to P_i \to P_{i+1} \ldots$ be an acyclic complex of projective (resp. injective) $R$-modules (not necessarily satisfying any boundedness conditions). Then $P^\bullet$ is homotopic to zero.*

Proof. We do only the projective case. Let $d_i$ be the map from $P_i$ to $P_{i+1}$, and let $M_i$ be the image of $P_i$. We have the exact sequence $* : 0 \to M_i \to P_{1+1} \to M_{i+1} \to 0$. This implies that $pd(M_i) \leq \max(pd(M_{i+1}) - 1, 0)$, and hence by induction and finite global dimension that $pd(M_i) = 0$, so $M_i$ is projective for all $i$. This implies that $*$ splits and so $P^\bullet$ is homotopic to zero.

**Lemma 5.2.** *Let $\mathcal{B}$ be an abelian category with enough projectives (resp. injectives) and closed under arbitrary direct sums (resp. products). Then given any chain complex $A^\bullet$*

*in $\mathcal{B}$ there exists a chain complex $P^\bullet$ of projectives (resp. $I^\bullet$ of injectives) and a quasi-isomorphism $f : P^\bullet \to A^\bullet$ (resp.$A^\bullet \to I^\bullet$).*

Proof. Take a Cartan-Eilenberg resolution of $A^\bullet$ and then apply the associated single complex functor, using sums or products as the case may be. (See [Weib], Exercise 5.7.1 and Lemma 5.7.2).

**Lemma 5.3.** *(Compare to [H], Proposition 4.7)) let $\mathcal{A}$ be the category of $R$-modules. Let $\mathcal{I}$ (resp.$\mathcal{P}$) be the (additive) subcategory of injective (resp. projective) $R$-modules. Then the natural functors*

$$\alpha : K(\mathcal{I}) \to D(R)$$

$$\beta : K(\mathcal{P}) \to D(R)$$

*are equivalences of categories. In particular, two complexes of projective (resp. injective) $R$-modules are homotopic if and only if they are quasi-isomorphic (equal in the derived category).*

Proof. We give the argument only for injectives. Let $Q_{is}$ be the multiplicative system of quasi-isomorphisms in the category of complexes of $R$-modules. We note that $K(\mathcal{I}) \cap Q_{is}$ is a multiplicative system in $K(\mathcal{I})$, by [H], Prop. 4.2], and we observe that lemma 5.2 implies that condition (ii) of [H], prop. 3.3] is satisfied for $K(\mathcal{I}) \subset K(\mathcal{A})$ and $Q_{is}$. Hence the natural functor

$$D(\mathcal{I}) \to D(R)$$

is fully faithful. On the other hand, Lemma 5.1 shows that every quasi-isomorphism in $K(\mathcal{I})$ is an isomorphism, hence $K(\mathcal{I})$ is naturally isomorphic to $D(\mathcal{I})$. Now Lemma 5.2 shows that every object of $D(R)$ is isomorphic to an object coming from $K(\mathcal{I})$ so $\alpha$ is an equivalence of categories.

We recall the definition of $K$-projective and $K$-injective complexes of $R$-modules from [S]:

**Definition 5.4.** *Let $P^\bullet$ (resp. $I^\bullet$) be a complex of projective (resp. injective) $R$-modules. $P^\bullet$ is $K$-projective (resp. $I^\bullet$ is $K$-injective) if for every acyclic complex $A^\bullet$ of $R$-modules, the complex $Hom^\bullet(P^\bullet, A^\bullet)$ (resp. $Hom^\bullet(A^\bullet, I^\bullet)$), is also acyclic.*

A *$K$-projective resolution* (resp. *$K$-injective resolution*) of a complex $A^\bullet$ of $R$-modules is a quasi-isomorphism $f : P^\bullet \to A^\bullet$ (resp, $g : A^\bullet \to I^\bullet$), with $P^\bullet$ (resp, $I^\bullet$) $K$-projective (resp. $K$-injective).

In [S] Spaltenstein shows that $K$-projective and $K$-injective resolutions of arbitrary complexes of $R$-modules always exist for any ring $R$, and are unique up to homotopy, so serve in the usual way to define $RHom$.

**Proposition 5.5.** *Let $R$ be a commutative ring of finite global dimension. Then any complex of projective (resp. injective) $R$-modules is $K$-projective (resp. $K$-injective).*

Proof. Let $P^\bullet$ be a complex of projective $R$-modules. Let $f : Q^\bullet \to P^\bullet$ be a $K$-projective resolution of $P^\bullet$. Since $f$ is a quasi-isomorphism the cone $C(f)$ is an acyclic complex of projective $R$-modules, hence homotopically trivial, by Lemma 5.1. Any homotopically trivial complex of projective modules is obviously $K$-projective, and hence $P^\bullet$ is $K$-projective, because it is part of a distinguished triangle where the other two vertices are $K$-projective. The proof for injectives is identical.

**Definition 5.6.** *Let $A^\bullet$ and $B^\bullet$ be complexes of $R$-modules. We define $RHom(A^\bullet, B^\bullet)$ as follows: Let $\pi : P^\bullet \to A^\bullet$ be a projective resolution of $A^\bullet$. Let $i : B^\bullet \to I^\bullet$ be an injective resolution of $B^\bullet$. Let $RHom(A^\bullet, B^\bullet)$ be either $Hom^\bullet(P^\bullet, B^\bullet)$ or $Hom^\bullet(A^\bullet, I^\bullet)$*

By Proposition 5.5 we know that these two definitions are naturally equivalent and depend only on the classes of $A^\bullet$ and $B^\bullet$ in the derived category $D(R)$. Of course this yields the usual definition under the usual boundedness hypotheses and Spaltenstein's definition in general.

**Definition 5.7.** *An $R-category$ $\mathcal{C}$ is a collection of objects $Ob(\mathcal{C})$ such that for all $A$ and $B$ in $Ob(\mathcal{C})$ we have an object $RHom(A, B)$ in $D(\mathbb{Z})$ such that*

*1: for all triples of objects $A, B, C$ in $Ob(\mathcal{C})$there exists maps $\rho_{A,B,C}$ in $D(\mathbb{Z})$ mapping $RHom(A,B) \otimes^L RHom(B,C)$ to $RHom(A,C)$ satisfying the associativity condition*

$$\rho_{A,C,D} \circ \rho_{A,B,C} = \rho_{A,B,D} \circ \rho_{B,C,D}$$

**Lemma 5.8.** *Let $\mathcal{C}$ be an abelian category with enough injectives. Then the left bounded derived category $D^+(\mathcal{C})$ is an $R$-category.*

Proof. Let $A^\bullet, B^\bullet$ and $C^\bullet$ be three complexes representing objects in $D^+(\mathcal{C})$. We choose a quasi-isomorphism of $B^\bullet$ (resp.$C^\bullet$)into a complex $I^\bullet$ (resp. $J^\bullet$ of injective objects, and compose the natural map from $Hom(A^\bullet, I^\bullet) \otimes^L Hom(I^\bullet, J^\bullet)$ to $Hom(A^\bullet, I^\bullet) \otimes Hom(I^\bullet, J^\bullet)$ with the natural map induced by composition of homomorphisms to $Hom(A^\bullet, J^\bullet)$. This is clearly associative.

**Proposition 5.9.** *Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories with enough injectives. Let $F$ be a covariant left exact functor from $\mathcal{C}$ to $\mathcal{D}$, and let $A$ and $B$ be objects in $D^+(\mathcal{C})$. Then $F$ is an $R$-functor, i.e. $F$ induces a map in $D(\mathbb{Z})$ from $RHom(A,B)$ to $RHom(RF(A), RF(B))$, which is compatible with the respective $R$-category structures.*

Proof. Straightforward.

**Proposition 5.10.** *Let $A^\bullet, B^\bullet$ and $C^\bullet$ be three objects in $D(R)$.*

*a) There is a natural isomorphism*

$$RHom_R(A^\bullet \otimes^L_R B^\bullet, C^\bullet) \simeq RHom_R(A^\bullet, RHom_R(B^\bullet, C^\bullet))$$

.

*b) There is a natural isomorphism*

$$Hom_{D(R)}(A^\bullet \otimes^L_R B^\bullet, C^\bullet) \simeq Hom_{D(R)}(A^\bullet, RHom_R(B^\bullet, C^\bullet))$$

.

Proof. a) is immediate by taking projective resolutions $P^\bullet$ of $A^\bullet$ and $Q^\bullet$ of $B^\bullet$ and recalling that $Hom^\bullet_R(P^\bullet \otimes_R Q^\bullet, C^\bullet)$ is naturally isomorphic to $Hom^\bullet_R(P^\bullet, Hom^\bullet_R(Q^\bullet, C^\bullet))$. Then b) follows by applying $H^0$ to a).

## §6. The duality theorem for non-singular curves.

Let $U$ be a smooth geometrically connected curve over a finite field $k$, let $\mathcal{S}h_U$ be the abelian category of Weil-étale sheaves on $U$, and let $\Gamma_U(F) = H^0_{\mathcal{W}}(U, F)$ . Let $j : U \to X$ be an open dense embedding of $U$ in a smooth projective curve $X$ over $k$, and let $F$ be an object of $\mathcal{S}h_X$. By Proposition 5.9, $\Gamma_X$ induces a map $R\Gamma_X$ in $D(\mathbb{Z})$ from $RHom_X(F, G_m)$ to $RHom_{D(\mathbb{Z})}(R\Gamma_X(F), R\Gamma_X(G_m))$.

Recall from Proposition 3.4 that $H^2_{\mathcal{W}}(X, G_m) = \mathbb{Z}$ and $H^q_{\mathcal{W}}(X, G_m) = 0$ for $q \geq 3$, which gives us a natural map in $D(\mathbb{Z})$ from $R\Gamma_X(G_m)$ to $\mathbb{Z}[-2]$. Now compose with the above map to get a map $\kappa, F$ from $RHom_X(F, G_m)$ to $RHom_{D(\mathbb{Z})}(R\Gamma_X(F), \mathbb{Z}[-2])$.

**Theorem 6.1.** *Let $F$ be either $j_!\mathbb{Z}$ or $j_!\mathbb{Z}/n\mathbb{Z}$. Then $\kappa, F$ is an isomorphism.*

Proof. We begin with the case $F = j_!\mathbb{Z}/n\mathbb{Z}$. Let $f : j_!\mathbb{Z}/n\mathbb{Z} \to I^\bullet$ and $g : G_m \to J^\bullet$ be injective resolutions. Recall from Section 2 that we have a functor $\psi$ mapping Weil-étale sheaves to étale sheaves which is left exact and takes injectives to injectives. Let $h : \psi(j_!\mathbb{Z}/n\mathbb{Z}) \to L^\bullet$ be an injective resolution. There is a map $\gamma : L^\bullet \to \psi(I^\bullet)$, unique up to homotopy, such that $\gamma h = \psi(f)$.

We first claim that there exists a commutative diagram in the derived category of abelian groups:

$$\begin{array}{ccccc}
Hom^\bullet_{WD}(j_!\mathbb{Z}/n\mathbb{Z}, J^\bullet) & \xrightarrow{(f^*)^{-1}} & Hom^\bullet_{WD}(I^\bullet, J^\bullet) & \xrightarrow{\Gamma} & Hom^\bullet_{D(\mathbb{Z})}(\Gamma(I^\bullet), \Gamma(J^\bullet)) \\
 & & \epsilon\Big\downarrow & & \delta\Big\downarrow \\
Hom^\bullet_{\acute{e}t}(\psi(j_!\mathbb{Z}/n\mathbb{Z}), \psi J^\bullet) & \longrightarrow & Hom^\bullet_{\acute{e}t}(\psi I^\bullet, \psi J^\bullet) & \longrightarrow & Hom^\bullet_{D(\mathbb{Z})}(\Gamma_{\acute{e}t}(\psi I^\bullet), \Gamma_{\acute{e}t}(\psi J^\bullet)) \\
 & & \rho\Big\downarrow & & \Big\downarrow \\
Hom^\bullet_{\acute{e}t}(\psi(j_!\mathbb{Z}/n\mathbb{Z}), \psi J^\bullet) & \xrightarrow{(h^*)^{-1}} & Hom^\bullet_{\acute{e}t}(L^\bullet, \psi J^\bullet) & \longrightarrow & Hom^\bullet_{D(\mathbb{Z})}(\Gamma_{\acute{e}t}(L^\bullet), \Gamma_{\acute{e}t}(\psi J^\bullet))
\end{array}$$

This makes sense because, since both $J^\bullet$ and $\psi J^\bullet$ are injective, $f^*$ and $h^*$ are quasi-isomorphisms. We next observe that $Hom^\bullet_{D(\mathbb{Z})}(\Gamma(I^\bullet), \Gamma(J^\bullet))$ is exactly $RHom_{D(\mathbb{Z})}(R\Gamma(j_!\mathbb{Z}/n\mathbb{Z}), R\Gamma(G_m))$ and $Hom^\bullet_{D(\mathbb{Z})}(\Gamma_{\acute{e}t}(L^\bullet), \Gamma_{\acute{e}t}(\psi J^\bullet))$ is exactly $RHom_{D(\mathbb{Z})}(R\Gamma_{\acute{e}t}(j_!\mathbb{Z}/n\mathbb{Z}), R\Gamma_{\acute{e}t} R\psi(G_m))$.

Since we may identify $Hom^{\bullet}_{WD}(j_!\mathbb{Z}/n\mathbb{Z}, J^{\bullet})$ with $\Gamma(U, J^{\bullet})_n$ and $Hom^{\bullet}_{\acute{e}t}(\psi(j_!\mathbb{Z}/n\mathbb{Z}), \psi J^{\bullet})$ with $\Gamma_{\acute{e}t}(U, \psi J^{\bullet})_n$, we see that the composite map $\rho\epsilon$ is an isomorphism in $D(\mathbb{Z})$.

We next look at the commutative diagrams:

$$\begin{array}{ccc} RHom_{\acute{e}t}(j_!\mathbb{Z}/n\mathbb{Z}, R\psi G_m) & \longrightarrow & RHom(R\Gamma_{\acute{e}t}(j_!\mathbb{Z}/n\mathbb{Z}), R\Gamma_{\acute{e}t}R\psi(G_m)) \\ \uparrow & & \uparrow \\ RHom_{\acute{e}t}(j_!\mathbb{Z}/n\mathbb{Z}, \psi G_m)) & \longrightarrow & RHom(R\Gamma_{\acute{e}t}(j_!\mathbb{Z}/n\mathbb{Z}), R\Gamma_{\acute{e}t}(\psi(G_m)) \end{array}$$

$$\begin{array}{ccc} RHom(R\Gamma_{\acute{e}t}(j_!\mathbb{Z}/n\mathbb{Z}), R\Gamma_{\acute{e}t}R\psi(G_m)) & \longrightarrow & RHom(R\Gamma_{\acute{e}t}(j_!\mathbb{Z}/n\mathbb{Z}), \mathbb{Z}[-2]) \\ \uparrow & & \uparrow \\ RHom(R\Gamma_{\acute{e}t}(j_!\mathbb{Z}/n\mathbb{Z}), R\Gamma_{\acute{e}t}(\psi(G_m)) & \longrightarrow & RHom(R\Gamma_{\acute{e}t}(j_!\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}/\mathbb{Z}[-3]) \end{array}$$

The first diagram is induced by the natural map from $\psi G_m$ to $R\psi G_m$, and is commutative by the R-functoriality of $R\Gamma$. Note that $R\Gamma = R\Gamma_{\acute{e}t}R\psi$.

The second diagram follows from the commutativity of the diagram

$$\begin{array}{ccc} R\Gamma(G_m) & \longrightarrow & \mathbb{Z} \\ \uparrow & & \uparrow \\ R\Gamma_{\acute{e}t}(G_m) & \longrightarrow & \mathbb{Q}/\mathbb{Z}[-1] \end{array}$$

This commutativity in turn follows from the natural identifications of $H^3_{\acute{e}t}(X, G_m)$ with $H^2(\hat{\mathbb{Z}}, Pic(\bar{X})) = \mathbb{Q}/\mathbb{Z}$, $H^2_{\mathcal{W}}(X, G_m)$ with $H^1(\mathbb{Z}, Pic(\bar{X})) = \mathbb{Z}$, and Lemma 1.2c.

Putting all this together, we get a commutative diagram:

$$\begin{array}{ccc} RHom_{\mathcal{W}}(j_!\mathbb{Z}/n\mathbb{Z}, G_m) & \xrightarrow{\kappa} & RHom_{D(\mathbb{Z})}(R\Gamma(j_!\mathbb{Z}/n\mathbb{Z}), \mathbb{Z}[-2]) \\ {\scriptstyle\lambda}\uparrow & & {\scriptstyle\mu}\uparrow \\ RHom_{\acute{e}t}(j_!\mathbb{Z}/n\mathbb{Z}, \psi(G_m)) & \xrightarrow{\nu} & RHom_{D(\mathbb{Z})}(R\Gamma_{\acute{e}t}(j_!\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}/\mathbb{Z}[-3]) \end{array}$$

We wish to complete the proof of the first part of the theorem ($\kappa$ is an isomorphism) by showing that $\lambda, \mu$ and $\nu$ are all isomorphisms.

Let $K^{\bullet}$ be an injective resolution of $\psi G_m$. Our map $\lambda$ is induced by a map $\rho$ from $K^{\bullet}$ to $\psi J^{\bullet}$, unique up to homotopy, which commutes with the obvious maps from $\psi G_m$

to $K^\bullet$ and $\psi J^\bullet$. We claim that the map induced by $\rho$ from $Hom_{ét}(j_!\mathbb{Z}/n\mathbb{Z}_{ét}, K^\bullet)$ to $Hom_{ét}(j_!\mathbb{Z}/n\mathbb{Z}_{ét}, \psi J^\bullet) = Hom_{\mathcal{W}}(\mathbb{Z}/n\mathbb{Z}, J^\bullet)$ is a quasi-isomorphism.

We have the two spectral sequences:

$$H^p_{ét}(X, \underline{Ext}^q_X(j!\mathbb{Z}/n\mathbb{Z})_{ét}, \psi G_m)) \Rightarrow Ext^{p+q}_{ét,X}((j_!\mathbb{Z}/n\mathbb{Z})_{ét}, \psi G_m)$$

$$H^p_{\mathcal{W}}(X, \underline{Ext}^q_X(j_!\mathbb{Z}/n\mathbb{Z}, G_m)) \Rightarrow Ext^{p+q}_{\mathcal{W},X}(j_!\mathbb{Z}/n\mathbb{Z}, G_m)$$

and a map between them induced by $\rho$. By Proposition 2.2(g) this map induces an isomorphism on the left-hand side of the spectral sequences and so also on the right-hand side, which completes the proof of the claim and shows that $\lambda$ is an isomorphism.

We next look at the map $\nu$. Let $\mathcal{A}$ be an abelian category with enough injectives and $\Gamma$ be a left exact functor from $\mathcal{A}$ to the category $\mathcal{A}b$ of abelian groups. Let $M$ and $N$ be objects of $\mathcal{A}$. Then the Yoneda product map from $Ext^p_{\mathcal{A}}(M, N)$ to $Hom_{\mathcal{A}b}(R^q\Gamma(M), R^{p+q}\Gamma(N))$ is described in terms of derived categories as follows:

Since $\Gamma$ induces an $R$-functor we have a map $R\Gamma$ from $RHom_{\mathcal{A}}(M, N)$ to $RHom_{\mathcal{A}b}(R\Gamma(M), R\Gamma(N)$. Let $S^\bullet$ and $T^\bullet$ be two complexes of abelian groups. By an easy variant of part of Exercise 3.6.1 of [Weib] we have a map:

$$h^p(RHom^\bullet(S^\bullet, T^\bullet)) \xrightarrow{\sigma_q} Hom(h^q(S^\bullet), h^{p+q}(T^\bullet))$$

The above Yoneda product map is obtained by composing $\sigma_q$ with $R\Gamma$, where $S^\bullet = R\Gamma(M)$ and $T^\bullet = R\Gamma(N)$. We now look at the commutative diagram:

$$\begin{array}{ccc} h^p(RHom_{D(\mathbb{Z})}(R\Gamma(j_!\mathbb{Z}/n\mathbb{Z}), R\Gamma(G_m))) & \xrightarrow{\beta_1} & h^p(RHom_{D(\mathbb{Z})}(R\Gamma(j_!\mathbb{Z}/n\mathbb{Z}), \mathbb{Q}/\mathbb{Z}[-3])) \\ \sigma_{3-p}\downarrow & & \sigma'_{3-p}\downarrow \\ Hom(H^{3-p}(j_!\mathbb{Z}/n\mathbb{Z}), H^3(G_m)) & \xrightarrow{\beta_2} & Hom(H^{3-p}(j_!\mathbb{Z}/n\mathbb{Z}), h^3(\mathbb{Q}/\mathbb{Z}[-3])) \end{array}$$

We have the Yoneda map from $Ext^p_{ét}(j_!\mathbb{Z}/n\mathbb{Z}, G_m)$ to $Hom(H^{3-p}(j_!\mathbb{Z}/n\mathbb{Z}), \mathbb{Q}/\mathbb{Z}))$ given by $\sigma_{3-p} \circ R\Gamma$. (Recall that $H^3_{ét}(G_m) = \mathbb{Q}/\mathbb{Z})$). It is immediate that $\sigma'_{3-p}$ and $\beta_2$ are isomorphisms. Since Deninger has shown [Den] that the Yoneda map is an isomorphism, it follows that $\nu = \beta_1 \circ R\Gamma$ is an isomorphism.

To show that $\mu$ is an isomorphism, it suffices to show that if $M^\bullet$ is any complex of abelian groups with torsion homology groups, then $RHom(M^\bullet, \mathbb{Q})$ is acyclic. This in turn follows immediately because $\mathbb{Q}$ is injective, and if $M$ is a torsion abelian group, $Hom(M, \mathbb{Q})$ is zero.

So we have shown that $\kappa, j_!\mathbb{Z}/n\mathbb{Z}$ is an isomorphism, and we want to completete the proof of the theorem by showing that $\kappa, j_!\mathbb{Z}$ is an isomorphism. We first point out the easy

**Lemma 6.2.** *Let $M$ and $N$ be finitely generated abelian groups and let $f$ be a map from $M$ to $N$ such that $f$ induces an isomorphism from $M/nM$ to $N/nN$ for every positive integer $n$. Then $f$ is an isomorphism.*

**Lemma 6.3.** *Let $M^\bullet$ and $N^\bullet$ be complexes of abelian groups and $g$ a map from $M^\bullet$ to $N^\bullet$. Assume:*

*1)The homology groups $h^i(M^\bullet)$ and $h^i(N^\bullet)$ are finitely generated for all $i$ and zero for $i$ large.*

*2) The map $g$ induces an isomorphism from $h^i(M^\bullet/nM^\bullet)$ to $h^i(N^\bullet/nN^\bullet$ for all integers $i$ and positive integers $n$. Then $g$ is a quasi-isomorphism of complexes.*

Proof. This is an straightforward descending induction on $i$, using Lemma 6.2.

**Lemma 6.4.** *Let $G$ be an object of $\mathcal{S}h_U$. There is a canonical isomorphism in $D(\mathbb{Z})$ between $RHom_X(j_!G, G_{m,X})$ and $RHom_U(G, G_{m,U})$.*

Proof. Because $j^*$ has the exact left adjoint $j_!$, $j^*$ takes injectives to injectives. Since $j^*$ is exact, it carries a resolution of $G_{m,X}$ to a resolution of $j^*G_{m,X} = G_{m,U}$. The first isomorphism then follows immediately from the adjointness of $j^*$ and $j_!$.

To complete the proof of Theorem 6.1, we need only point out that the cohomology groups in question are finitely generated and zero for large $i$ by Theorem 3.3, Proposition 3.4, and Lemma 6.4.

**Theorem 6.5.** *$R\Gamma_U(G_m)$ is naturally isomorphic to $RHom_{D(\mathbb{Z})}(R\Gamma_X(j_!\mathbb{Z}), \mathbb{Z}[-2])$*

Proof. Since $R\Gamma_U(F)$ is the same as $RHom_U(\mathbb{Z}, F)$, this follows immediately from Theorem 6.1 and Lemma 6.4.

## §7. The computation of some cohomology groups

In this section we will compute the Weil-étale cohomology of $G_m$ and $j_!\mathbb{Z}$ on a smooth curve $V$.

**Theorem 7.1.** *Let $X$ be a projective smooth geometrically connected curve over the finite field $k$. Then*

*a) $H^0_{\mathcal{W}}(X, G_m) = k^*$, $H^1_{\mathcal{W}}(X, G_m) = Pic(X)$, $H^2_{\mathcal{W}}(X, G_m) = \mathbb{Z}$, and $H^i_{\mathcal{W}}(X, G_m) = 0$ for $i \geq 3$.*

*b)$H^0_{\mathcal{W}}(X, \mathbb{Z}) = \mathbb{Z}$, $H^1_{\mathcal{W}}(X, \mathbb{Z}) = Hom(Pic(X), \mathbb{Z}) = \mathbb{Z}$, $H^2_{\mathcal{W}}(X, \mathbb{Z}) = Ext(Pic(X), \mathbb{Z})$ which is the Pontriagin dual of the finite group $Pic_0(X)$, $H^3_{\mathcal{W}}X, \mathbb{Z}) = Ext(k^*, \mathbb{Z})$ which is the Pontriagin dual of the finite group $k^*$, and $H^i_{\mathcal{W}}(X, \mathbb{Z}) = 0$ for $i \geq 4$.*

*Let $V = X\text{-}S$ be a quasiprojective smooth curve over the finite field $k$, with $X$ projective, smooth and geometrically connected, and $S$ finite and non-empty. Let $j : V \to X$ be the natural open immersion and $i : S \to X$ be the natural closed immersion. Then*

*c) $H^0_{\mathcal{W}}(U, G_m) = U(V) =$ the units of $V$. We have the exact sequence:*

$$0 \to Pic(V) \to H^1_{\mathcal{W}}(V, G_m) \to Hom(\coprod_S \mathbb{Z}/\mathbb{Z}, \mathbb{Z}) \to 0$$

*and $H^i_{\mathcal{W}}(V, G_m) = 0$ for $i \geq 2$.*

*d) $H^0_{\mathcal{W}}(X, j_!\mathbb{Z}) = 0$, $H^1_{\mathcal{W}}(X, j_!\mathbb{Z}) = \coprod_S \mathbb{Z}/\mathbb{Z}$. We have the exact sequence:*

$$0 \to Ext(Pic(V), \mathbb{Z}) \to H^2_{\mathcal{W}}(X, j_!\mathbb{Z}) \to Hom(U(V), \mathbb{Z}) \to 0$$

*and $H^3_{\mathcal{W}}(X, j_!\mathbb{Z}) = Ext(U(V), \mathbb{Z}) =$ the Pontriagin dual of $k^*$.*

Proof. We have proven a) as part of Proposition 3.4. For c) we claim that $H^i_{\mathcal{W}}(X, j_*G_m)$ is naturally isomorphic to $H^i_{\mathcal{W}}(U, G_m)$. We have the two spectral sequences following from Proposition 2.1:

$$H^p(\mathbb{Z}, H^q_{\acute{e}t}(\bar{X}, \bar{j}_* G_m)) \Rightarrow H^{p+q}_{\mathcal{W}}(X, j_* G_m)$$

$$H^p(\mathbb{Z}, H^q_{\acute{e}t}(\bar{U}, \bar{G}_m)) \Rightarrow H^{p+q}_{\mathcal{W}}(U, G_m)$$

Since $R^i \bar{j}_* G_m = 0$ for $i \geq 1$, the Leray spectral sequence degenerates, and we conclude that $H^p_{\acute{e}t}(\bar{X}, \bar{j}_* G_m) = H^p_{\acute{e}t}(\bar{U}, G_m)$. Hence the two above spectral sequences agree on their $E_2$-terms, and hence in the limit, which proves the claim.

From the exact sequence of sheaves

$$0 \to G_m \to j_* G_m \to i_* \mathbb{Z} \to 0$$

we get, using the above claim, the exact sequence

$$0 \to H^0_{\mathcal{W}}(X, G_m) \to H^0_{\mathcal{W}}(U, G_m) \to H^0_{\mathcal{W}}(S, \mathbb{Z}) \to H^1_{\mathcal{W}}(X, G_m) \to$$

$$H^1_{\mathcal{W}}(U, G_m) \to H^1_{\mathcal{W}}(S, \mathbb{Z}) \to H^2_{\mathcal{W}}(X, G_m) \to H^2_{\mathcal{W}}(U, G_m) \to 0$$

which easily yields

$$0 \to Pic(U) \to H^1_{\mathcal{W}}(U, G_m) \to Hom(\coprod_S \mathbb{Z}/\mathbb{Z}, \mathbb{Z}) \to 0$$

and $H^2_{\mathcal{W}}(U, G_m) = 0$, thus proving c).

We next recall from Weibel ([Weib], Exercise 3.6.1) that for $A^\bullet$ any complex of abelian groups, we have the exact sequence

$$0 \to Ext(h^{1-p}(A^\bullet), \mathbb{Z}) \to h^p(RHom(A^\bullet, \mathbb{Z})) \to Hom(h^{-p}(A^\bullet), \mathbb{Z}) \to 0$$

Now d) follows immediately from c), together with the duality theorem (Theorem 6.1) and Lemma 6.4.

## §.8 Values of zeta-functions at zero

In this section we will prove the conjecture given in the introduction for curves, for smooth projective varieties, and a modified form for smooth surfaces. Let $k = \mathbb{F}_q$ be a finite field. Let $\hat{\mathbb{Z}}$ be the Galois group of $\bar{k}$ over $k$, and let $\phi$ be the Frobenius element of $k$, which sends $x$ to $x^q$ and topologically generates $\hat{\mathbb{Z}}$. Let $\mathbb{Z}$ denote the subgroup of $\hat{\mathbb{Z}}$ generated by $\phi$. Let $\theta$ in $H^1_{\mathcal{W}}(k,\mathbb{Z}) = H^1(\mathbb{Z},\mathbb{Z}) = Hom(\mathbb{Z},\mathbb{Z})$ be the homomorphism which sends $\phi$ to 1.

We also denote by $\theta$ the pullback of $\theta$ to $H^1_{\mathcal{W}}(X,\mathbb{Z})$, where $X$ is any scheme over $k$. For any sheaf (or complex of sheaves) $F$ on the Weil-étale site of $X$, there is a natural pairing $F \otimes \mathbb{Z}$ to $F$ induced by $x \otimes n \to nx$. This pairing induces a map $\cup\theta$ from $H^i_{\mathcal{W}}(X,F)$ to $H^{i+1}_{\mathcal{W}}(X,F)$. Observe that since $\theta$ lies in $H^1$, $\theta \cup \theta = 0$, so cupping with $\theta$ makes the cohomology groups (or hypercohomology groups) $H^i_{\mathcal{W}}(X,F)$ into a complex.

We now restate the conjecture:

**Conjecture 8.1.** *Let $U$ be a geometrically connected quasi-projective variety over $k$, and let $j : U \to X$ be an open dense immersion of $U$ in a projective variety $X$. Then*

*a) The cohomology groups $H^i_{\mathcal{W}}(X, j_!\mathbb{Z})$ are finitely generated abelian groups, independent of the choice of open immersion $j$.*

*b) The alternating sum of the ranks of the cohomology groups $H^i_{\mathcal{W}}(X, j_!\mathbb{Z})$ is equal to zero.*

*c) The order of the zero of the zeta-function $Z(U,t)$ at $t = 1$ is given by the "secondary Euler characteristic" $\sum(-1)^i i r_i$, where $r_i = rank(H^i_{\mathcal{W}}(X, j_!\mathbb{Z}))$.*

*d) The homology groups $h^i_{\mathcal{W}}(X, j_!\mathbb{Z})$ of the complex $(H^\bullet_{\mathcal{W}}(X, j_!\mathbb{Z}), \theta)$ are finite.*

*Let $Z^*(U,1)$ be $lim_{t\to 1} Z(U,t)(1-t)$.*

*e) $Z^*(U,1)$ is equal, up to sign, to the alternating product $\chi(j_!\mathbb{Z})$ of the orders of these homology groups: $\prod_i o(h^i_{\mathcal{W}}(X, j_!\mathbb{Z}))^{(-1)^i}$.*

**Theorem 8.2.** *Conjecture 8.1 is true if $U$ is projective and smooth, or if $U$ is a smooth surface, or if $U$ is a curve. (If $U$ is a smooth surface, we must also take the projective surface $X$ containing $U$ to be smooth).*

We begin with the case where $U$ is projective and smooth, so $X = U$ and $j$ is the identity.

**Lemma 8.3.** *Let $U$ be projective and smooth. Then Theorem 8.2 is true for $U$ and the cohomology groups $H^i_{\mathcal{W}}(U, \mathbb{Z})$ are finite for $i \geq 2$.*

Proof of lemma. We start with the formula of Milne ([M2], Theorem 0.4a). :

$$Z^*(X,1) = o((H^2_{ét}(X,Z)_{cotor}) \prod_{i\geq 3}(o(H^i_{ét}(X,\mathbb{Z}))^{(-1)^i}$$

Note that we have switched here from Milne's use of $\zeta(X,s)$ to $Z(X,t)$, where $t = q^{-s}$, and corrected the obvious misprint ($q^{1-s}$ should be $q^{-s}$).

We should point out here that the formula of Milne combines the cohomological description of the zeta-function due to M. Artin and Grothendieck, a deep theorem of Gabber ([G]), Deligne's proof of the Riemann hypothesis for varieties over finite fields [Del], and some sophisticated p-adic computations due to Milne. Milne of course also shows that the groups $H^i_{ét}(X, \mathbb{Z})$ are finite for $i \geq 3$ and that the cotorsion quotient group of $H^2_{ét}(X, \mathbb{Z})$ is finite. In fact he shows that $H^2_{ét}(X, \mathbb{Z})$ is the direct sum of $\mathbb{Q}/\mathbb{Z}$ and a finite group.

A comparison of the two spectral sequences of Proposition 2.2f) shows that the natural map from $H^p_{ét}(X, \mathbb{Z})$ to $H^p_{\mathcal{W}}(X, \mathbb{Z})$ is an isomorphism for $p \geq 3$. We also have the exact sequence:

$$0 \to H^2(\hat{\mathbb{Z}}, H^0_{ét}(\bar{X}, \mathbb{Z})) \to H^2_{ét}(X, \mathbb{Z}) \to H^2_{\mathcal{W}}(X, \mathbb{Z}) \to 0$$

which identifies $H^2_{\mathcal{W}}(X, \mathbb{Z})$ with $H^2_{ét}(X, \mathbb{Z})_{cotor}$, and of course shows that its order is finite.

To show the identity of Milne's Euler characteristic with ours, it only remains to verify that the map from $H^0_{\mathcal{W}}(X, \mathbb{Z})$ to $H^1_{\mathcal{W}}(X, \mathbb{Z})$ given by cup product with $\theta$ is an isomorphism. But this cup product may be identified with the cup product map from $H^0(\mathbb{Z}, H^0_{ét}(\bar{X}, \mathbb{Z}))$ to $H^1(\mathbb{Z}, H^0_{ét}(\bar{X}, \mathbb{Z}))$, which is tautologically an isomorphism. So we have proved parts d) and e) of the theorem, and b) is an immediate consequence of d). Also, a) and c) follow immediately from Theorem 3.2

**Lemma 8.4.** *Let $U = X - Z$. Let $j$ be the open immersion of $U$ in $X$ and let $i$ be the closed immersion of $Z$ in $X$. Assume that Theorem 8.2 holds for $Z$ and that $X$ is projective and smooth. Assume also that the cohomology groups of $U$ with compact support $H^i_{\mathcal{W}}(X, j_!\mathbb{Z})$ are finitely generated and independent of the choice of $j$. Then Theorem 8.2 holds for $U$.*

Proof. It follows from Lemma 8.3 that Theorem 8.2 also holds for $X$ and the cohomology groups $H^i_{\mathcal{W}}(X,\mathbb{Z})$ are finite for $i \geq 2$. We have the exact sequence of Weil-étale sheaves on $X$:

$$0 \to j_!\mathbb{Z} \to \mathbb{Z} \to i_*\mathbb{Z} \to 0$$

Let $\alpha_i$ be the induced map from $H^i_{\mathcal{W}}(X, j_!\mathbb{Z})$ to $H^i_{\mathcal{W}}(X,\mathbb{Z})$ and $\beta_i$ be the induced map from $H^i_{\mathcal{W}}(X,\mathbb{Z})$ to $H^i_{\mathcal{W}}(X, i_*\mathbb{Z})$. Let $g_i$ be the the cup product map $(\cup\theta)$ from $H^i_{\mathcal{W}}(X,\mathbb{Z})$ to $H^{i+1}_{\mathcal{W}}(X,\mathbb{Z})$, $f_i$ the cup product map from $H^i_{\mathcal{W}}(X, j_!\mathbb{Z})$ to $H^{i+1}_{\mathcal{W}}(X, j_!\mathbb{Z})$, and $h_i$ the cup product map from $H^i_{\mathcal{W}}(X, i_*\mathbb{Z})$ to $H^{i+1}_{\mathcal{W}}(X, i_*\mathbb{Z})$. Let $A_i$ be the image of $\alpha_i$, $B_i$ be the image of $\beta_i$, and $C_i$ be the image of $\gamma_i$.

Let $\epsilon_i$ be the map induced by $f_i$ from $A_i$ to $A_{i+1}$, let $\delta_i$ be the map induced by $h_i$ from $B_i$ to $B_{i+1}$), and let $\rho_i$ be the map induced by $f_i$ from $C_i$ to $C_{i+1}$. Then we have the commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & H^0_{\mathcal{W}}(X,\mathbb{Z}) & \xrightarrow{\beta_0} & B_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle \delta_0} & & \\
0 & \longrightarrow & A_1 & \longrightarrow & H^1_{\mathcal{W}}(X,\mathbb{Z}) & \xrightarrow{\beta_1} & B_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle \delta_1} & & \\
0 & \longrightarrow & A_2 & \longrightarrow & H^2_{\mathcal{W}}(X,\mathbb{Z}) & \xrightarrow{\beta_2} & B_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}$$

Since we are assuming that $H^i_{\mathcal{W}}(X,\mathbb{Z})$ is finite for $i \geq 2$, we see that $A_i$ and $B_i$ are finite for $i \geq 2$. Let $H^i(\epsilon) = Ker(\epsilon_i)/Im(\epsilon_{i-1})$, and similarly for $H^i(\delta)$, $H^i(\rho)$, $H^i(f)$,$H^i(g)$, and

$H^i(h)$. Then $H^i(\epsilon)$ and $H^i(\delta)$ are finite for $i \geq 2$. It follows from the above commutative diagram that we have the long exact sequence

$$0 \to H^0(g) \to H^0(\delta) \to H^1(\epsilon) \to H^1(g) \to H^1(\delta) \to \dots$$

Similarly, we have the exact sequences

$$0 \to H^0(\delta) \to H^0(h) \to H^0(\rho) \to H^1(\delta) \to H^1(h) \to \dots$$

$$0 \to H^0(\rho) \to H^1(f) \to H^1(\epsilon) \to H^1(\rho) \to H^2(f) \to \dots$$

Since we are assuming the conjecture is true for $X$ and for $Z$ it follows that $H^i(g)$ and $H^i(h)$ are finite for all $i$. It then follows without difficulty that all the groups in the three above exact sequences are finite.

If $\lambda$ is any one of $f, g, h, \delta, \epsilon$ or $\rho$, define $\chi(\lambda)$ to be $\prod o(h^i(\lambda))^{(-1)^i}$. Then the three exact sequences yield $\chi(g) = \chi(\epsilon)\chi(\delta)$, $\chi(h) = \chi(\delta)\chi(\rho)$, and $\chi(\epsilon) = \chi(f)\chi(\rho)$, from which it follows that $\chi(g) = \chi(f)\chi(h)$. Since we also have that $Z^*(X,1) = Z^*(U,1)Z^*(Z,1)$, we have proven b), d) and e) of the Theorem for $U$, and we have assumed a) as part of the hypothesis. It remains to prove c).

**Lemma 8.5.** . *Let $U$ be as in Lemma 8.4. Then a) The order of the zero of $Z(U,t)$ at $t = 1$ is equal to $-\sum_i (-1)^i rank((H^i_{ét}(\bar{X}, j_!\mathbb{Z}))^{\mathbb{Z}})$. b) The groups $H^i_{ét}(\bar{X}, j_!\mathbb{Z})$ are semisimple at zero (a $\mathbb{Z}$-module $M$ is semis-simple at zero if the natural map from $M^{\mathbb{Z}}$ to $M_{\mathbb{Z}}$ is an isomorphism modulo torsion.)*

Proof. Since part c) of theorem 8.2 is true for $X$ and for $Z$, and taking $\mathbb{Z}$-invariants is exact modulo torsion, we get part a) of the lemma. The fact that the cohomology groups $H^i_{ét}(\bar{X}, i_*\mathbb{Z})$ are semi-simple at zero automatically implies (since $H^i_{ét}(\bar{X}, \mathbb{Z})$ is torsion for $i \geq 1$) that the groups $H^i_{ét}(\bar{X}, j_!\mathbb{Z})$ are also semi-simple at zero.

The proof of Lemma 8.4 now is concluded by the observation that a) and b) of Lemma 8.5 formally imply part c) of Theorem 8.2.

Now we return to the proof of the original theorem. Lemmas 8.3 and 8.4 immediately yield the theorem for smooth curves. Then an argument essentially identical to that used in the proof of Lemma 8.4 yields the result for all curves, by comparing an arbitrary curve to a smooth open dense subset. Finally the result for any smoth surface follows immediately by using Lemma 3.1, Lemma 8.3 and Lemma 8.4.

## References

[Del] Deligne, P. La conjecture de Weil II, I.H.E.S. Publ. Math. 52 (1980) 137-252

[Den] Deninger, C. On Artin-Verdier duality for function fields, Math. Z. 188 (1984) 91-100

[G] Gabber, O. Sur la torsion dans la cohomologie l-adique d'une variété, C.R. Acad. Sci. Paris Sér I Math. 297 (1983) 179-182

[H] Hartshorne, R. Residues and Duality, SLN 20 Springer-Verlag New York 1966

[M1] Milne, J.S. Etale Cohomology, Princeton University Press, Princeton NJ 1980

[M2] Milne, J.S. Values of zeta functions of varieties over finite fields, Amer. J. Math. 108 (1986) 292-360

[S] Spaltenstein, N. Resolutions of unbounded complexes, Compositio Math. 65 (1988) 121-154

[Weib] Weibel, C.A. An Introduction to Homological Algebra, Cambridge Studies in Advanced Mathematics 38 Cambridge University Press, Cambridge, England 1994

[Weil] Weil, A. Sur la théorie du corps de classes, J. Math. Soc. Japan 3 (1951) 1-35