Stable Billiard Paths on Polygons

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April 29, 2013

Contents

Chapter 1

Preface

This is an honor thesis of Yilong Yang, a senior math student at Brown University. This thesis is supposed to be mainly expository and partly original. This thesis is written under the supervision of Richard Schwartz.

In this thesis I plan to first introduce some general tools for studying billiards and flat structure, from the most basic setup of billiards to some more sophisticated tools like holonomy representation. These will be the contents for Chapter 2 and 3.

Then I will use these tools to present the proofs of known facts in the current literature. In particular, I shall present Patrick Hooper's proofs that there are no stable periodic billiard path in right triangles, and neither in isosceles triangles with base angle $\frac{\pi}{2n}$ where $n \geq 2$ is a power of 2. I shall also mention some corollaries about billiard path in rhombi and rectangles. These will be the first four sections of Chapter 4.

Finally, I will present my own work that a parallelogram with one angle $\frac{\pi}{4}$ and modulus 1 has no stable periodic billiard path. This will be in the last section of Chapter 4.

For prerequisites, Chapter 2 assumes only basic planar geometry. For most of Chapter 3 and 4, the reader should know what is a geodesic and a flat metric structure, and along with some basic knowledge in algebraic topology. In particular, the reader is supposed to be familiar with fundamental groups, covering spaces, deck transformation, singular homology theory, smooth manifolds, Riemannian metric on manifolds and geodesics. The discussion about Veech dichotomy and some part of Chapter 4 will require some knowledge of hyperbolic geometry.

The knowledge assumed for this thesis can be found on most textbooks on the subject of algebraic topology [1], smooth manifolds [2], differential geometry [3], hyperbolic geometry [4] and Riemann surfaces [5]. Any fact used outside of these sources will be either proven, or given a reference.

I want to thank my great advisors and friends Professors Richard Schwartz and Patrick Hooper who refer me to this amazing field and offer me constant support. Their program McBilliard and McBilliard2 are amazing sources of inspiration. They can be found on Hooper's website `http://wphooper.com/visual/web_mcb/` and `http://wphooper.com/visual/mcb2/`.

Chapter 2

Introduction to Mathematical Billiards

In this chapter I will try to present a basic setup of billiards. For more detailed introduction, see the book Geometry and Billiards by Serge Tabachnikov [6].

2.1 Basic Setup

A mathematical billiard consists of a domain (the billiard table) and a mass point (the billiard ball) that moves freely and without fraction in the domain. The mass point shall move in straight line until it hits the boundary of the domain. Then the movement of the ball will be reflected according to the law of reflection, i.e. angle of incidence equals angle of reflection. See Figure 2.1.1. The study of billiards is to figure out the nature of this motion, i.e. the trajectory of the billiard ball. We shall make the following formal definitions.

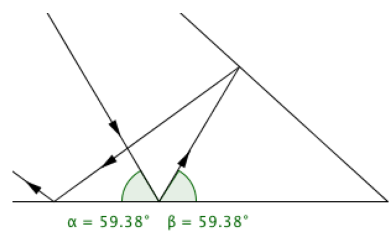

Figure 2.1.1: The trajectory of a bouncing billiard ball

Definition 2.1.1. A ***billiard table*** is a domain $T \in \mathbb{R}^2$ bounded by a simple closed piecewise differentiable curve γ. In this case γ is called the ***boundary curve***.

Definition 2.1.2. An ***orientable line segment*** is a smooth curve $\ell : [0,1] \to \mathbb{R}^2$ with zero second derivative. Its ***starting point*** is $\ell(0)$ and ***ending point*** is $\ell(1)$, and its ***vertices*** are the starting point and the ending point of it.

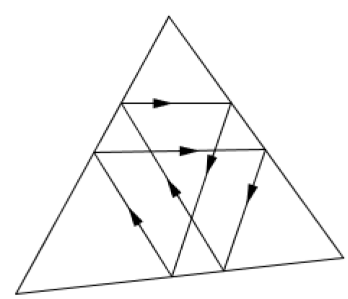

Figure 2.1.2: A periodic billiard path

Definition 2.1.3. A ***billiard path*** is a sequence of oriented line segments $\{s_i\}_{i\in\mathbb{Z}}$ inside of a billiard table T with boundary curve γ, such that the ending point s_i is the starting point of s_{i+1}, and all starting and ending points of these line segments are nonsingular points on γ. Finally, for each $i \in \mathbb{Z}$, let v be the ending point of s_i. Then the angles s_i and s_{i+1} make with the tangent line of γ at v are complementary, i.e. the motion from s_i to s_{i+1} obeys the law of reflection.

Definition 2.1.4. A billiard path is called ***periodic*** if the sequence $\{s_i\}$ is periodic. See Figure 2.1.2.

In this thesis we are interested in the case where the billiard table T is a polygon. For the sake of rigorousness we shall also give a definition of polygons here. This definition shall agree with all our knowledge about polygons in basic geometry.

Definition 2.1.5. A ***polygon*** is a domain $T \in \mathbb{R}^2$ bounded by a simple closed piecewise linear curve γ. An n***-gon*** is a polygon T whose boundary curve γ has exactly n linear pieces. These linear pieces will be called the ***edges*** of the polygon, and the singular points will be called the ***vertices*** of the polygon.

A major conjecture of the field is the following:

Conjecture 1. *Every polygon admits a periodic billiard path.*

However, proving this turns out to be far more difficult than expected. In fact, whether every triangle admits a periodic billiard path is still an open problem. The best known result so far is that every triangle with no angle more than 100 degree will have a periodic billiard path [7],[8]. The target of this honor thesis is to try attacking this conjecture with tools from differential geometry about geodesics, and algebraic topology about covering spaces and homology theory. The main idea would be to represent a billiard path on a billiard table as a geodesic on certain flat Riemannian surface, and construct a translation surface as a covering space of that flat surface.

2.2 Orbit Types and Unfolding

Definition 2.2.1. A ***labeling*** of an n-gon is a bijection between the set of edges of the n-gon to the set $\mathbb{Z}/n\mathbb{Z} = \{1, 2, 3, ..., n\}$, where adjacent edges (i.e. edges sharing a vertex) are sent to adjacent numbers in $\mathbb{Z}/n\mathbb{Z}$. A polygon with a labeling is called a ***labeled polygon***.

In the future, we shall treat all polygons as labeled.

Definition 2.2.2. For an n-gon $T \in \mathbb{R}^2$ and any billiard path $\{s_i\}$ in it, let w_i be the label of the edge containing the ending point of s_i for each $i \in \mathbb{Z}$. Then $\{w_i\}$ is a sequence of labels. This is called the ***orbit type*** of the billiard path.

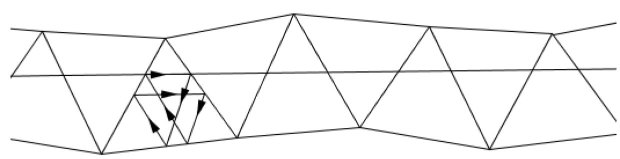

Figure 2.2.1: The unfolding of the triangle in Figure 2.1.2 according to the orbit type 123123

For example, the orbit type of the path in Figure 2.1.2 is 123123. Clearly the orbit type of a periodic billiard path is periodic. We will show that the converse is also true, see Proposition 2.2.6. We will also show that two billiard paths are "almost the same" if they have the same orbit type, see Proposition 2.2.5.

To prove the two propositions, we need an important tool, the unfolding. When the billiard ball in a polygon hits the boundary, instead of reflecting the motion of the ball, we can reflect the polygon about the edge it hits, and then allow the billiard ball to go straight through. In this way, we keep reflecting the polygon and the billiard path will be a straight line through this sequence of polygons. This sequence of polygons forms an unfolding. See Figure 2.2.1.

To be more formal, we shall give the following definition.

Definition 2.2.3. Given a polygon $T \in \mathbb{R}^2$ and any sequence of labels of edges $\{w_i\}_{i \in \mathbb{Z}}$, let T_0 be the polygon T, and for each i, we let T_{i+1} be the polygon obtained by reflecting T_i about the edge with label w_i. This way we inductively construct a sequence of polygons $\{T_i\}$, which is called the ***unfolding corresponding to the label sequence***. Take their union in $\mathbb{R}^2$, we have $D = \bigcup_{i \in \mathbb{Z}} T_i$ which is the ***unfolding domain***. If $\{w_i\}$ happens to be the orbit type for a billiard path $\{s_i\}$ of T, then we also say this is the ***unfolding corresponds to the billiard path***.

Definition 2.2.4. Let $\{s_i\}$ be a billiard path in T with corresponding unfolding $\{T_i\}$. Then each T_j has a corresponding billiard path $\{s_{i,j}\}$. Then the ***unfolding representation of*** $\{s_i\}$ is the union $L = \bigcup_{i \in \mathbb{Z}} s_{i,i}$, which would be a straight line contained in the unfolding domain D.

Proposition 2.2.5. *In a polygon T, if two billiard paths $\{s_i\}$ and $\{\ell_i\}$ have the same orbit type, then s_i is parallel to ℓ_i for all $i \in \mathbb{Z}$.*

Proof. Suppose the orbit type is $\{w_i\}$. Let $\{T_i\}$ be the corresponding unfolding, and let D be the unfolding domain. WLOG we can rotate and translate everything so that the unfolding representation of $\{s_i\}$ is the straight line coincide with the x-axis. Let $L \subset D$ be the unfolding representation of $\{\ell_i\}$.

Now T_i is the same polygons with the same size for all i, so let $d > 0$ be the diameter of T_i for all i. Suppose L is not horizontal. Then it can be parameterized by $x = ay + b$ for some $a, b \in \mathbb{R}$. Then we can find a point (x_0, y_0) on L with $y_0 > d$ and $x_0 = ay_0 + b$. Then as $L \in D$, $(x_0, y_0) \in T_i$ for some $i \in \mathbb{Z}$. However, as T_i has diameter d and it intersects with the positive x-axis, all points in T_i will have y coordinates less than or equal to d, contradiction. So we conclude that L is parallel to the x-axis. Then our statement is true. $\square$

The above proposition shows that an orbit type determines the billiard path "up to translation". So we can classify billiard paths first by their orbit types.

Proposition 2.2.6. *In a polygon T, a billiard path is periodic iff its orbit type is periodic.*

Proof. The necessity is clear. To see sufficiency, let $\{s_i\}$ be any billiard path with periodic orbit type $\{w_i\}$ with minimal period p. Now let $k = p$ if p is even and $k = 2p$ if p is odd, so that k is always even. k will be referred to as the minimal even period in the future. Let $\{T_i\}$ be the unfolding corresponding to $\{w_i\}$, D be the unfolding domain, and L be the unfolding representation of $\{s_i\}$. Now for each T_i, let c_i be its centroid. As k is even, T_0 and T_k should have the same orientation, i.e. we can obtain T_k from T_0 by a translation plus rotation. Let $t : \mathbb{R}^2 \to \mathbb{R}^2$ be the translation, and $r : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation centered on c_k such that $f = r \circ t$ will send T_0 to T_k. Then by periodicity, f^n would send T_0 to T_{nk}.

We connect the points $..., c_{-2k}, c_{-k}, c_0, c_k, c_{2k}, c_{3k}, ...$ by line segments. Suppose r is not a rotation by a multiple of 2π, then $\angle c_0c_kc_{2k}$ is not a multiple of 2π. Then the points c_0, c_k, c_{2k} will determine a circle S. (In case $\angle c_0c_kc_{2k}$ is an odd multiple of 2π, c_0 and c_2k will coincide, so S will be the circle with diameter c_0c_k.) Now because f is a rigid motion and $f^n(c_0c_k) = c_{nk}c_{(n+1)k}$ and $f^n(\angle c_0c_kc_{2k}) = \angle c_{nk}c_{(n+1)k}c_{(n+2)k}$, we can see that c_{nk} lies on S for all $n \in \mathbb{Z}$. Now let o be the center of S, and let $d = \sup\{|p - o| : p \in \bigcup_{i=0}^{k} T_i\}$, which exists as $\bigcup_{i=0}^{k} T_i$ is compact. Then by periodicity we have $d = \sup\{|p - o| : p \in \bigcup_{i\in\mathbb{Z}} T_i = D\}$. Let S' be the closed ball centered at o with radius d, then $D \subset S'$ must be bounded. However, the straight line L is contained in D and cannot be bounded, contradiction. So, r must be a rotation by a multiple of 2π. We conclude that T_0 and T_k differ only by a translation.

Now s_0 ends in the edge w_0, and s_k ends in the edge $w_k = w_0$ in the same direction. Rotate and scale everything so that the edge w_0 of T is the line segment from $(0, 0)$ to $(1, 0)$. Let s_0 ends in $(a, 0)$ and s_k ends in $(b, 0)$, and suppose $a \neq b$. WLOG let $b > a$. Then by periodicity of the unfolding, s_{2k} would ends in $(b + (b - a), 0)$, and s_{nk} would ends in $(a + n(b - a), 0)$. For n large enough, we would have $a + n(b - a) > 1$, then s_{nk} would be out of the polygon T_{nk}, contradiction. So, we can only have $a = b$. So s_0 and s_k would be line segments in T ending in the same spot with the same direction. So the billiard path would repeat itself ever since. So the billiard path is periodic. $\square$

Corollary 2.2.7. *If a billiard path is periodic, let $\{T_i\}$ be the corresponding unfolding, and let p be the minimal even period. Then T_p can be obtained from T_0 by a translation, and this translation is in the direction of the unfolding representation L of the billiard path.*

2.3 Space of Labeled n-gons and Stable Periodic Billiard Path

Given a particular polygon, there are many ways to label its edges, and different ways of labeling will give the same billiard path different orbit types. So to study billiard paths by the orbit type, it would be convenient to have a space of labeled n-gons.

Definition 2.3.1. The ***space of labeled n-gons*** is $\widetilde{P}_n = \{n\text{-gons in } \mathbb{R}^2 \text{ with a labeling}\}$.

Proposition 2.3.2. *$\widetilde{P}_n$ can be seen as an open subset of $\mathbb{R}^{2n}$.*

Proof. Consider the map $f : \widetilde{P}_n \to \mathbb{R}^{2n}$ such that for each labeled polygon T, let $v_i \in \mathbb{R}^2$ be the vertex between edges i and edges $i+1$, then $f(T) = (v_1, ..., v_n) \in \mathbb{R}^{2n}$. Since each polygon is determined by its vertices, and the ordering of its vertices determines its labeling, we see that this map is injective. So $\widetilde{P}_n$ can be seen as a subset of $\mathbb{R}^{2n}$.

Now if the collection of n vertices $v = (v_1, ..., v_n)$ form a labeled polygon, let $r = \frac{1}{2}\min_{i,j}|v_i - v_j|$. Let B_i be the open ball in $\mathbb{R}^2$ with center v_i and radius r, and consider the open set $R = \Pi B_i$. For any $p = (p_1, ..., p_n) \in R$, then $p_i \in B_i$ for all i, and consider the polygonal curve γ by joining $p_1p_2, ..., p_np_1$. This is a closed piecewise linear curve. Now if v_iv_{i+1} and v_jv_{j+1} are disjoint line segments, then the vertices of these two line segments are at least $2r$ units apart, so points on one of the line segments will be at least $2r$ apart from points on the other. So p_ip_{i+1} and p_jp_{j+1} cannot intersect. So γ indeed determines a polygon T. We give the edge with vertices p_n, p_1 the label 1, and give the edge with vertices p_i, p_{i+1} the label $i+1$. Then $p \in \widetilde{P}_n$. So $R \subset \widetilde{P}_n$. So $\widetilde{P}_n$ is open. $\square$

However, this space is too large for our purpose. For example, congruent labeled polygons will have the same configuration of billiard paths.

Definition 2.3.3. A function $f : \mathbb{R}^2 \to \mathbb{R}^2$ that is the composition of a rigid motion and a scaling is called a ***congruence maps***. For any unlabeled polygon T and T', if $f : T \to T'$ is a congruence map, then f would induce a bijection between edges of T and edges of T'. Let $\ell : \{\text{edges of } T\} \to \mathbb{Z}/n\mathbb{Z}$ be any labeling for T, then we will have an ***induced labeling*** for T', defined by $f_*\ell(a) = \ell \circ f(a)$ for each edge a of T'.

Definition 2.3.4. Two labeled polygons $T, T' \in \widetilde{P}_n$, with labeling ℓ, ℓ' respectively, are said to be ***congruent*** if there exists a congruence map $f : T \to T'$ such that $f_*\ell = \ell'$.

Proposition 2.3.5. *Congruence is an equivalence relation.*

Proof. It is reflexive because all $T \in \widetilde{P}_n$ is congruent to itself through the identity map. If T is congruent to T' through map f, and T' is congruent to T'' through map g, then T is congruent to T'' through map $g \circ f$. Finally, as rigid motions are invertible with inverse again a rigid motion, and the same is true for scaling, we see that a congruence map will have an inverse a congruence map. So if T is congruent to T' through map f, then T' is congruent to T through map f^{-1}. $\square$

Proposition 2.3.6. *If $T, T' \in \widetilde{P}_n$ are two congruent polygons, and $w = \{w_i\}$ is certain orbit type. Then T has a billiard path with orbit type w iff T' has a billiard path with orbit type w.*

Proof. Let $\{s_i\}$ be a billiard path for T with orbit type w, then $\{f(s_i)\}$ would be a billiard path for T'. The ending point of $f(s_i)$ would lie on the edge whose preimage on T has label w_i. Then this edge will also have label w_i. So $\{f(s_i)\}$ will have orbit type w as well. The other direction follows from the fact that congruence is an equivalence relation. $\square$

The above proposition says that we can work only on congruence classes of labeled polygons and ask if they have billiard paths of certain orbit type. So the main objects of interests are the following.

Definition 2.3.7. The ***space of labeled n-gons modulo congruence*** is $\mathcal{P}_n = \widetilde{P}_n/\sim$, where $\sim$ is the congruence relation among n-gons, and we endorse $\mathcal{P}_n$ with the quotient topology.

Proposition 2.3.8. *$\mathcal{P}_n$ can be seen as an open subset $\mathbb{R}^{2n-4}$.*

Proof. For each congruence class, pick any representative with vertices $(v_1, ..., v_n)$. Then we translate so that $v_1 = (0, 0)$, and rotate and scale so that $v_2 = (1, 0)$, and we finally reflect so that v_3 is in the upper half plane. This gives a map from $\mathcal{P}_n$ to an open subset of $\mathbb{R}^{2n-4}$. It is trivial to check that this map is a well-defined embedding. $\square$

Definition 2.3.9. Given an orbit type $w = \{w_i\}$, its ***orbit tile*** $T(w) \subset \mathcal{P}_n$ is the set of all congruence classes of labeled n-gons which have a billiard path with orbit type w. An orbit type is ***stable*** if its orbit tile is an open set. A periodic billiard path is ***stable*** if its orbit type is stable.

Now we can restate Conjecture 1.

Conjecture 2. *Given $n \geq 3$, $\bigcup_{periodic\ orbit\ types\ w} T(w) = \mathcal{P}_n$.*

Observe Figure 2.2.1 again. Clearly the unfolding domain has two "boundaries", an upper one and a lower one, both piecewise linear. The following definition and proposition formalize this observation.

Definition 2.3.10. Given a polygon T and a periodic orbit type $\{w_i\}$ with minimal even period p, let $\{T_i\}$ be the corresponding unfolding. For each T_i, let $bdry_i$ be the union of all its vertices and all its edges with edge-label different from w_{i-1} or w_i. The ***boundary of the unfolding domain*** D is the union of all $bdry_i$.

Proposition 2.3.11. *The boundary of the unfolding domain is always the union of two unique piecewise linear curve.*

Proof. For T_0, clearly $bdry_0$ has two connected components, both are piecewise linear curves (here we treat a single vertex as a piecewise linear curve). Let us denote these as u_0 and ℓ_0. Now suppose $\cup_{i=-n}^{n} bdry_i$ is the union of two disjoint piecewise linear curve, let the one contain u_0 be u_n and let the one contain ℓ_0 be ℓ_n. Then for T_{n+1}, $bdry_{n+1}$ also has two connected components, both piecewise linear, and they are connected to u_n, ℓ_n through the two vertices of the w_n edge of T_n respectively. So we can extend u_n and ℓ_n to include these two components respectively. Note that as no piece in $bdry_{n+1}$ can contain both vertices of w_n, this extension is unique. Similarly we also extend them uniquely to include the two pieces in $bdry_{-n-1}$. This way we obtain u_{n+1} and ℓ_{n+1}. Then we are done by induction. □

Definition 2.3.12. Given two piecewise linear curve $\gamma, \gamma' \subset \mathbb{R}^2$, we say they are ***separable*** if there is a straight line $L \subset \mathbb{R}^2$ such that γ and γ' are contained in the two distinct connected components of $\mathbb{R}^2 - L$.

Lemma 2.3.13 (Periodic Billiard Path Existence Lemma)**.** *Given any polygon T and any periodic orbit type $w = \{w_i\}$ with minimal even period p and unfolding $\{T_i\}$, then T admits a billiard path with orbit type w iff the two piecewise linear curves in the boundary of the unfolding domain are separable.*

Proof. First let us prove sufficiency. Let L be the unfolding representation of the billiard path $\{s_i\}$ whose orbit type is w. Then L is a straight line contained in the unfolding domain D. Since L only touch the edge w_{i-1} and w_i of T_i, it is disjoint from the two piecewise linear curves in the boundary of the unfolding domain, and since $L \subset D$, the two curve of D must be on different side of L, i.e. L separates the two piecewise linear curves in the boundary of the unfolding domain. Then we are done.

Now for necessity, let L be the line separating the two piecewise linear curves in the boundary of the unfolding domain D. Then clearly $L \subset D$, and for each T_i, L intersects T_i only with its w_{i-1} and w_i edges. Now clearly T_i and T are congruent as labeled polygons, so let $f_i : T_i \to T$ be a congruence map. Let $t_i = L \cap T_i$, and $s_i = f_i(t_i)$ for all i. I claim that $\{s_i\}$ is a periodic billiard path in T with orbit type w_i. Indeed, as f_i are label preserving, each s_i start in edge w_{i-1} and ends in edge w_i. Since T_i and T_{i+1} differs only by reflection $r : T_{i+1} \to T_i$ about w_i edge, we see that $f_{i+1} = f_i \circ r$. Now as r is a reflection, and t_i and t_{i+1} are on the same line L and they connect on the a single point on w_{i+1}, we see that t_i and $r(t_{i+1})$ hit the same point on the w_{i+1} edge of T_i, and their angles with w_{i+1} edge are complementary. So the same must be true for $f_i(t_i) = s_i$ and $f_i(r(t_{i+1})) = f_{i+1}(t_{i+1}) = s_{i+1}$. So $\{s_i\}$ is a periodic billiard path with orbit type w. □

The above lemma is nicely exemplified by Figure 2.2.1.

Now we can move on the discussion of stable periodic billiard path and the unstable ones. Intuitively, stable billiard path are those that survives any small perturbation of the polygon. So it is more effective to use stable periodic orbits to cover $\mathcal{P}_n$. However, there are many polygons without stable periodic billiard path, e.g. all right triangles [9]. So an understanding of the unstable ones is also important.

Proposition 2.3.14. *Given an unstable periodic orbit type with nonempty orbit tile, assuming the relation* $\sum_{i=1}^{n} \theta_i = (n-2)\pi$ *where* θ_i *are all the inner angles. There exists a unique nontrivial linear relation over* $\mathbb{R}$ *on inner angles such that any* n*-gon whose congruence class is in that orbit tile will satisfy this linear relation. Furthermore, the coefficients of this relation can be taken to be in* $\mathbb{Z}$*. On the other hand, for a stable periodic orbit type with nonempty orbit tile, there is no such linear relation.*

Proof. Fix an arbitrary periodic orbit type. For any n-gon T in its orbit tile, let θ_i be the angle between edges i and $i+1$ for all $i \in \mathbb{Z}/n\mathbb{Z}$. We will use $[s,t]$ to denote $\sum_{i\in\{s,s+1,...,t-1\}\subset\mathbb{Z}/n\mathbb{Z}} \theta_i$. The i-th vertex would be the vertex for the angle θ_i.

Let $w = \{w_1, ..., w_{2p}\}$ be a single minimal even period for the given periodic orbit type, and let $\{T_i\}$ be the unfolding for the orbit type. For each $k \in \mathbb{Z}^+$, T_{2k+1} is obtained from T_{2k-1} by reflecting through the edge w_{2k-1} and then reflecting through the edge w_{2k}. So from T_{2k-1} to T_{2k+1} would be a translation plus a rotation of degree $2[w_{2k-1}, w_{2k}]$. Now since T_1 to T_{2p+1} is just a translation as in Corollary 2.2.7, we have $\sum_{k=1}^{p} 2[w_{2k-1}, w_{2k}] = 0$. This is a linear relation with even integer coefficients on inner angles, and it is satisfied by all n-gons in the orbit tile of the given orbit type.

If this is trivial or equivalent to the relation that the sum of inner angles is $(n-2)\pi$, then for all n-gons T, let $\{T_i\}$ be the corresponding unfolding, then from T_0 to T_2p is a translation. Let P be any polygon whose congruence class is in the orbit tile. Let D be its unfolding domain. Then by the periodic billiard path existence lemma, the boundary of the unfolding domain is the union of two separable piecewise linear curves. Let L be such a separating line, and let $r > 0$ be the minimal distance from any point of the boundary of the unfolding domain to the line L.

Let Q be a polygon such that the i-th vertex of it is less than $\epsilon > 0$ away from the i-th vertex of P for all i, where ϵ is a small value to be determined. Note that the set of all such n-gons forms an open neighborhood of P. We translate, rotate and scale Q to obtain Q' such that the centroid of Q' is the same as P, and in the unfolding the centroid of Q'_{2p} is the same as P_{2p}. Note that the amount of translation and rotation for ϵ small enough can be made arbitrarily small. So the i-th vertex of Q' can be made arbitrarily near to the i-th vertex of P. Now once corresponding vertices of P_0, Q'_0 can be made arbitrarily near, then the corresponding vertices of P_1, Q'_1 can also be made arbitrarily near, and so forth. So we set $\epsilon > 0$ so small that the corresponding vertices of P_i, Q'_i are less than r away for all $i \in \{0, 1, ..., 2p-1\}$. This is possible since we only required this for finitely many i.

Now by periodicity, corresponding vertices of P_i, Q'_i are less than r away for all $i \in \mathbb{Z}$. Then the two piecewise linear curves in the boundary of unfolding domain for Q' will never touch the line L, and L will be in this unfolding domain. Then by periodic billiard path existence lemma, Q' has a periodic billiard path with the given orbit type, and thus does Q. So, the orbit tile for the orbit type is open. So the orbit type is stable.

Now if the linear relation $\sum_{k=1}^{p} 2[w_{2k-1}, w_{2k}] = 0$ is nontrivial and not equivalent with the relation that the sum of inner angles is $(n-2)\pi$. Then this linear relation gives a hypersurface of $\mathcal{P}$, and as the orbit tile is a subset of this hypersurface, it cannot be open. So the orbit type is unstable. However, by similar arguments as above we can see that the orbit tile is an open subset of this hypersurface.

Finally, suppose there is any other linear relation over $\mathbb{R}$ on inner angles satisfied by all polygons in the orbit type. If the orbit type is stable, this means that the orbit tile is contained in a hypersurface, and thus not open, contradiction. If the orbit type is unstable, then this means that the orbit tile is contained in two distinct hypersurfaces. Then their intersection cannot be open in each of the hypersurfaces, contradiction. So we are done. □

Definition 2.3.15. The linear relation in the above proposition is called the ***canonical linear relation*** for the orbit type.

Definition 2.3.16. A finite sequence of labels $(w_1, ..., w_p)$, where $w_i \in \{1, ..., n\}$, is said to be ***well-balanced*** if for each $k \in \{1, ..., n\}$, $\#\{w_i : i \text{ is even and } w_i = k\} = \#\{w_i : i \text{ is odd and } w_i = k\}$.

For example, the sequence 123123 is well-balanced, but 12341234 is not.

Proposition 2.3.17. *A periodic orbit type is stable iff a single minimal even period of it is well-balanced.*

Proof. Let $f(\theta_1, ..., \theta_n) = 0$ be the canonical linear relation for the orbit type $w = \{w_i\}$ with minimal even period $2p$, and let $w' = \{w_1, ..., w_{2p}\}$. We shall show that, assuming the relation $\sum_{i=1}^{n} \theta_i = (n-2)\pi$, w' is well-balanced iff $f \equiv 0$.

Now for any $a, b, c \in \mathbb{Z}/n\mathbb{Z}$, we clearly have by definition $[a, b]+[b, c] = [a, c]+S$ where S is a multiple of the sum $\sum_{i=1}^{n} \theta_i$. This sum is a multiple of π, so $2S$ is a multiple of 2π, and thus $2S = 0$. So $2[a, b]+2[b, c] = 2[a, c]$. As a result, if w' is well balanced, we have $f \equiv 0$.

On the other hand, suppose $f \equiv 0$. Let $[a, b]$ be an interval in f, i.e. $[a, b] = [w_{2k-1}, w_{2k}]$ for some k. Now θ_{b-1} is contained in even number of intervals, and so does θ_b, but $[a, b]$ does not contain θ_b any more. So, there exists $[b, c]$ an interval of f. Because f has finitely many intervals, keep doing this, we can eventually find an interval ending in a, and all these intervals we find sums up to 0. Now we throw these intervals away from f as they sum up to 0. if there is no other interval left, then clearly w' is well balanced. Otherwise we pick any interval left, and repeat the above process. As f has finitely many intervals, eventually there will be no interval left, and we see that w' is well-balanced. $\square$

Corollary 2.3.18. *If an orbit type has odd minimal period, then it is stable.*

Chapter 3

Billiards, Geodesics and Homology

3.1 The Fundamental Correspondence between Geodesics and Billiard Paths.

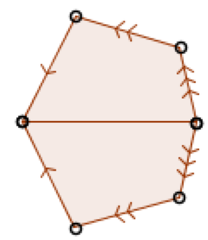

Figure 3.1.1: The double of a quadrilateral

The idea of billiards is to imitate the motion of light. We know light always travels through some form of geodesics. Therefore, one should expect some relation between billiard paths and geodesics. This chapter serves to illustrate this relation and some of its immediate consequences.

We start by building a flat Riemannian manifold corresponding to each polygon, the double of the polygon.

Definition 3.1.1. Given an n-gon P, its ***double*** $\mathcal{D}P$ is the union of two copies of P with boundary identified in the canonical way, and with vertices removed. See Figure 3.1.1. This is topologically a sphere with n punctures.

Definition 3.1.2. Given a polygon P, there is a natural ***folding map*** $\phi : \mathcal{D}P \to P$ that send each copy of P in $\mathcal{D}P$ to P by the identity map. The preimage of each edge of P is called an ***edge*** of $\mathcal{D}P$.

Proposition 3.1.3. *The double $\mathcal{D}P$ of a polygon P has a natural flat Riemannian manifold structure.*

Proof. The fact that $\mathcal{D}P$ is a smooth manifold is trivial. So we only need to construct the Riemannian metric.

Let the P_1, P_2 be the two copies of P without vertices in $\mathcal{D}P$. Let P_1', P_2' be the interior of P_1, P_2 respectively. Then for each edge w_i of P, let $U_i = P_1' \cup P_2' \cup w_i'$ where $w_i' \subset \mathcal{D}P$ is the edge of $\mathcal{D}P$ corresponding to w_i. Clearly these U_i are open subsets of $\mathcal{D}P$. However, if we reflect P about edge w_i to get Q, and we take the interior V of $P \cup Q$, then this open subset of $\mathbb{R}^2$ is clearly homeomorphic to U_i. Let $\phi_i : U_i \to V$ be this homeomorphism. Then we can pull back the flat metric structure from V to U_i.

Now we have a flat metric structure for all U_i. For any i, j, $U_i \cap U_j = P_1' \cup P_2'$, and the metric on U_i restricted to this intersection is clearly just the one induced by the metric on P. So the metric structure on these U_i agrees with each other, and this gives us a flat metric structure on $\mathcal{D}P$. □

The target of this section is the following theorem:

Theorem 3.1.4 (Fundamental correspondence between billiard paths and geodesics)**.** *For a polygon P, there is a one-to-one correspondence between complete geodesics on $\mathcal{D}P$ and billiard paths on P, such that: (1) A billiard path on P is periodic iff its corresponding geodesic on $\mathcal{D}P$ is closed. (2) A periodic billiard path on P is stable iff its corresponding closed geodesics on $\mathcal{D}P$ is null homologous.*

We shall break it into two parts.

Proposition 3.1.5. *For a polygon P, there is a one-to-one correspondence between complete geodesics on $\mathcal{D}P$ and billiard paths on P, such that a billiard path on P is periodic iff its corresponding geodesic on $\mathcal{D}P$ is closed.*

Proof. Step 1: Let us first find a complete geodesic for each billiard path. Fix a billiard path $\{s_i\}$ in P. Let P_1, P_2 be the two distinct copies of P without vertices in $\mathcal{D}P$, and let $j_1 : P_1 \to \mathcal{D}P, j_2 : P_2 \to \mathcal{D}P$ be the inclusion maps. Now $\{s_i\}$ can also be seen as billiard paths on P_1, P_2, so we can have geodesic segments $\{j_i(s_i)\}$ on $\mathcal{D}P$ where $j_i = j_1$ when i is odd, and $j_i = j_2$ when i is even. I claim that their union $\gamma = \bigcup_{i\in\mathbb{Z}} j_i(s_i)$ form a complete geodesics.

Now for each i, $j_i(s_i)$ is a geodesic segment ending in an edge in one copy of P in $\mathcal{D}P$. $j_{i+1}(s_{i+1})$ is a geodesic segment starting in the same edge at the same point. So all these s_i will connect to each other, and their union γ is a piecewise geodesic curve. Now suppose s_i ends in edge w_i, and we construct U_i as in Proposition 3.1.3, then the interior of the curve $j_i(s_i) \cup j_{i+1}(s_{i+1})$ would be a straight line segment in U_i. So γ satisfies the geodesic condition at the ending points of each $j_i(s_i)$ as well. So γ is a geodesic. To show that γ is a complete geodesic, we fix a point p, the starting point of s_0. Then we want to show that γ is bi-infinite from p, i.e. $\sum_{i\geq 0} length(s_i) = \sum_{i<0} length(s_i) = \infty$.

Suppose $\sum_{i\geq 0} length(s_i) < \infty$. Then we know $\lim_{i\to\infty} length(s_i) = 0$. Then $\forall \epsilon > 0 \exists N \in \mathbb{Z}^+$ such that $length(s_i) < \epsilon$ for all $i \geq N$. Now let D be the set of disjoint pair of edges of P. This is clearly finite. Set $\epsilon < \frac{1}{2}\min_{(w,w')\in D} dist(w, w')$ where $dist(w, w')$ is the minimal distance between any points on w and any points on w'. This is positive for $(w, w') \in D$ as w, w' are disjoint. Because D is finite, $\epsilon > 0$. Now we find the corresponding $N \in \mathbb{Z}^+$ such that $length(s_i) < \epsilon$ for all $i \geq N$. Suppose the line segment s_N in P has vertices on edge e and edge e', then as $length(s_N) < \epsilon < \min_{(w,w')\in D} dist(w, w')$, edges e and e' must share a vertex. Further more, as $length(s_i) + length(s_{i+1}) < 2\epsilon < \min_{(w,w')\in D} dist(w, w')$ for all $i \geq N$, we see that for all $i \geq N$, if s_i ends in edges e and e', then so does s_{i+1}. So by induction, all s_i ends in e, e' for $i \geq N$. Now let $\{T_i\}$ be the unfolding for $\{s_i\}$ and let L be the unfolding representation for $\{s_i\}$. Because s_i ends in e, e' for all $i \geq N$, from T_i to T_{i+1} after T_N is just reflection about e or e' alternatively. So, let θ be the inner angle between e, e', the motion from T_i to T_{i+2} after T_N is a simple rotation of degree 2θ around the vertex v between e, e'. Now for any integer $M > N$ with same parity as N, then T_M is a rotation around v with degree $(M - N)\theta$, and the straight line L goes through all T_i between T_N, T_M. So we must have $(M - N)\theta < \pi$. But let $M \to \infty$, we have $\theta = 0$, contradiction. So $\sum_{i\geq 0} length(s_i) = \infty$. Similarly we have $\sum_{i<0} length(s_i) = \infty$, so γ is a complete geodesic.

Step 2: Now for each complete geodesic γ on $\mathcal{D}P$, we can treat it as a function $\gamma : \mathbb{R} \to \mathcal{D}P$. Let $C = \{c \in \mathbb{R} : \gamma(c) \text{ is on an edge of } \mathcal{D}P\}$. Now for each point $p \in C$, $\gamma(p)$ is on an edge of $\mathcal{D}P$. Let U a small neighborhood of $\gamma(p)$ containing on other edges than the one containing $\gamma(p)$. Then $\gamma^{-1}(U)$ is a neighborhood of p. Furthermore, as the geodesic γ cannot touch two points on the same edge without going through other edges, we conclude that $\gamma^{-1}(U) \cap C = \{p\}$. So C is discrete. It is also closed because the union of all edges is closed in $\mathcal{D}P$ and C is the preimage of this union. So C is countable. Further more, because maximal geodesics in P have finite length and γ has infinite length, each time γ enters a copy of P, it must leave it eventually, so C cannot have upper bound or lower bound. So, if we index the element of C by their order, i.e. $C = \{c_i\}_{i\in\mathbb{Z}}$ with $c_i > c_j$ iff $i > j$, then we can define $s'_i = [c_i, c_{i+1}] \subset \mathbb{R}$, and $\bigcup s'_i = \mathbb{R}$. Let $\phi : \mathcal{D}P \to P$ be the folding map, and define $s_i = \phi \circ \gamma(s'_i)$, then I claim that $\{s_i\}$ is a billiard path in P. If so, then the map $\gamma \to \{s_i\}$ here is clearly an inverse of the correspondence in Step 1. So the correspondence between billiard paths and complete geodesics is indeed one-to-one.

For all i, s_i is clearly an oriented line segments, and its vertices are clearly on edges of P. The ending point of s_i is $\phi \circ \gamma(c_{i+1})$, which is exactly the starting point of s_{i+1}. Finally, let w_i be the edge containing the ending point of s_i, and we construct the corresponding open subset U_i of $\mathcal{D}P$ as in Proposition 3.1.3. Then as the interior of $\gamma(s'_i) \cup \gamma(s'_{i+1})$ is a part of γ, it is a geodesic in U_i. By the definition of the metric structure on U_i, after the folding map, the motion from s_i to s_{i+1} will automatically satisfy the law of reflection. So $\{s_i\}$ is indeed a billiard path.

Step 3: For a periodic billiard path $\{s_i\}$ with minimal even period $2k$, let γ be the corresponding geodesic in $\mathcal{D}P$ as constructed in Step 1. Then $j_0(s_0)$ is a geodesic segment in $P_2 \subset \mathcal{D}P$. Now as $2k$ is even, $j_{2k}(s_{2k})$ is also a geodesic segment in P_2. Finally, because $j_{2k} = j_0$ by definition and $s_0 = s_{2k}$, we have $j_0(s_0) = j_{2k}(s_{2k})$, so γ closed up.

Now suppose γ is a closed geodesic in $\mathcal{D}P$. Then the sequence $\{s_i'\}$ as in Step 2 is clearly periodic. So the billiard path $\{s_i\} = \{\phi \circ \gamma(s_i')\}$ is periodic. So, we have proven property (1) of the correspondence. $\square$

For the second part, we will need some more sophisticated tools introduced in the next section.

3.2 Holonomy Representation

We shall generalize the ideal of unfolding by constructing the universal covering space for the double of a polygon, and introduce the holonomy representation as a tool. For a more detailed introduction, see reference [10].

For any $n \geq 3$, let G be the free product of n copies of $\mathbb{Z}/2\mathbb{Z}$, and let the generators for these $\mathbb{Z}/2\mathbb{Z}$ be $g_1, ..., g_n$. Let P be any labeled n-gon. Then let $g_i(P)$ be the polygon obtained by reflecting P about its edge with label i. Now fix an n-gon P, and let P' be P without vertices. For each $g \in G$, if $g = a_1 a_2 ... a_k$ is its expression in generators $g_1, ..., g_n$, then $P_g = a_1 \circ ... \circ a_k(P')$. Let $G(P) = \bigsqcup_{g \in G} P_g$. Now we identify an edge of P_g with an edge of $P_{g'}$ iff the edges identified have the same label i, and $g' = g_i \cdot g$. After this identification, let's denote the result $\widetilde{\mathcal{D}P}$. See Figure 3.2.1.

Definition 3.2.1. In the construction above, the ***height*** of $g \in G$ is the minimal integer k such that we can express g as product of generators $g = a_1 ... a_k$. g has height 0 iff g is the identity element.

Proposition 3.2.2. *Each $g \in G$ with height $k > 0$ has a unique minimal expression $g = a_1 ... a_k$ in generators.*

Proof. Suppose $g = a_1 ... a_k = b_1 ... b_k$ are two minimal expression. Then $a_1 ... a_k b_k^{-1} ... b_1^{-1} = e$ the identity. But then as $a_1 ... a_k$ and $b_1 ... b_k$ are minimal the only cancellation has to be $a_k b_k^{-1} = 0$. After cancellation, for the same reason we must have $a_{k-1} b_{k-1}^{-1} = 0$, and so on. After finitely many steps, we have $a_i = b_i$ for all i. $\square$

Intuitively, the $\widetilde{\mathcal{D}P}$ constructed above is obtained by keep reflecting (unfolding) P in all possible ways, and glue these resulting polygons through their edge of reflection. Thus this is indeed a generalization of the unfolding. Now each reflection change orientation. We can define a map $\pi : \widetilde{\mathcal{D}P} \to \mathcal{D}P$ that maps positively oriented copies of P in $\widetilde{\mathcal{D}P}$ to one copy of P in $\mathcal{D}P$, and maps negatively oriented copies of P in $\widetilde{\mathcal{D}P}$ to the other copy of P in $\mathcal{D}$. This is indeed a covering map. Now we want to show that it's the universal cover.

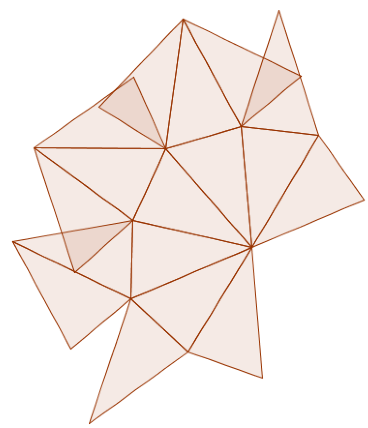

Figure 3.2.1: Part of the universal covering space of the double of a triangle, where triangles are treated as having no overlap. Vertices should be treated as thrown away.

Proposition 3.2.3. *In the construction above, $\widetilde{\mathcal{D}P}$ is the universal covering space of $\mathcal{D}P$. It has a natural flat Riemannian manifold structure induced by its covering of $\mathcal{D}P$.*

Proof. $\widetilde{\mathcal{D}P}$ is clearly a covering space of $\mathcal{D}P$ by construction, and the second statement is trivial. So we only need to show that $\widetilde{\mathcal{D}P}$ is simply connected. Let γ be any closed curves with base point $v \in P_e \subset \widetilde{\mathcal{D}P}$, where e is the identity element of G, i.e. P_e is P itself without vertices. Define the height k of γ be the maximal height of $g \in G$ such that $\gamma \cap int(P_g) \neq \varnothing$, where int means the interior. Such k is not infinity because a close curve is compact. We shall perform induction on k, the height of γ.

Let $g \in G$ be the one with height equal k, such that $\gamma \cap int(P_g) \neq \varnothing$. Now if γ is disjoint from the edges of P_g, then by continuity γ is entirely contained in P_g. So, because P_g is simply connected, $g = e$ and γ is null-homotopic. Note that this is the case $k = 0$.

If γ exits or enters P_g through an edge with label i of P_g, then γ intersects with $int(P_{g_i \cdot g})$. If $g = a_1 ... a_k$ is a minimal expression of g in generators, then $g_i \cdot g = g_i a_1 ... a_k$. But then as $g_i \cdot g$ cannot have height $k+1$, we have $g_i = a_1^{-1} = a_1$, so such i is unique. In other words, γ only exits or enters P_g through the edge with label i, and not with any other edge. Now let α be any connected component of $\gamma \cap P_g$. Then α must start from some point in i edge, and end in some point in i edge. As P_g is simply connected, α is homotopic to a curve α' starting from the same point and ending in the same point, but entirely contained in the i edge. Do this for all connected components of $\gamma \cap P_g$, we obtain a homotopy from γ to a closed curve γ' which has no intersection with $int(P_g)$. Keep doing this for all $g \in G$ with height equal k and $\gamma \cap int(P_g) \neq \varnothing$, we end up with γ homotopic to a closed loop γ' of height $k-1$. Then we are done by induction hypothesis. $\square$

Now we shall generalize the ideal of unfolding domain.

Proposition 3.2.4. *There is a locally isometric immersion of $\widetilde{\mathcal{D}P}$ into $\mathbb{R}^2$.*

Proof. Recall that P as a polygon can be seen as a subset of $\mathbb{R}^2$. Then this will induce an embedding of each P_g into $\mathbb{R}^2$ by performing the reflection required by g on $P \subset \mathbb{R}^2$. Furthermore, P_g and $P_{g\cdot g_i}$ are clearly embedded to share the edge i. So this induce a locally isometric immersion of $\widetilde{\mathcal{D}P}$ into $\mathbb{R}^2$. $\square$

Definition 3.2.5. The above immersion is called the developing map.

Corollary 3.2.6. *Given a polygon P and a billiard path $\{s_i\}$, let γ be the corresponding complete geodesic on $\mathcal{D}P$. Let $\widetilde{\gamma}$ be the lifting of γ to a geodesic on $\widetilde{\mathcal{D}P}$, and let $\{P_i\}_{i\in\mathbb{Z}}$ be the sequence of copies of P passed through by $\widetilde{\gamma}$ on $\widetilde{\mathcal{D}P}$. Then $dev(\bigcup_{i\in\mathbb{Z}} P_i)$ is the unfolding domain corresponding to the billiard path $\{s_i\}$.*

Proof. This is just by the definition of the developing map and the unfolding domain. $\square$

Given a periodic billiard path $\{s_i\}$ with minimal even period k in a polygon P, let $\{P_i\}$ be the unfolding. Then the rigid motion transforming P_0 to P_k is a translation. Now we want to generalize this idea and construct a map from closed curves on $\mathcal{D}P$ to orientation preserving isometries on R^2, i.e. rotation plus translation.

Lemma 3.2.7. *Let $Aut(\widetilde{\mathcal{D}P})$ be the group of deck transformations of $\widetilde{\mathcal{D}P}$ over $\mathcal{D}P$. Then for any $f \in Aut(\widetilde{\mathcal{D}P})$, $f(P_e) = P_g$ for some $g \in G$ with even height. Here e is the identity of G.*

Proof. Let P_0, P_1 be the two copies of P without vertices in $\mathcal{D}P$. If P_g is in the preimage of P_0, then all P_h adjacent to P_g will be in the preimage of P_1. The same is true the other way. Now WLOG suppose P_e is in the preimage of P_0. For any P_g in the preimage of P_0, from P_0 to P_g, we must go through even number of edges. So $g = a_1 ... a_k$ is an expression of g in generators with k even. Now all cancellations happen in pairs, so in the minimal expression of g in generators, the length of the expression is still an even number. So g has even height. Now f is a deck transformation, so it must send P_e to some P_g in the preimage of P_0, so g has even height. $\square$

Proposition 3.2.8. *There is a group homomorphism $hol : \pi_1(\mathcal{D}P) \to Isom^+(\mathbb{R}^2)$, such that for all $a \in \pi_1(\mathcal{D}P)$, the following diagram commutes:*

$$\begin{tikzcd}
\widetilde{\mathcal{D}P} \arrow[r, "\psi(a)"] \arrow[d, "dev"] & \widetilde{\mathcal{D}P} \arrow[d, "dev"] \\
\mathbb{R}^2 \arrow[r, "hol(a)"] & \mathbb{R}^2
\end{tikzcd}$$

Here dev is the developing map, ψ is the isomorphism from $\pi_1(\mathcal{D}P)$ to the deck transformations of the universal cover $\widetilde{\mathcal{D}P}$.

Proof. For any $a \in \pi_1(\mathcal{D}P)$, as $\psi(a)$ is a deck transformation from $\widetilde{\mathcal{D}P}$ to itself, it will send P_e to P_g for some $g \in G$ with even height. Here e is the identity in G as usual. Since g has even height, let $g = a_1 ... a_k$ be the minimal expression for g in generators with k even. Then recall that each generator g_i corresponds to a reflection about the edge i, so g corresponds to k reflections. As k is even, g corresponds to an element in $Isom^+(\mathbb{R}^2)$. Let this element be $hol(a)$. This expression is unique as the minimal expression of g in generators is unique. So $hol : \pi_1(\mathcal{D}P) \to Isom^+(\mathbb{R}^2)$ is well-defined. Also note that by the definition of the developing map, we have $dev(P_{g \cdot h}) = hol(a)dev(P_h)$ for all $h \in G$.

Now for each $h \in G$, we have $dev \circ \psi(a)(P_h) = dev(P_{g \cdot h}) = hol(a) \circ dev(P_h)$. Because for each $h \in G$, the developing map restricted to P_h is an isometric embedding of P_h into $\mathbb{R}^2$, we have a commutative diagram:

$$\begin{tikzcd} P_h \arrow[r, "\psi(a)"] \arrow[d, "dev"] & P_{g \cdot h} \arrow[d, "dev"] \\ \mathbb{R}^2 \arrow[r, "hol(a)"] & \mathbb{R}^2 \end{tikzcd}$$

Now because $\widetilde{\mathcal{D}P} = \bigcup_{h \in G} P_h$, we have the commutative diagram for $\widetilde{\mathcal{D}P}$ as desired.

Finally, we need to check group homomorphism. For any $a, b \in \pi_1(\mathcal{D}P)$, suppose $\psi(a)$ sends P_e to P_g and $\psi(b)$ send P_e to P_h. Then $\psi(a \cdot b) = \psi(a) \circ \psi(b)$ will send P_e to $P_{g \cdot h}$. Now the composition of reflections in g is $hol(a)$, and the composition of reflections in h is $hol(b)$. So the composition of reflections in $g \cdot h$ is $hol(a) \cdot hol(b)$. So $hol(a \cdot b) = hol(a) \cdot hol(b)$. □

Definition 3.2.9. The map $hol : \pi_1(\mathcal{D}P) \to Isom^+(\mathbb{R}^2)$ is the ***holonomy map*** of $\mathcal{D}P$.

Now we go further to examine the first homology group on $\mathcal{D}P$, and complete Step 4 in Theorem 3.1.4. We know $H_1(\mathcal{D}P, \mathbb{Z})$ is the abelianization of $\pi_1(\mathcal{D}P)$.

Proposition 3.2.10. *The abelianization of $Isom^+(\mathbb{R}^2)$ is the group on unit circle S^1.*

Proof. We know an element of $Isom^+(\mathbb{R}^2)$ is a translation plus rotation. So we have a map $ab : Isom^+(\mathbb{R}^2) \to S^1$ by forgetting the translation, i.e. $ker(ab)$ is the normal subgroup of all translations. Now because S^1 is an abelian group, we know the commutator subgroup C of $Isom^+(\mathbb{R}^2)$ must be contained in the kernel of ab. On the other hand, any $f \in ker(ab)$, then f is a translation. Let r_0 be the rotation by π around the origin of $\mathbb{R}^2$, and let f' be half of the translation of f. Then we have $f = r_0^{-1} f'^{-1} r_0 f' \in C$. So $C = ker(ab)$, and ab is indeed the abelianization. □

Definition 3.2.11. The ***holonomy representation for homology*** is the homomorphism $hol_{ab} : H_1(\mathcal{D}P, \mathbb{Z}) \to S^1$ descended through abelianization from the holonomy $hol : \pi_1(\mathcal{D}P) \to Isom^+(\mathbb{R}^2)$. i.e. if we let ab denote the abelianization map, we have a commutative diagram:

$$\begin{tikzcd} \pi_1(\mathcal{D}P) \arrow[r, "hol"] \arrow[d, "ab"] & Isom^+(\mathbb{R}^2) \arrow[d, "ab"] \\ H_1(\mathcal{D}P, \mathbb{Z}) \arrow[r, "hol_{ab}"] & S^1 \end{tikzcd}$$

Given an n-gon P, we know $\mathcal{D}P$ is topologically a sphere with n-punctures, each corresponding to a vertex of P. We can label vertices of P according to its labeling of edges, i.e. vertex i is the one between edge i and edge $i+1$. So, we can label punctures of $\mathcal{D}P$. In the following discussion, we shall let α_i be the homology class represented by a simple closed loop around the puncture i, such that each α_i starts inside P_0, enters P_1 through the edge i, and then exits P_1 through the edge $i+1$, and returns to its original position in P_0. We know $H_1(\mathcal{D}P, \mathbb{Z}) \simeq \bigoplus^{n-1} \mathbb{Z}$, and any $n-1$ of $\alpha_1, ..., \alpha_n$ would generate it.

Proposition 3.2.12. *For a polygon P, let θ_i be the inner angle for the vertex i. Then the holonomy representation for homology is defined by $\alpha_i \mapsto 2\theta_i$.*

Proof. Let P_0, P_1 be the two copies of P without vertices forming $\mathcal{D}P$, and let $\widetilde{\mathcal{D}P}$ be the covering space with P_e covering P_0. Then for each i, α_i is lifted to a curve $\widetilde{\alpha_i}$ from P_e to $P_{g_{i+1}\cdot g_i}$, so $hol(\widetilde{(\alpha_i)})$ is the reflection about edge i, then about edge $i+1$. This is a rotation around vertex i of P with radius $2\theta_i$. So we are done. □

Proposition 3.2.13. *For a periodic billiard path $\{s_i\}$ on a polygon P, let γ be the corresponding closed geodesic on $\mathcal{D}P$. Then $hol_{ab}(\gamma) = 0$. Furthermore, let $f(\theta_1, ..., \theta_n)$ be the canonical linear relation for the orbit type of $\{s_i\}$. Then $\gamma = \frac{1}{2}f(\alpha_1, ..., \alpha_n)$.*

Proof. Let $\widetilde{\gamma}$ be the geodesic segment by lifting of the geodesic γ from $\mathcal{D}P$ to $\widetilde{\mathcal{D}P}$, starting from a point in P_e and ending in P_g for some $g \in G$. Then by Corollary 3.2.6 and Corollary 2.2.7, the rigid motion transforming P_e to P_g is a translation. In particular, treating γ as a member of $H_1(\mathcal{D}P, \mathbb{Z})$, then $hol_{ab}(\gamma) = 0$.

Now let $\gamma = g(\alpha_1, ..., \alpha_n)$ for some linear relation g over $\mathbb{R}$. Then $0 = hol_{ab}(\gamma) = hol_{ab}(g(\alpha_1, ..., \alpha_n)) = g(hol_{ab}(\alpha_1), ..., hol_{ab}(\alpha_n)) = 2g(\theta_1, ..., \theta_n)$. This is a linear relation on inner angles with coefficients in $\mathbb{Z}$. Suppose it is trivial or equivalent to $\sum_{i=1}^n \theta_i = (n-2)\pi$, then g is either trivial or equivalent to the sum of its arguments. As $\sum_{i=1}^n \alpha_i = 0$ in the homology class, we have $\gamma = g(\alpha_1, ..., \alpha_n) = 0$, so it's null-homologous. Let $\{w_i\}$ be the orbit type of $\{s_i\}$ with minimal even period k. Then $w_1, ..., w_k$ is also the sequence of edges we hits by going along γ once on $\mathcal{D}P$. Note that this finite sequence $w_1, ..., w_k$ determines the homology class of γ, and it is well-balanced iff γ is null homologous. So if γ is null-homologous, the billiard path $\{s_i\}$ is stable, and so $f = 2g \equiv 0$ assuming $\sum_{i=1}^n \theta_i = (n-2)\pi$.

Suppose $2g(\theta_1, ..., \theta_n) = 0$ is not trivial or equivalent to $\sum_{i=1}^n \theta_i = (n-2)\pi$. Then this can only be the unique non-trivial canonical linear relation. So $f = 2g$. □

We finally have enough tools to prove the second part of the fundamental correspondence.

Proposition 3.2.14. *For a polygon P with a periodic billiard path $\{s_i\}$, let γ be the corresponding closed geodesic on $\mathcal{D}P$. Then γ is null-homologous iff $\{s_i\}$ is stable.*

Proof. Let $\{s_i\}$ be a periodic billiard path on P, and let γ be its corresponding closed geodesic on $\mathcal{D}P$. If the billiard path is not stable, then the canonical linear relation is not trivial, so γ is not null-homologous. On the other hand, if the billiard path is stable, then the canonical linear relation is trivial, so γ is null homologous. □

3.3 The Minimal Translation Surface

The universal cover of $\mathcal{D}P$ is a really large and complicated space. We shall construct a more elegant covering space for $\mathcal{D}P$, the minimal translation surface, which is easier to study. Then we shall introduce some basic properties of translation surfaces.

Definition 3.3.1. A ***translation surface*** is a flat Riemannian manifold obtained by taking the union of polygonal subsets of $\mathbb{R}^2$ with edges glued together pairwise by translation, such that each geodesic is a collection of line segments in the same direction. We throw away vertices as usual. See Figure 3.3.1

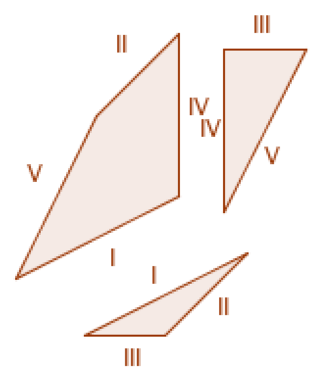

Figure 3.3.1: A translation surface. Edges with the same label are identified with translation, and vertices shall be treated a thrown away.

Translation surfaces are naturally Riemannian manifolds (in fact a Riemann surface with a chosen holomorphic 1-form). Their Riemannian metric is obtained by pulling back metrics on the polygons forming it. When we refers to a translation surface, we are referring to this Riemannian manifold structure, i.e. if two translation surfaces differ only by rigid motions, we will treat them as the same translation surface.

Definition 3.3.2. Let $ab : \pi_1(\mathcal{D}P) \to H_1(\mathcal{D}P, \mathbb{Z})$ be the abelianization map. Then the ***minimal translation surface*** for P is $MT(P) = \widetilde{\mathcal{D}P}/ker(hol_{ab} \circ ab)$.

Informally, the minimal translation surface is the surface we obtain if we identify all pairs of polygons in $\widetilde{\mathcal{D}P}$ if they differ only by translation. See Figure 3.3.2 for an example. It's therefore clearly a translation surface. In fact, it is the smallest one that covers $\mathcal{D}P$. We have the following proposition:

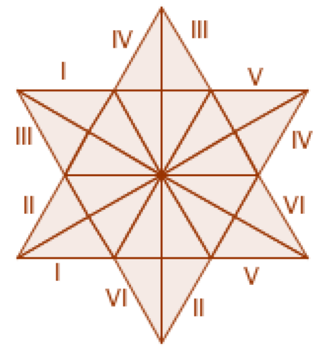

Figure 3.3.2: The minimal translation surface for the triangle with angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}$

Proposition 3.3.3. *If X is a connected translation surface that is also a covering space of $\mathcal{D}P$, then X is a covering space of $MT(P)$.*

Proof. We know $\widetilde{\mathcal{D}P}$ is the universal cover, so it is a covering space for X as well. Let $\phi_X : \widetilde{\mathcal{D}P} \to X$ and $\phi_{MT} : \widetilde{\mathcal{D}P} \to MT(P)$ be the covering maps. Let G be the free product of n copies of $\mathbb{Z}/2\mathbb{Z}$, and let $\{P_g\}_{g \in G}$ be the polygons making $\widetilde{\mathcal{D}P}$ as usual. Then if P_g, P_h are identified by ϕ_X, then as X is a translation surface, we must have that P_g, P_h differs only by translation. So ϕ_{MT} descends to a covering map $\phi : X \to MT(P)$. $\square$

Proposition 3.3.4. *Any closed geodesic in $\mathcal{D}P$ will lift to a closed geodesic on $MT(P)$.*

Proof. Let γ be a closed geodesic on $\mathcal{D}P$, then $hol(\gamma)$ is a translation. So lift γ to $\widetilde{\gamma}$ on $\widetilde{\mathcal{D}P}$ based inside P_e, we know by Corollary 3.2.13 that $\widetilde{\gamma}$ end in a polygon P_g which differs from P_e only by translation. So P_e and P_g will be identified in $MT(P)$. So γ lifts to a closed geodesic in $MT(P)$. $\square$

The main approach to study stability of billiards in this thesis is to find connections between geodesics on the minimal translation surface, and see how these connections affect their homology class down in $\mathcal{D}P$. The connections between geodesics here are mainly through affine automorphism of the minimal translation surface, which we shall now describe.

Definition 3.3.5. An ***affine diffeomorphism*** for a translation surface X is an orientation preserving diffeomorphism $\phi : X \to X$ to itself with constant derivative. These maps form the ***group of affine diffeomorphism*** of X, denoted by $\mathrm{Aff}(X)$.

Definition 3.3.6. The ***area*** of a translation surface $Area(X)$ is the sum of area of all the polygons forming it. This is clearly the area determined by the Riemannian metric structure on X.

Proposition 3.3.7. *The derivative of any affine diffeomorphism will be a constant matrix in $SL(2, \mathbb{R})$.*

Proof. Let A be the derivative of an affine diffeomorphism ϕ. Then as ϕ is an orientation preserving diffeomorphism, by definition $\det(A) > 0$. Now $Area(X) = Area(\phi(X)) = \det(A)Area(X)$, so as $Area(X) \neq 0$, we have $\det(A) = 1$. So $A \in SL(2, \mathbb{R})$. $\square$

Definition 3.3.8. The subgroup $\Gamma(X) = \{A \in SL(2, \mathbb{R}) : A \text{ is the derivative of some } \phi \in \mathrm{Aff}(X)\}$ of $SL(2, \mathbb{R})$ is the ***Veech group*** of X.

Now recall from hyperbolic geometry that the group $SL(2, \mathbb{R})$ acts on the hyperbolic plane $\mathbb{H}^2$. So we have the following definition:

Definition 3.3.9. A subgroup Γ of $SL(2, \mathbb{R})$ is a ***lattice*** if the quotient $\mathbb{H}^2/\Gamma$ has finite hyperbolic area.

Finally we have our major theorem connecting the Veech group of $MT(P)$ with billiards on P. Note that this is a paraphrase, as the original statement would require knowledge of ergodic theory. For a proof of this theorem, see reference [11].

Theorem 3.3.10 (Veech Dichotomy)**.** *Let X be a translation surface, and suppose $\Gamma(X)$ is a lattice. Then for any direction θ in $\mathbb{R}^2$, either all complete geodesics on X with direction θ is closed, or all complete geodesics on X with direction θ are dense on X.*

Corollary 3.3.11. *Given a polygon P with $\Gamma(MT(P))$ a lattice, and given a direction θ in $\mathbb{R}^2$. Then either all billiard paths $\{s_i\}$ with some s_i in direction θ are periodic, or all billiard paths $\{s_i\}$ with some s_i in direction θ are dense in P.*

We end this section by introducing one more tool to study translation surfaces, the cylinder decomposition and Dehn twists.

Definition 3.3.12. A ***cylinder*** is a parallelogram with one pair of parallel edges identified by translation, with vertices removed. The other pair of edges will become the ***upper boundary*** and ***lower boundary*** depending on the labeling of the parallelogram (edge with smaller $i \in \{1, 2, 3, 4\}$ as a label is the upper one). The ***modulus*** of the cylinder is the base-altitude ratio of the parallelogram forming it, where the base is either the upper or the lower boundary. A ***cylinder decomposition in direction*** θ of a translation surface X is a collection of cylinders $\{C_i\}_{i\in\mathbb{Z}/k\mathbb{Z}}$ such that all upper and lower boundaries of these cylinders have direction θ, and if we identify the lower boundary of each C_i with the upper boundary of C_{i+1} by translation, we recover our translation surface X. A cylinder decomposition is ***commensurable*** if the moduli of these C_i are rational multiples of each other.

Definition 3.3.13. A matrix $A \in SL(2,\mathbb{R})$ is ***parabolic*** if $\mathrm{tr}A = \pm 2$.

Proposition 3.3.14. *$A \in SL(2,\mathbb{R})$ is parabolic iff it is conjugate to either $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & k \\ 0 & -1 \end{pmatrix}$ for some $k \in \mathbb{R}$ in $SL(2,\mathbb{R})$.*

Proof. The necessity is clear, because trace of a matrix is invariant under conjugation. For sufficiency, let λ_1, λ_2 be its eigenvalues, then $\lambda_1 + \lambda_2 = 2$ and $\lambda_1\lambda_2 = 1$.

Suppose they are complex conjugates. Let $\lambda_1 = a + bi$ with $a, b \in \mathbb{R}$ and $b \neq 0$. Then $a^2 + b^2 = \lambda_1\lambda_2 = 1$, so $a^2 < 1$. Then $-2 < 2a < 2$, and $\lambda_1 + \lambda_2 \neq \pm 2$, contradiction. So $\lambda_1, \lambda_2 \in \mathbb{R}$. Then $\lambda_1\lambda_2 = 1$ and $\lambda_1 + \lambda_2 = \pm 2$ yield $\lambda_1 = \lambda_2 = \pm 1$. Now let J be the Jordan normal form of A and let $A = PJP^{-1}$ be the corresponding decomposition. Then J must be either $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & k \\ 0 & -1 \end{pmatrix}$ for some $k \in \mathbb{R}$ in $SL(2,\mathbb{R})$. Furthermore, because $\det P \neq 0$, we can let $P' = \frac{1}{\det P}P$. Then $A = P'JP'^{-1}$ with $P', P'^{-1} \in SL(2,\mathbb{R})$. So we are done. $\square$

Corollary 3.3.15. *A parabolic matrix $A \neq \pm I$ has a unique eigenvalue and any two eigenvectors are multiples of each other by a real number.*

Proposition 3.3.16. *Let P be a parallelogram making a cylinder C. Let v be the vector representing the identified pair of edges of P, and let u be the vector representing the other pair. Let A be the matrix such that $u \mapsto u$, $v \mapsto v + ku$ for some $k \in \mathbb{Z}$. Then $\pm A \in SL(2,\mathbb{R})$ and they are parabolic. And there is an orientation preserving diffeomorphism $\phi : C \to C$ with constant derivatives A and fix the boundaries of C, and another orientation preserving diffeomorphism $\phi' : C \to C$ with constant derivatives $-A$ and swapping the boundaries of C.*

Proof. Let's first prove the statement about matrix A. Find $B \in GL(2,\mathbb{R})$ such that $u \mapsto (1, 0)$ and $v \mapsto (0, 1)$. This is possible because, as u, v makes a parallelogram, u, v are linearly independent. Consider BAB^{-1}. This matrix maps $(1, 0)$ to $(1, 0)$ and maps $(0, 1)$ to $(k, 1)$. So this is $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, and it's conjugate to A. So $\pm A \in SL(2,\mathbb{R})$ and are parabolic.

Let A_n be the matrix in the proposition when $k = n$. Then clearly $A_n = (A_1)^n$ and $-A_n = -A_0(A_1)^n$ for all n, so it is enough to prove the proposition for A_1 and $-A_0$.

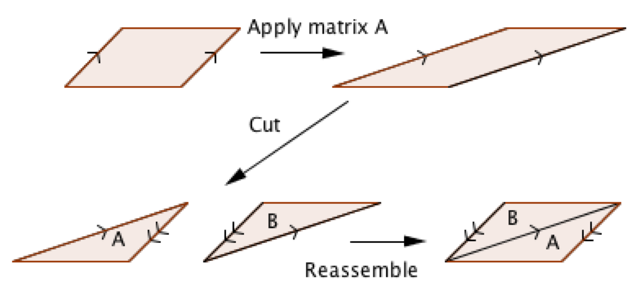

Figure 3.3.3: The Dehn twist on a cylinder formed by the parallelogram with vertices $(0, 0), (2, 0), (1, 1), (3, 1)$. Here the matrix A is parabolic and maps $(1, 0)$ to $(1, 0)$ and maps $(0, 1)$ to $(2, 1)$.

Now treat the parallelogram P as a subset of $\mathbb{R}^2$. By applying a rigid motion and scaling, we can assume that $u = (1, 0)$, $v = (x, y)$, and P has centroid on the origin. Then P is symmetric about the origin. Now $-A_0$ corresponds to the rotation of π around the origin, so apply $-A_0$ to P will clearly fix P and reverse the direction of u, v. So $-A_0$ induces an orientation preserving diffeomorphism ϕ_0' on C with derivative $-A_0$, and it swaps the upper and the lower boundaries of C.

Now applying a congruence map, we can assume that P has vertices $(0, 0), (1, 0), (x, y), (x + 1, y)$. Clearly $y \neq 0$, otherwise P will not be a parallelogram. Then $A_1 = \begin{pmatrix} 1 & \frac{1}{y} \\ 0 & 1 \end{pmatrix}$. So the parallelogram $A_1(P)$ will have vertices $(0, 0), (1, 0), (x + 1, y), (x + 2, y)$. Then by cutting and reassembling, we see that $A_1(P)$ and P actually form the same cylinder, See Figure 3.3.3. So A_1 induces an orientation preserving diffeomorphism ϕ_1 on C with derivative A_1 and fixing the boundaries of C. So we are done. □

Definition 3.3.17. The above diffeomorphism is the ***Dehn Twist*** of the cylinder. See Figure 3.3.3.

The following proposition is used in the proof of Veech Dichotomy, but it is also important for our purpose.

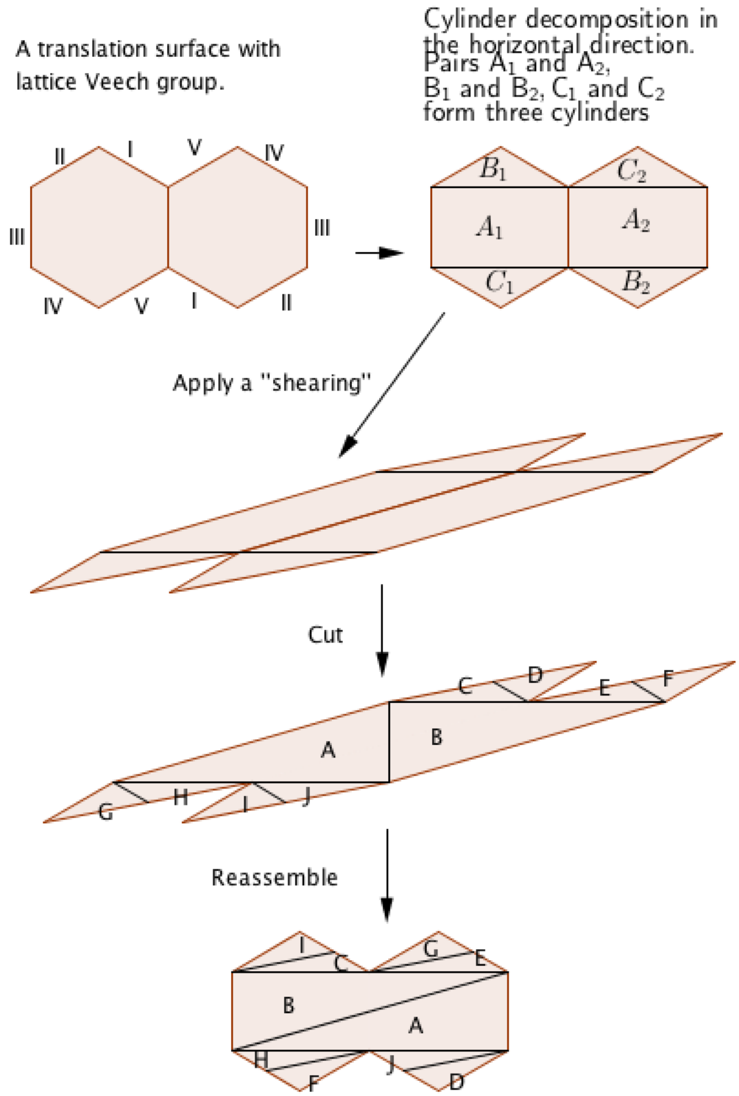

Figure 3.3.4: A Dehn twist for a translation surface

Proposition 3.3.18. *Let X be a translation surface with $\Gamma(X)$ a lattice. Then a matrix $A \in \Gamma(X)$ with $A \neq \pm I$ is parabolic iff there is a commensurable cylinder decomposition $\{C_i\}$ in the direction of an eigenvector of A, and the affine diffeomorphism $\phi \in Aff(X)$ with derivative A induces Dehn twists on each of these cylinders.*

Definition 3.3.19. By abuse of notation we also call the above affine diffeomorphism a ***Dehn twist*** for the translation surface X.

For an example, see Figure 3.3.4

Chapter 4

Some Applications

The most efficient way to cover up $\mathcal{P}_n$, the space of labeled n-gons up to congruence, is to use stable periodic billiard paths, as their orbit tiles are open. Therefore it is important to understand which polygon does not admit a stable periodic billiard path, which this chapter tries to explore.

I will show that right triangles have no stable periodic billiard paths, and no isosceles triangles with a base angle $\frac{\pi}{2n}$ with $n \geq 2$ a power of 2 has stable periodic billiard paths. Then as a corollary we have that no rhombi with one angle $\frac{\pi}{n}$ with $n \geq 2$ a power of 2 has stable periodic billiard paths. Both proofs about triangles are mainly following proofs of Patrick Hooper.

Then I shall show that no rectangles have stable periodic billiard paths. If the reader is willing to assume that squares have no stable periodic billiard paths, then this section is independent from section 4.1 and 4.2.

Finally, I shall present my own work proving that parallelograms with one angle $\frac{\pi}{4}$ and modulus 1 has no periodic billiard paths. This is independent from other sections as well.

All the proofs for the main theorems in the following sections have roughly the same general approach:
Step 1: We start by calculating at the minimal translation surface, and try to find a nice representation of the surface if possible. We shall also calculate the Veech group and the group of affine automorphisms.
Step 2: We try to use the affine automorphisms and the Veech group to build relations between closed geodesics and reduce the problem to finitely many cases. We do this usually through some form of enumeration theorems like Theorem 4.2.10 and 4.5.5, or by pairing up geodesics like Proposition 4.1.7.
Step 3: Finally we use homology and cohomology to obtain criteria for stability, and solve the problem by computing those finitely many cases left.

This is the main strategy used in Section 4.1, 4.2 and 4.5.

4.1 Right Triangles

The main theorem here is the following one:

Theorem 4.1.1 (Hooper [9])**.** *Right triangles have no stable periodic billiard path.*

To prove this theorem, we shall first focus on the right triangles whose non-right angles are irrational multiples of π. We shall first give an explicit description of its minimal translation surface.

Proposition 4.1.2. *Let T be a right triangle whose non-right angles are irrational multiples of π. Reflecting T about one of its legs, we obtain an isosceles triangle, and then reflecting this triangle about its base, we obtain a rhombus R_0. Let θ be an angle of R_0. Let R_i be R_0 rotated counterclockwise by $i\theta$ about the vertex at θ for all $i \in \mathbb{Z}$. These rhombi $\{R_i\}_{i\in\mathbb{Z}}$, with vertices and centroids removed, form a unique translation surface, and this surface is the minimal translation surface. $MT(T)$. See Figure 4.1.1.*

Proof. Let (a, a') and (b, b') be the two pairs of parallel edges of R_0. Then we can label edges of R_i accordingly such that it's compatible with the rotation. Now R_0 and R_1 have one edge in common, so WLOG suppose it is a of R_0 and b of R_1. Then a of R_i and b of R_{i+1} will be the same edge for all $i \in \mathbb{Z}$. As the two non-right angles of T are irrational multiples of π, we see that θ is an irrational multiple of π. In particular, R_i, R_j cannot differ only by translation when $i \neq j$. So only edges b, b' of R_{i+1} and a' of R_i will be in the same direction as a of R_i. We identify a of R_i with b of R_{i+1}, and a' of R_i with b' of R_{i+1}. We do this for all $i \in \mathbb{Z}$. This way, each edge of all R_i are identified to some other edge, and we obtain a connected translation surface X.

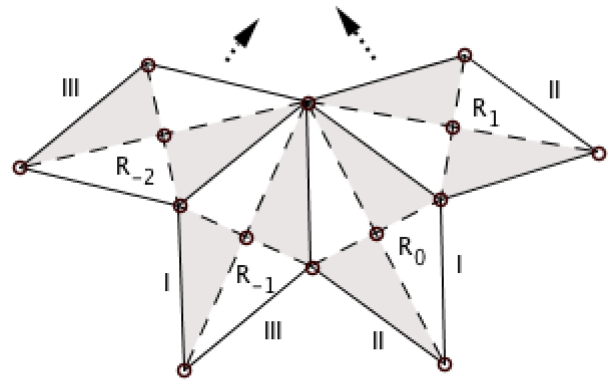

Figure 4.1.1: The minimal translation surface for a right triangle T. Edges with same label are identified by translation.

Now each R_i contains four copies of T by construction, each copy contains a unique edge of R_i. So we color the ones containing to a, a' of R_i black, and the rest white. This way we have a coloring scheme where no adjacent copies of T on X have the same color. See Figure 4.1.1. Let $\phi : X \to \mathcal{D}T$ be the map mapping black copies of T on X to T_0 of $\mathcal{D}T$, and white copies on X to T_1 of $\mathcal{D}T$, where T_0, T_1 are the two copies of T making $\mathcal{D}T$. Then clearly ϕ is a well-defined covering map. So X is a covering space of $\mathcal{D}T$. Then X must also be the covering space of $MT(T)$ by the universal property of minimal translation surfaces.

Suppose T', T'' are two copies of T in X that get identified by the covering map $X \to MT(T)$. Then T', T'' differs only by translation. WLOG say T' is the copy of T in R_0 containing a. Then T'' has an edge in the same direction of a. There are only three other possibilities, and none differs from T' by translation. So $T' = T''$. So $X = MT(T)$. □

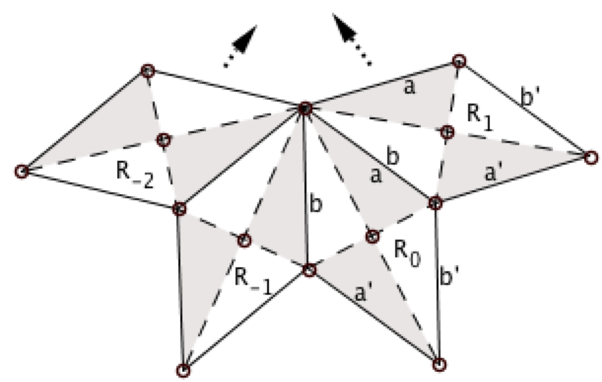

Figure 4.1.2: The edges a, a', b, b' of R_0 and R_1

Proposition 4.1.3. *The group of deck transformations $Deck(MT(T)) \simeq \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Z}$ is generated by rotation counterclockwise by θ around the common vertex of all R_i, and $\mathbb{Z}/2\mathbb{Z}$ is generated by rotating each rhombus 2π around its centroid.*

Proof. Adopt the black and white coloring scheme as in last Proposition. The generator r of $\mathbb{Z}$ and the generator i of $\mathbb{Z}/2\mathbb{Z}$ are clearly deck transformations, as they map the black copies to black copies, and white copies to white copies. Furthermore, they clearly commute. So we only need to show that they generate all of $Deck(MT(T))$.

Let f be a Deck transformation. Let $T' \subset R_0$ be the copy of T containing edge a of R_0. Then $T'' = f(T')$ must also be a black copy of T, so it is in some R_j and contains edge a or a' of that R_j. Suppose it contains edge a. Then $r^{-j} \circ f(T') = T'$. Then by the uniqueness of the lifting property, we have $r^{-j} \circ f = id$ the identity map. So $f = r^j$. Now suppose T'' contains a'. Then $r^{-j} \circ i \circ f(T') = T'$, so $f = i \circ r^j$. So r, i indeed generate $Deck(MT(T))$. □

Proposition 4.1.4. *For any closed geodesic γ on $MT(T)$, γ and $i(\gamma)$ are disjoint.*

Proof. i acts on the set of all directions on $MT(T)$ by a rotation of π. So γ and $i(\gamma)$ are in opposite direction. Then γ and $-i(\gamma)$ will be in the same direction, where $-i(\gamma)$ would means the same curve as $i(\gamma)$ but travels in the opposite direction. If $\gamma \cap i(\gamma) \neq \varnothing$, then $\gamma, -i(\gamma)$ will have at least one point in common, and they are geodesics in the same direction. So $\gamma = -i(\gamma)$ and $-\gamma = i(\gamma)$. Now as γ is a simple closed loop, it is topologically a circle. And as i maps γ to $-\gamma$, i would induce an orientation reversing automorphism of the circle, which must have 2 fixed points. However, nontrivial deck transformation has no fixed point, contradiction. So γ and $i(\gamma)$ are disjoint. $\square$

Proposition 4.1.5. *A closed geodesic γ intersects with opposite edges of R_j an equal number of times for all $j \in \mathbb{Z}$.*

Proof. If γ is disjoint with R_j, then we are done. Suppose otherwise, and WLOG assume $\gamma(0) \in R_j$. Let α be the direction of γ. If α is the direction of one pair of parallel edges of R_j, then γ must intersects that pair of edges an equal number of times, and γ will be disjoint from the other pair, so we are done. So suppose α is not the direction of either pair of edges of γ. Then given α, there is a unique edge where γ enters R_j from R_{j-1}, and a unique edge where γ enters R_j from R_{j+1}, and a unique edge where γ exits R_j to R_{j-1}, and a unique edge where γ exits R_j to R_{j-1}. Now R_j connects with R_{j+1} through one pair of its parallel edges a, a', and connects with R_{j-1} through the other pair b, b'. WLOG let a, b be edges where γ can enter R_j. If a connected component of $\gamma \cap R_j$ enters R_j from a and exists through a', or enters R_j from b and exits through b', then we are fine. Let m be the number of connected components enters from a and exits through b', and let n be the number of connected components enters from b and exits through a'. I claim that $m = n$, which will clearly prove our statement.

In fact, a geodesic segment enters R_j from a and exits through b' iff it goes from R_{j+1} to R_j to R_{j-1}, and a geodesic segment enters R_j from b and exits through a' iff it goes from R_{j-1} to R_j to R_{j+1}. As γ is a closed geodesics, the numbers of above two cases must agree. So $m = n$. $\square$

Corollary 4.1.6. *For a closed geodesic γ, $\gamma \cup i(\gamma)$ intersects each edge of R_j an even number of times.*

Proposition 4.1.7. *For any closed geodesic γ, γ and $i(\gamma)$ cuts $MT(T)$ into at least two pieces, one of which is a twice punctured cylinder with boundary γ and $i(\gamma)$.*

Proof. We shall introduce a new coloring scheme on $MT(T)$. For each R_j, we first cut it into pieces along $\gamma, i(\gamma)$. Recall that all R_j on $MT(T)$ share a vertex. Let the piece in R_j containing this common vertex be white, and then we color the rest of R_j such that no two adjacent pieces have the same color. We do this for all R_j. It's easy to check that this becomes a coloring scheme where $MT(T)$ are divided into white pieces and a black pieces, and the boundaries of these pieces are γ and $i(\gamma)$. See Figure 4.1.3.

Now all vertices of R_j are contained in the white pieces by Corollary 4.1.6. We fill in the punctures on the centroids of these rhombi for now. I claim that the black pieces are all connected and they form a cylinder.

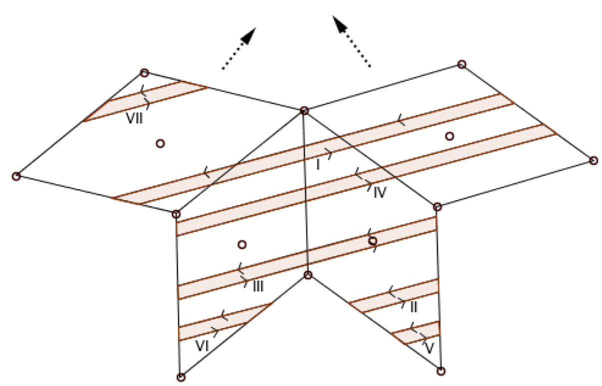

Figure 4.1.3: The black piece is a cylinder in $MT(T)$. The geodesic γ is labeled by Roman numerals by order.

Note that γ is a closed geodesic. So the sequence of rhombi it passes will be periodic. Let this sequence be $\{R_{i_n}\}_{n\in\mathbb{Z}}$, and let the corresponding connected piece of γ in R_{i_n} be γ_n. Then for each γ_n inside R_{i_n}, one side of it will be colored white, and the other side will be colored black. Let the unique connected black piece touching γ_n in R_{i_n} be B_n. Then the sequence $\{B_n\}_{n\in\mathbb{Z}}$ must be periodic as γ_n is periodic. Furthermore every connected black pieces in any rhombus R_i must be touching some γ_n for some $n \in \mathbb{Z}$, so every black piece must be one of $\{B_n\}$. Let $\{B_1, ..., B_p\}$ be a single period, then the whole black piece of $MT(T)$ is $\bigcup_{n=1}^{p} B_n$. Now each B_n will be a polygonal piece with a part of γ and a part of $i(\gamma)$ as a pair of its edges, and for the other two edges, one is glued to B_{n+1} and the other is glued to B_{n-1}. It follows that $\bigcup_{n=1}^{p} B_n$ is a cylinder $\mathcal{C}$. Now clearly $i(\mathcal{C}) = \mathcal{C}$, and i swaps its boundaries. Such a map must have two fixed points on $\mathcal{C}$. But fixed points on $\mathcal{C}$ must not be on $MT(T)$. So the two fixed points must be centroids of some rhombi. So throw away the centroids, $\mathcal{C}$ will be a twice punctured cylinder. $\square$

Proposition 4.1.8. *If the two non-right angles of a right triangle T are irrational multiples of π, then T has no stable periodic billiard path.*

Proof. Let $\{s_i\}$ be any periodic billiard path on the triangle T, and let γ' be the corresponding closed geodesic on $\mathcal{D}T$, and lift this geodesic to a closed geodesic γ on $MT(T)$. We know $\gamma, i(\gamma)$ bounds a twice punctured cylinder on $MT(T)$, and the punctures corresponds to the centroids of two rhombi. Let α_1, α_2 be two properly oriented small circular disk around the punctures so that $\gamma + i(\gamma) + \alpha_1 + \alpha_2$ will be the boundary of the cylinder with two open disk removed around the two punctures. Note that α_1, α_2 will either both be counterclockwise, or both clockwise. Then we know this sum is null-homologous. Let $[\alpha]$ be the homology class for any closed loop α. Then $[\gamma] + [i(\gamma)] + [\alpha_1] + [\alpha_2] = 0$.

Now let $p : MT(T) \to \mathcal{D}T$ be the covering map, and let α_0 be a properly oriented small circular disk around the vertex of $\mathcal{D}T$ corresponding to the right angle of the triangle, such that $p([\alpha_1]) = p([\alpha_2]) = [\alpha_0]$. As $p(\gamma) = \gamma'$, and i is a deck transformation, we have $p([\gamma]) = p([i(\gamma)]) = [\gamma']$. So we have $2[\gamma'] + 2[\alpha_0] = 0$, and so $[\gamma'] = -[\alpha_0] \neq 0$. Then by Theorem 3.1.4, the periodic billiard path cannot be stable. □

Proof of Theorem 4.1.1. All right triangles up to congruence can be parameterized by its smaller non-right angle, which take values in $(0, \frac{\pi}{2})$. This corresponds to a curve $\mathcal{R} \subset \mathcal{P}_3$. Now suppose a right triangle has a stable periodic billiard path, then let U be the orbit tile. Then $U \cap \mathcal{R}$ is an open subset of $(0, \frac{\pi}{2})$. So there exists a triangle T with non-right angles irrational multiples of π in U, and therefore it has a stable periodic billiard path, contradiction. □

Corollary 4.1.9. *Let T' be an isosceles triangle with angles irrational multiple of π. Let $\{s_i\}$ be a periodic billiard path on T'. Then there exists a periodic billiard path $\{s'_i\}$ of the same orbit type on T', such that it passes through the midpoint of the base of T' twice in its single minimal even period.*

Proof. Let T be the right triangle such that T and its reflection along one of its legs will form the isosceles triangle T'. Then note that $MT(T)$ and $MT(T')$ are the same surface except that, on $MT(T')$, the centroids of rhombi R_i are no longer removed. Now for any periodic billiard path on T', lift it to a close geodesic γ on $MT(T')$. If γ and $i(\gamma)$ intersect, then we know in fact $-\gamma = i(\gamma)$, and i would have two fixed points on γ. The only fixed points of i are centroids of the rhombi, so γ passes through two such centroids. So the periodic billiard path on T' passes through the midpoint of the base of T' twice in its single minimal even period. Now suppose γ and $i(\gamma)$ are distinct. Then we know they bound a cylinder with two centroids of rhombi in it. We know this cylinder is the union of a family of geodesics in same direction as γ, and each of them is either disjoint from the two centroids, or containing both. So there must exist γ' closed geodesic in the cylinder that passes through two centroids. Then fold it back to billiard path, it has the same orbit type as the original billiard path, and it would pass through the midpoint of the base of T' twice. □

4.2 Isosceles Triangles

The main theorem here is the following one:

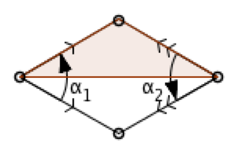

Figure 4.2.1: The double of V_3 and the generators of the homology

Theorem 4.2.1 (Hooper, Schwartz[12]). *An isosceles triangle has a stable periodic billiard path iff its base angles are not $\frac{\pi}{2n}$ for n a power of 2.*

We shall not prove the whole theorem. However, we shall discuss it for isosceles triangles with base angle $\frac{\pi}{2n}$ for $n \in \mathbb{Z}$ and $n \geq 2$. In particular, we will prove that when n is a power of 2, this isosceles triangles will have no stable periodic billiard path.

Definition 4.2.2. We call isosceles triangle with base angle $\frac{\pi}{2n}$ the ***Veech point*** V_n for all $n \in \mathbb{Z}$ and $n \geq 2$.

Theorem 4.2.3 (Hooper). *The isosceles triangle V_n has a stable periodic billiard path iff n is a power of 2.*

In fact when $n = 2$, V_2 is a right triangle, so we know it has no periodic billiard path. So from now on we can in fact assume $n \geq 3$.

Here is an outline for the proof:

Step 1: We start by showing that $MT(V_n)$ is a translation surface obtained from two $2n$-gons, calculate its Veech group, and calculate how the covering map $\phi : MT(V_n) \to \mathcal{D}V_n$ acts on homology and cohomology.

Step 2: We find $2n - 1$ geodesics as the standard geodesics, and establish an enumeration theorem to show that any geodesic can be mapped to one of these standard geodesics by an affine automorphism. We use this fact to build a stability criteria on cohomology.

Step 3: When n is a power of 2, we will show a pattern on cohomology such that the stability criteria can never by achieved. So V_n will have no stable periodic billiard path.

Step 4: When n is not a power of 2, we will find explicit geodesics on $MT(V_n)$ that satisfy the stability criteria. So V_n in this case will have stable periodic billiard paths.

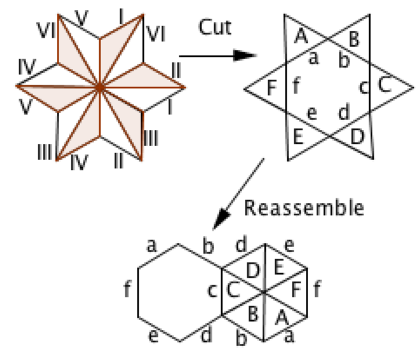

Figure 4.2.2: The minimal translation surface for V_3. Roman numerals and small letters are labels of edges, and capital letters are labels for the pieces after cutting.

We shall first need a description about the double of V_n and its minimal translation surface.

We know the double $\mathcal{D}V_n$ is topologically a sphere with three puncture, so $H_1(\mathcal{D}V_n, \mathbb{Z}) = \mathbb{Z}^2$. Let α_1, α_{-1} be loops each winding counterclockwise once around a distinct puncture corresponding to a base angle of V_n. See Figure 4.2.1. Clearly α_1, α_{-1} generate the homology group. Now we can proceed to minimal translation surface.

Proposition 4.2.4. *$MT(V_n)$ is a translation surface formed by two regular $2n$-gons.*

Proof. Reflecting V_0 about its base, we obtain a rhombus R_0. Let $\theta = \frac{\pi}{n}$ be the angle of the rhombus which corresponds to a base angle of V_n, and let the vertex there be P. Then we can rotate R_0 counterclockwise around the vertex P by $i\theta$ to obtain R_i for all $i \in \mathbb{Z}/2n\mathbb{Z}$. Now let a, b be the edges adjacent to P in R_0, and let a', b' be the edges parallel to a, b respectively in R_0. We glue edges a, a' of R_i to edges b, b' of R_{i+1} respectively for all $i \in \mathbb{Z}/2n\mathbb{Z}$. This is clearly $MT(V_n)$, because if you reflect any copy of V_n in this surface, you would obtain another copy of V_n in it up to translation, and such copy is unique. Recall that $\mathcal{D}V_n$ will have two copies of V_n, so we color the isosceles triangles on $MT(V_n)$ covering the first copy black, and the ones covering the second copy white. Now cut and reassemble, we do not change the surface, and we shall get two regular $2n$-gons forming $MT(V_n)$. See Figure 4.2.2. $\square$

Definition 4.2.5. We define the ***direction*** of a geodesic on $MT(V_n)$ to be the angle it makes with the diagonal of $R_0 \subset MT(V_n)$ which corresponds to the base of V_n.

Theorem 4.2.6 (Veech)**.** *The Veech group $\Gamma(MT(V_n))$ is a hyperbolic (n, ∞, ∞) triangle group. It is generated by two of its parabolic elements, which fix the direction of angle 0 or $\frac{\pi}{2n}$. The affine automorphism corresponding to each of the two parabolic elements is a single Dehn twist in each cylinder of the cylinder decomposition in the respective direction.*

For a proof of this theorem, see reference [13].

Now we shall find a basis $\{\beta_1, \beta_{-1}, \gamma_{1-n}, ..., \gamma_{n-1}\}$ for the homology group of $MT(V_n)$, which by counting Euler characteristics is isomorphic to $\mathbb{Z}^{2n+1}$. See

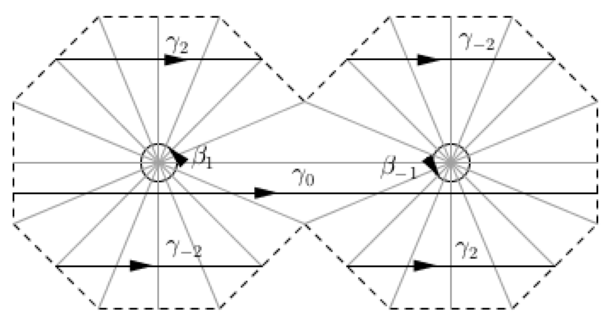

Figure 4.2.3: β_1, β_{-1} and γ_k for even k on $MT(V_4)$.

Figure 4.2.3 and 4.2.4. Here β_1, β_{-1} are counterclockwise simple closed loop around the center of each regular $2n$-gon respectively. γ_n with n even are all in the horizontal order, and $m > n$ iff γ_m has higher y-coordinate in the left polygon than γ_n. Here we choose γ_0 to be right below the centers of the two polygon. Similarly for the ones with n odd, they are in the direction of $\frac{\pi}{2n}$, and we order the index so that in the left polygon, one with larger y-coordinate will have higher index. We can arrange them such that γ_k only intersect with γ_{k+1} and γ_{k-1} for all k.

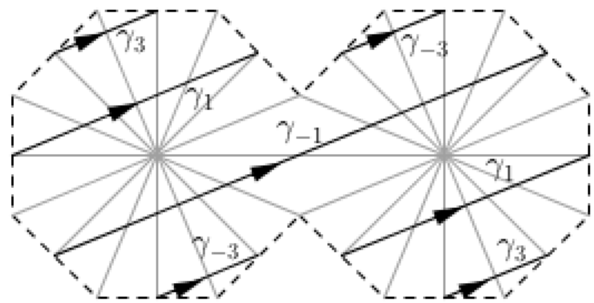

Figure 4.2.4: γ_k for odd k on $MT(V_4)$.

Proposition 4.2.7. $\{\beta_1, \beta_{-1}, \gamma_{1-n}, ..., \gamma_{n-1}\}$ *is indeed a basis for* $H_1(MT(V_n), \mathbb{Z})$.

Proof. First, label all edges of the two polygon forming $MT(V_n)$. Note that edges that are identified have the same label, and we treat these labels as members of $\mathbb{Z}/(2n)\mathbb{Z}$. Now let ℓ_i be a simple loop around the hole between edge i and $i+1$, see Figure 4.2.5 and 4.2.6. Clearly $\{\ell_1, ..., \ell_{2n}, \beta_1, \beta_{-1}\}$ generates the whole of $H_1(MT(V_n), \mathbb{Z})$. (They are not a basis though, because their sum is 0).

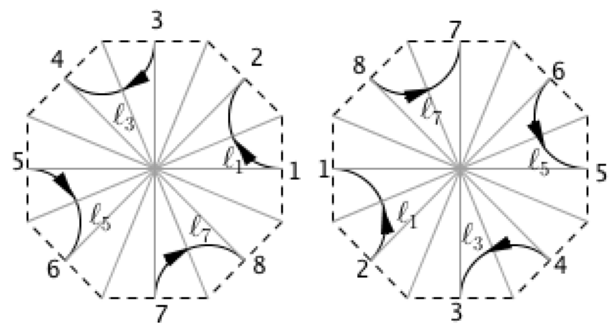

Figure 4.2.5: ℓ_k for odd k on $MT(V_4)$.

Now express $\{\beta_1, \beta_{-1}, \gamma_{1-n}, ..., \gamma_{n-1}\}$ in the basis above. Then

$$\begin{aligned}
\gamma_{n-1} &= -\ell_{\lceil \frac{n+1}{2} \rceil} \\
\gamma_{1-n} &= \ell_{\lceil \frac{n+1}{2} \rceil + n} \\
\gamma_k - \gamma_{k-1} &= \ell_{\lceil \frac{n+1}{2} \rceil + (-1)^k \lceil \frac{n-k}{2} \rceil} \text{ for all } k \in n-1, ..., 2, -1, ..., 2-n \\
\gamma_1 - \gamma_0 + \beta_1 &= \ell_1 \\
\gamma_0 - \gamma_{-1} - \beta_{-1} &= -\ell_n
\end{aligned}$$

So, elements of $\{\beta_1, \beta_{-1}, \gamma_{1-n}, ..., \gamma_{n-1}\}$ generates all of $\{\ell_1, ..., \ell_{2n}, \beta_1, \beta_{-1}\}$, and therefore they generates all of $H_1(MT(V_n), \mathbb{Z}) \simeq \mathbb{Z}^{2n+1}$. So, this must be a basis. $\square$

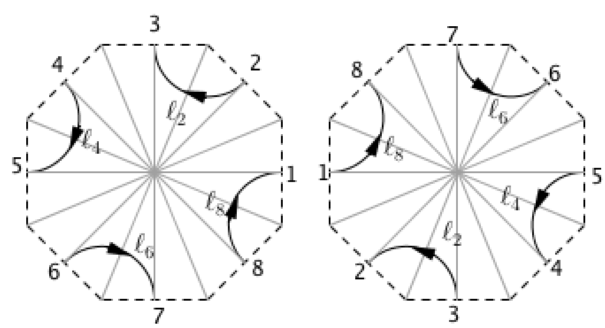

Figure 4.2.6: ℓ_k for even k on $MT(V_4)$.

With this basis, we can find the dual basis for cohomology $\{\beta_1^*, \beta_{-1}^*, \gamma_{1-n}^*, ..., \gamma_{n-1}^*\}$.

Proposition 4.2.8. *If we let* $\phi : MT(V_n) \to \mathcal{D}V_n$ *be the natural covering map, then*

$$\begin{aligned}
\phi_*(\beta_i) &= 2n\alpha_i \\
\phi_*(\gamma_k) &= (n+k)\alpha_1 - (n+k)\alpha_{-1} \qquad \textit{when } k > 0 \\
\phi_*(\gamma_k) &= -(n-k)\alpha_1 + (n-k)\alpha_{-1} \quad \textit{when } k < 0 \\
\phi_*(\gamma_0) &= n\alpha_1 + n\alpha_{-1}
\end{aligned}$$

Proof. In the above proof, we have $\{\ell_1, ..., \ell_{2n}, \beta_1, \beta_{-1}\}$ generating the whole of $H_1(MT(V_n), \mathbb{Z})$. Here clearly $\phi_*(\beta_i) = 2n\alpha_i$, and $\phi_*(\ell_i) = \alpha_1 - \alpha_{-1}$.

Here we shall adopt the convention that $\gamma_n = \gamma_{-n} = 0$. Then $\gamma_k - \gamma_{k-1} = \ell_j$ for some j depending on k for all $k \in \{n, n-1, ..., 2, -1, -2, ..., 1-n\}$. As a result, we have $\phi_*(\gamma_k) = (n+k)\alpha_1 - (n+k)\alpha_{-1}$ when $k > 0$, and $\phi_*(\gamma_k) = -(n-k)\alpha_1 + (n-k)\alpha_{-1}$ when $k < 0$.

Let ρ be the affine automorphism of $MT(V_n)$ which rotate everything by π. This is clearly a deck transformation of the covering $\phi : MT(V_n) \to \mathcal{D}V_n$, so

$$\begin{aligned}
&\phi_*(\rho(\gamma_k)) = (\phi \circ \rho)_*(\gamma_k) = \phi_*(\gamma_k) \\
\text{Then } &\phi_*(\gamma_k + \rho(\gamma_k)) = 2\phi_*(\gamma_k) \\
\text{Then } &\gamma_0 + \rho(\gamma_0) = \beta_1 + \beta_{-1} \\
\text{Then } &2\phi_*(\gamma_0) = 2n\alpha_1 + 2n\alpha_{-1} \\
\text{Then } &\gamma_0 = n\alpha_1 + n\alpha_{-1}.
\end{aligned}$$

□

Definition 4.2.9. We define $\phi_1^*, \phi_{-1}^* \in H^1(MT(V_n), \mathbb{Z})$ to be the element such that $\phi_*(h) = \phi_1^*(h)\alpha_1 + \phi_{-1}^*(h)\alpha_{-1}$.

Then we can use Proposition 4.2.8 to express ϕ_1^*, ϕ_{-1}^* in the dual basis $\{\beta_1^*, \beta_{-1}^*, \gamma_{1-n}^*, ..., \gamma_{n-1}^*\}$.

Now we choose the basis $\{\beta_1^*, \beta_{-1}^*, \gamma_{1-n}^*, ..., \gamma_{n-1}^*\}$ precisely because γ_i is a closed geodesic for each i. In fact, we have the following theorem.

Theorem 4.2.10 (Enumeration Theorem)**.** *If γ is a closed geodesic on $MT(V_n)$, then there exists $\gamma' \in \{\gamma_{1-n}, ..., \gamma_{n-1}\}$ and $\psi \in Aff(MT(V_n))$ such that $\psi(\gamma)$ is homologous to γ'.*

Proof. Let θ be the direction of γ. Then as γ is a closed geodesic, by Veech Dichotomy all geodesics in direction of θ are closed. As a result, there exists a parabolic element $C \in \Gamma(MT(V_n))$ such that C has θ as a fixed direction. Let A, B be the parabolic elements generating $\Gamma(MT(V_n))$ with eigendirection $0, \frac{\pi}{2n}$.

Now as $\Gamma(MT(V_n))$ is the (n, ∞, ∞) triangle group, we know C is either conjugate to a power of A, or conjugate to a power of B. So exists $D \in \Gamma(MT(V_n))$ such that $DCD^{-1} = A^k$ or B^k for some $k \in \mathbb{Z} - \{0\}$. Then D must map direction θ to 0 or $\frac{\pi}{2n}$. Let ψ be an affine automorphism with derivative D. Then $\psi(\gamma)$ must be in the direction of 0 or $\frac{\pi}{2n}$, and therefore homologous to one of $\pm\gamma_{1-n}, ..., \pm\gamma_{n-1}, \pm(\gamma_0 - \beta_1 - \beta_{-1})$. Now, $\gamma_0 - \beta_1 - \beta_{-1}$ can be obtained from $-\gamma_0$ by a rotation of 2π on $MT(V_n)$ which is clearly an affine automorphism. Furthermore, $-\gamma_k$ can be obtained from γ_k by flipping $MT(V_n)$ left-side-right, which is also an affine automorphism. So we conclude that, eventually, we can choose $\psi \in \text{Aff}(MT(V_n))$ such that $\psi(\gamma)$ is homologous to one of $\{\gamma_{1-n}, ..., \gamma_{n-1}\}$. □

So we need to see how the elements of $\text{Aff}(MT(V_n))$ act on homology and cohomology.

Proposition 4.2.11. *$Aff(MT(V_n))$ is generated by an involution σ (element of order 2), the right Dehn twist τ_e in direction 0 (γ_k in this direction will have k even), and the right Dehn twist τ_o in direction $\frac{\pi}{2n}$ (γ_k in this direction will have k odd).*

Proof. We have a natural map $d : \text{Aff}(MT(V_n)) \to \Gamma(MT(V_n))$ by taking derivative. Then any affine automorphism ψ will have derivative $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, i.e. ψ is a "translation" on $\text{Aff}(MT(V_n))$. Recall that $MT(V_n)$ is made up by two $2n$-gons, so the only possibilities are either ψ is the identity, or ψ is the map that swaps the two $2n$-gons, in which case we call this affine automorphism σ. σ is clearly an involution. Now, we know the derivative of τ_e, τ_o generates the Veech group $\Gamma(MT(V_n))$. Therefore σ, τ_e, τ_o generates $\text{Aff}(MT(V_n))$. □

Now by checking one by one, we have the following results on homology:

$$\begin{aligned}
\sigma_*(\beta_i) &= \beta_{-i} \\
\sigma_*(\gamma_k) &= \gamma_{-k} \\
\tau_{o*}(\beta_i) &= \beta_i \\
\tau_{o*}(\gamma_k) &= \gamma_k \text{ for odd } k \\
\tau_{o*}(\gamma_k) &= \gamma_k + \gamma_{k-1} + \gamma_{k+1} \text{ for even } k \\
\tau_{e*}(\beta_i) &= \beta_{-i} \\
\tau_{e*}(\gamma_k) &= \gamma_k \text{ for even } k \\
\tau_{e*}(\gamma_k) &= \gamma_k + \gamma_{k-1} + \gamma_{k+1} \text{ for odd } k \neq \pm 1 \\
\tau_{e*}(\gamma_k) &= -\beta_{-k} + \gamma_k + \gamma_{k-1} + \gamma_{k+1} \text{ for } k = \pm 1.
\end{aligned}$$

On cohomology, we have the following results:

$$\begin{aligned}
\sigma^*(\beta_i^*) &= \beta_{-i}^* \\
\sigma^*(\gamma_k^*) &= \gamma_{-k}^* \\
\tau_o^*(\beta_i^*) &= \beta_i^* \\
\tau_o^*(\gamma_k^*) &= \gamma_k^* \text{ for even } k \\
\tau_o^*(\gamma_k^*) &= \gamma_{k-1}^* + \gamma_k^* + \gamma_{k+1}^* \text{ for odd } k \\
\tau_e^*(\beta_i^*) &= \beta_{-i}^* - \gamma_{-i}^* \\
\tau_e^*(\gamma_k^*) &= \gamma_k^* \text{ for odd } k \\
\tau_e^*(\gamma_k^*) &= \gamma_{k-1}^* + \gamma_k^* + \gamma_{k+1}^* \text{ for even } k.
\end{aligned}$$

Here s^*, t_o^*, t_e^* are pull backs.

With all the above discussion, we can have a very elegant criteria for stability of billiard paths.

Lemma 4.2.12 (Stability Lemma)**.** *For any periodic billiard paths on V_n, let γ be the corresponding closed geodesic on $MT(V_n)$. Then $\gamma = w(\gamma_k)$ for some $w \in Aff(MT(V_n))$ and $k \in \{1-n, ..., n-1\}$. Then the path is stable iff for $w^*(\phi_1^*), w^*(\phi_{-1}^*)$, when expressed in the basis $\{\beta_1^*, \beta_{-1}^*, \gamma_{1-n}^*, ..., \gamma_{n-1}^*\}$, has coefficient 0 for γ_k^*.*

Proof. Recall that $\phi : MT(V_n) \to \mathcal{D}V_n$ is the natural covering map. Then

$$\begin{aligned}
&\text{The path is stabel} \\
\iff &\phi(\gamma) \text{ is null homologous} \\
\iff &\phi_1^*(\gamma) = \phi_{-1}^*(\gamma) = 0 \\
\iff &\phi_1^*(w(\gamma_k)) = \phi_{-1}^*(w(\gamma_k)) = 0 \\
\iff &w^*(\phi_1^*)(\gamma_k) = w^*(\phi_{-1}^*)(\gamma_k) = 0 \\
\iff &w^*(\phi_1^*), w^*(\phi_{-1}^*), \text{ when expressed in the basis } \{\beta_1^*, \beta_{-1}^*, \gamma_{1-n}^*, ..., \gamma_{n-1}^*\}, \text{ has coefficient 0 for } \gamma_k^*.
\end{aligned}$$

□

Now we can finally begin our proof of Theorem 4.2.3. We shall break it down into several propositions.

Proposition 4.2.13. *Let w be an affine automorphism of $MT(V_n)$. Then there exists odd integers r, s such that $w^*(\phi_i^*) \equiv \sum_{j=1-n}^{n-1} ic(j)\gamma_j^* \mod 2n$. Here $c(j) = r(j+n)$ if j is odd, and $c(j) = s(j+n)$ if j is even.*

Proof. The proof is by induction. When w is the identity, then set $r = s = 1$, the statement is clearly true.

Suppose the statement is true for w_0 with $r = r_0, s = s_0$. Then for $\sigma \circ w_0$, we can let $r = -r_0, s = -s_0$, and then the statement will be true. For $\tau_o^{\pm 1} \circ w_0$, set $r = r_0, s = s_0 \mp 2r_0$, and for $\tau_e^{\pm 1} \circ w_0$, set $r = r_0 \mp 2s_0, s = s_0$. The by computation the statement is true in both cases. □

Proposition 4.2.14. *For n a power of 2, V_n has no stable periodic billiard path.*

Proof. Assume $n = 2^m$. For all $j \in \{1-n, ..., n-1\}$, we have $1 \leq j+n \leq 2n-1$. So $j+n \not\equiv 0 \mod 2^{m+1}$. Since r, s in the above proposition are odd, we have $c(j) \not\equiv 0 \mod 2n$ for all j. So the stability lemma implies that no billiard path is stable. □

Finally, we quote the following proposition without proof. The proof can be done by a (somewhat complicated) direct calculation, and can be found in the joint paper by Schwartz and Pat [12]. These propositions plus the above one will complete the proof of theorem 4.2.3.

Proposition 4.2.15. *Suppose $n \geq 3$ is odd. Then a closed geodesic in the homology class $(\tau_e \circ \tau_o^{-1})^{\frac{n-1}{2}} \circ \tau_e^{\frac{n-3}{2}}(\gamma_{n-2})$ in $MT(V_n)$ projects to a stable periodic billiard path in V_n via the folding map $MT(V_n) \to V_n$.*

Proposition 4.2.16. *Suppose n is even and not a power of 2. Then $n = 2^a b$ for an odd integer $b \geq 3$ and an integer $a \geq 1$. Let $\psi = (\tau_e \circ \tau_o^{-1})^{\frac{n}{2}} \circ \tau_o^{\frac{b-1}{2}}$. Then a closed geodesic in the homology class $w(\gamma_{n-2^{a+1}})$ projects to a stable periodic billiard path in V_n via the folding map $MT(V_n) \to V_n$.*

4.3 Rhombi

The same as last section, we shall not discuss the general case of rhombi. The rhombi of our interests will be those with one angle equal $\frac{\pi}{n}$ with $n \geq 2$. We shall show the following theorem.

Theorem 4.3.1. *A rhombi R_n with on angle $\frac{\pi}{n}$ has no stable periodic billiard path when n is a power of 2.*

Proof. If we reflect the isosceles triangle V_n along its base, then we would obtain R_n. Therefore, there is a natural covering map $p : \mathcal{D}R_n \to \mathcal{D}V_n$. Suppose R_n has a stable periodic billiard path. Then we can find γ a null-homologous closed geodesic on $\mathcal{D}R_n$. Then $p(\gamma)$ will be a null homologous closed geodesic on $\mathcal{D}V_n$. So $p(\gamma)$ will project to a stable periodic billiard path on V_n. Then n must not be a power of 2. This concludes our proof. □

Corollary 4.3.2. *A square has no stable periodic billiard path.*

4.4 Rectangle

Here we shall made a conjecture:

Conjecture 3. *All parallelogram with one angle $\frac{\pi}{2n}$ for n a power of 2 will have no stabel periodic billiard path.*

We shall only prove a very special case of it here. The first case is of course when $n = 1$, in which case the parallelogram must be a rectangle.

Theorem 4.4.1. *A rectangle R has no stable periodic billiard path.*

We begin by looking its minimal translation surface.

Proposition 4.4.2. *$MT(R)$ is isometric to a torus with a flat metric and with four punctures.*

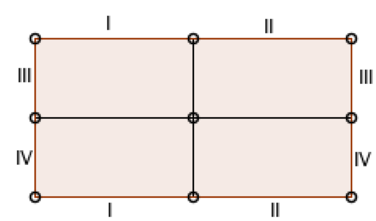

Figure 4.4.1: The minimal translation surface for a rectangle

Proof. One can clearly see that $\pi_1(\mathcal{D}R)/ker(hol_{ab} \circ ab)$ has only four elements, which can be represented by the group generated by vertical reflection and horizontal reflection (i.e. $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$). So the minimal translation surface would looks like four rectangle glued up together and throwing away vertices. See Figure 4.4.1. Edges identified by translation have the same label in the figure. Note that the gluing pattern is exactly how a torus is obtained, and the metric is clearly a flat metric. So our statement is true. □

Definition 4.4.3. Choose an edge of a parallelogram as the base. The modulus of the parallelogram is the ratio of its base and its height. See Figure 4.4.2. Note that by different choice of base edge, we have two moduli for each parallelogram.

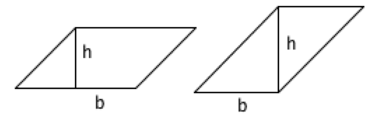

Figure 4.4.2: The two moduli for a parallelogram. Here b is the base and h is the height, and their ratio is the modulus.

Corollary 4.4.4. *A billiard path in a rectangle is periodic iff its slope is a rational multiple of the modulus, and it's trajectory is dense in the rectangle iff its slope is an irrational multiple of the modulus.*

By Corollary 4.3.2, we already know that a square has no stable periodic billiard path. Now we have the following observation.

Definition 4.4.5. Let $e_m : \mathbb{R}^2 \to \mathbb{R}^2$ be the stretching map defined by the matrix $\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}$ for each $m \in \mathbb{R}^+$. This map acts as a stretching on the plane horizontally by a factor of m.

Proposition 4.4.6. *Let S be a square, and let R be any rectangle with a modulus m. WLOG we can assume the sides of R has length 1,m respectively. Then e_m induces a one-to-one correspondence between closed geodesics on $\mathcal{D}S$ and on $\mathcal{D}R$, and also between null-homologous closed geodesics on $\mathcal{D}S$ and on $\mathcal{D}R$.*

Proof. We know that the plane $\mathbb{R}^2$ is a covering space for the torus, and therefore $\mathbb{R}^2 - \mathbb{Z}^2$ is a covering space for $MT(S)$ in the natural way. See Figure 4.4.3. Let $L_m = e_m(\mathbb{Z}^2)$, then similarly $\mathbb{R}^2 - L_m$ is a covering space of $MT(R)$. So the map $e_m : \mathbb{R}^2 - \mathbb{Z}^2 \to \mathbb{R}^2 - L_m$ descends to a map $e'_m : MT(S) \to MT(R)$, and further to a map $e''_m : \mathcal{D}S \to \mathcal{D}R$. Note that we also have a map $e_{\frac{1}{m}} : \mathbb{R}^2 - L_m \to \mathbb{R}^2 - \mathbb{Z}^2$ as the inverse of e_m. Then this map similarly induces the maps $e'_{\frac{1}{m}}, e''_{\frac{1}{m}}$ which are inverses of e'_m, e''_m respectively.

Then clearly e''_m and $e''_{\frac{1}{m}}$ gives the desired bijection between closed geodesics on $\mathcal{D}S$ and on $\mathcal{D}R$. Furthermore, as they induces isomorphism on homology class, this also gives the desired bijection between null-homologous closed geodesics on $\mathcal{D}S$ and on $\mathcal{D}R$. □

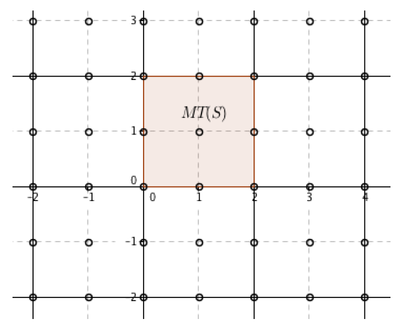

Figure 4.4.3: The covering of $\mathbb{R}^2 - \mathbb{Z}^2$ over $MT(S)$

Corollary 4.4.7. *Rectangles have no stable periodic billiard path.*

4.5 Parallelogram

Now we can move on to parallelogram with one angle $\frac{\pi}{4}$. We shall do only one case here.

Theorem 4.5.1. *A parallelogram P_1 with one angle $\frac{\pi}{4}$ and modulus 1 will have no stable periodic billiard path.*

Here is an outline for the proof:

Step 1: We compute $MT(P_1)$, and compute the action of the covering map $\phi : MT(P_1) \to \mathcal{D}P_1$ on homology and cohomology. In particular, we shall find a stability criteria: there are elements $\alpha^*, \beta^*, \omega^*$ of cohomology group $H^1(MT(P_1))$ such that $\phi(\gamma)$ is null-homologous iff $\alpha^*(\gamma) = \beta^*(\gamma) = \omega^*(\gamma) = 0$ for a closed loop γ.

Step 2: We do the cylinder decompositions of $MT(P_1)$ in direction $(1, 0), (1, 1), (2, 1)$ and find a geodesic in each cylinder of each cylinder decomposition. These are the ten standard geodesics. Then we build

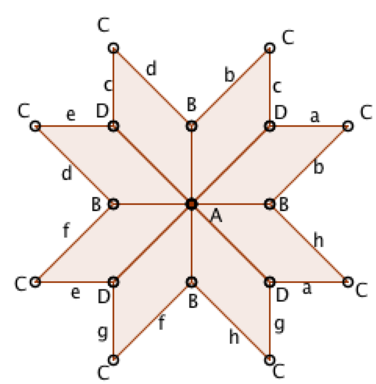

Figure 4.5.1: The minimal translation surface for P_1. It has four holes A, B, C, D, and the edges with the same labels are identified via translation.

an enumeration theorem showing that any geodesic on $MT(P_1)$ can be mapped to one of these standard geodesics by an affine automorphism.

Step 3: We show that for any geodesic γ and any affine automorphism ψ, if $\phi(\psi(\gamma))$ is null-homologous in $\mathcal{D}P_1$, then $\beta^*(\gamma)$ is even and $\alpha^*(\gamma) = 2\beta^*(\gamma)$, which none of the ten standard geodesics satisfies.

Step 4: Combine Step 2 and 3, we concludes that there are no stable periodic billiard paths.

We shall first study the minimal translation surface. See Figure 4.5.1. To build this surface, we start with the parallelogram P_1, and let the edges adjacent to an angle $\frac{\pi}{4}$ be a, b, and let a', b' be the edges parallel to a, b respectively in P_1. Now we reflect P_1 about a and b alternatively. Then we are back to the original parallelogram after exactly eight reflections. This would give us parallelograms $P_1, ..., P_8$, where the indices lives in $\mathbb{Z}/8\mathbb{Z}$. Now for each i, the edges a, a' of P_i are glued to the edges a, a' of P_{i+1} respectively, and the edges b, b' of P_i are glued to the edges b, b' of P_{i-1} respectively. This way we obtain the minimal translation surface.

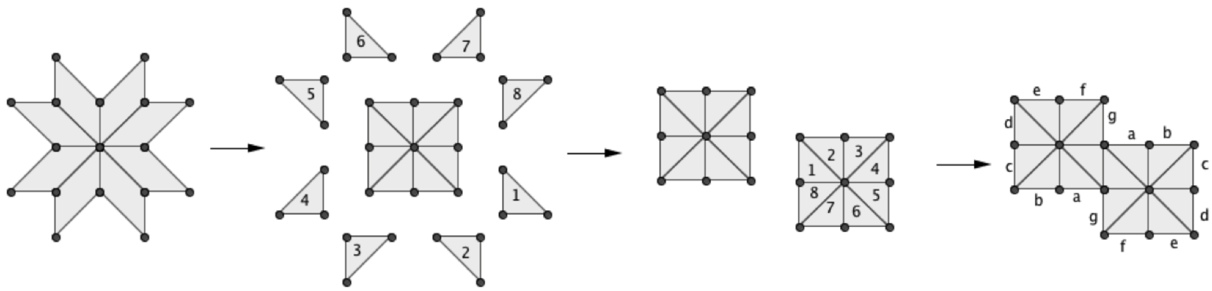

Figure 4.5.2: Cut and reassemble of $MT(P_1)$. The numbers are labels for the pieces. In the right most figure, edges with the same letter label are glued together by translation.

However, this is not the best way to represent this surface. We can cut and reassemble and obtain a better version. We start from the representation in Figure 4.5.1. Then we cut each P_i along their shorter diagonal. and by some rearrangement shown in Figure 4.5.2, we obtain a double square glued together in a weird way. Note that this is topologically a double octagon like the minimal translation surface for V_4 in section 4.2, except that the octagon here is not regular, and has four flat angles and thus looks like a square.

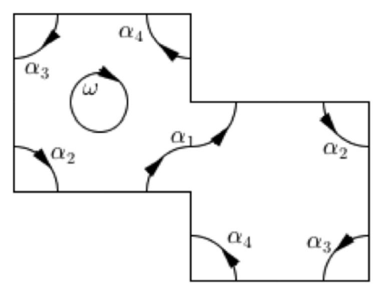

Figure 4.5.3: The generators ω and α_i

One advantage of this new representation is the following result.

Proposition 4.5.2. *Any close geodesic on $MT(P_1)$ will have rational slope.*

Proof. With the new representation, we see that there is a natural map from $MT(P_1)$ to the double of a square. Then the closed geodesic will be mapped to a periodic billiard path of the square. However, such path must have rational slope. So we are done. □

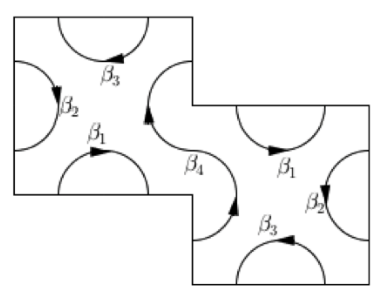

Figure 4.5.4: The generators β_i

Now we shall proceed to find generators for homology group. A basis for $H_1(MT(P_1)) \simeq \mathbb{Z}^9$ is $\{\alpha_1, \alpha_2, \alpha_3, \ \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4, \omega\}$. See Figure 4.5.3 and 4.5.4. We shall treat their index as in $\mathbb{Z}/4\mathbb{Z}$. One can prove that they are indeed a basis for homology by using simplicial homology. A basis for $H_1(\mathcal{D}P_1) \simeq \mathbb{Z}^3$ is $\{\lambda_1, \lambda_2, \lambda_3\}$. See Figure 4.5.5.

Now let $\phi : MT(P_1) \to \mathcal{D}P_1$ be the natural covering map. Then we have

$$\phi_*(\omega) = 4\lambda_1$$
$$\phi_*(\omega + \sum \alpha_i + \sum \beta_i) = 4\lambda_2$$
$$\phi_*(\omega + \sum \beta_i) = -4\lambda_3$$

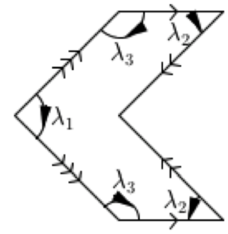

Figure 4.5.5: The generators of $\mathcal{D}P_1$

Proposition 4.5.3. *$\phi_*(\alpha_i) = \lambda_2 + \lambda_3, \phi_*(\beta_i) = -\lambda_1 - \lambda_3$ for all i.*

Proof. Let r be the counterclockwise rotation by $\frac{\pi}{2}$ of both square forming $MT(P_1)$ around their center respectively. Clearly r is an affine automorphism and a deck transformation. Furthermore, $r(\alpha_i) = \alpha_{i-1}, r(\beta_i) = \beta_{i-1}$ for all i. As a result,

$$\phi_*(\alpha_i) = \phi_*(r(\alpha_{i+1})) = (\phi \circ r)_*(\alpha_{i+1}) = \phi_*(\alpha_{i+1})$$

for all i. So $\phi_*(\alpha_i) = \phi_*(\alpha_j)$ for all i, j, and the same is true for β_i by similar arguments. Then

$$\begin{cases} \phi_*(\omega) = 4\lambda_1 \\ \phi_*(\omega + \sum \alpha_i + \sum \beta_i) = 4\lambda_2 \\ \phi_*(\omega + \sum \beta_i) = -4\lambda_3 \end{cases}$$
$$\implies \begin{cases} \phi_*(\omega) = 4\lambda_1 \\ \phi_*(\omega) + 4\phi_*(\alpha_i) + 4\phi_*(\beta_i) = 4\lambda_2 \text{ for each } i \\ \phi_*(\omega) + 4\phi_*(\beta_i) = -4\lambda_3 \text{for each } i \end{cases}$$
$$\implies \begin{cases} \phi_*(\omega) = 4\lambda_1 \\ \phi_*(\alpha_i) = \lambda_2 + \lambda_3 \text{ for all } i \\ \phi_*(\beta_i) = -\lambda_1 - \lambda_3 \text{ for all } i \end{cases}$$

□

Now let $\{\alpha_1^*, \alpha_2^*, \alpha_3^*, \alpha_4^*, \beta_1^*, \beta_2^*, \beta_3^*, \beta_4^*, \omega^*\}$ be the dual basis for cohomology. Then we can define $\alpha^* = \sum \alpha_i^*, \beta^* = \sum \beta_i^*$. And we have the following lemma:

Lemma 4.5.4 (Stability Criteria)**.** *Any loop γ on $MT(P_1)$, $\phi(\gamma)$ is null-homologous in $\mathcal{D}P_1$ iff $\alpha^*(\gamma) = \beta^*(\gamma) = \omega^*(\gamma) = 0$.*

Proof. Let γ be any loop. Then $\gamma = \sum a_i\alpha_i + \sum b_i\beta_i + k\omega$ for some integers a_i, b_i, k. Then

$$\begin{aligned} \phi_*(\gamma) &= (\sum a_i)(\lambda_2 + \lambda_3) + (\sum b_i)(-\lambda_1 - \lambda_3) + 4k\lambda_1 \\ &= \alpha^*(\gamma)(\lambda_2 + \lambda_3) + \beta^*(\gamma)(-\lambda_1 - \lambda_3) + 4\omega^*(\gamma)\lambda_1 \\ &= (4\omega^*(\gamma) - \beta^*(\gamma))\lambda_1 + \alpha^*(\gamma)\lambda_2 + (\alpha^*(\gamma) - \beta^*(\gamma))\lambda_3 \end{aligned}$$

Now we have:

$$(4\omega^*(\gamma) - \beta^*(\gamma))\lambda_1 + \alpha^*(\gamma)\lambda_2 + (\alpha^*(\gamma) - \beta^*(\gamma))\lambda_3 = 0$$
$$\iff \begin{cases} \alpha^*(\gamma) = 0 \\ \alpha^*(\gamma) - \beta^*(\gamma) = 0 \\ 4\omega^*(\gamma) - \beta^*(\gamma) = 0 \end{cases}$$
$$\iff \alpha^*(\gamma) = \beta^*(\gamma) = \omega^*(\gamma) = 0$$

□

Now what we need is something similar to the enumeration theorem for isosceles triangles. To build this theorem, we can start by doing the cylinder decomposition of $MT(P_1)$ in different directions. The ones with direction $(1,0),(1,1),(2,1)$ are shown in Figure 4.5.6, 4.5.7 and 4.5.8. In Figure 4.5.6, A and B form a cylinder, and C and D form a cylinder. In Figure 4.5.7 and 4.5.8, polygons forming the same cylinder have the same label. Within each cylinder of each cylinder decomposition is a family of closed geodesics with the same homology class. We pick a representative from each, and we obtain 10 geodesics $\gamma_1,...,\gamma_{10}$. For example, γ_1,γ_2 are shown in Figure 4.5.9.

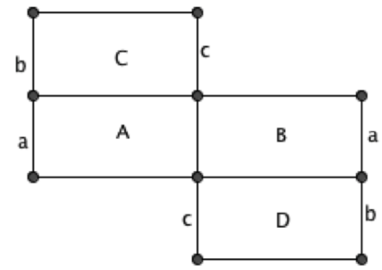

Figure 4.5.6: The cylinder decomposition in the (1,0) direction

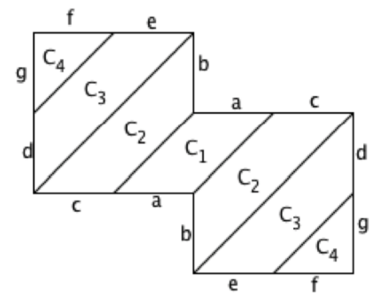

Figure 4.5.7: The cylinder decomposition in the (1,1) direction

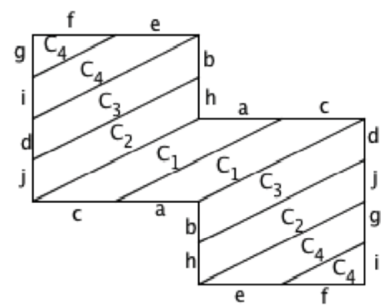

Figure 4.5.8: The cylinder decomposition in the (2,1) direction

Let τ_0 be the Dehn twist in horizontal direction with derivative $\begin{pmatrix}1 & 4\\ 0 & 1\end{pmatrix}$, and let τ_1 be the Dehn twist in $(1,1)$ direction with derivative $\begin{pmatrix}-2 & 3\\ -3 & 4\end{pmatrix}$, and let r be the rotation as before. They generate a subgroup $\langle\tau_0,\tau_1,r\rangle \subset \mathrm{Aff}(MT(P_1))$.

Theorem 4.5.5 (Enumeration Theorem)**.** *For any closed geodesic γ, there exists an affine automorphism $\psi \in \langle\tau_0,\tau_1,r\rangle$ such that $\psi(\gamma)$ is homologous to one of $\pm\gamma_1,...,\pm\gamma_{10}$.*

This theorem is a corollary of Proposition 4.5.7. However, before we go into the proof of that Proposition, we need the following theorem in hyperbolic geometry. For a proof of the theorem and definition of relevant terms, see the book on hyperbolic manifolds by Ratcliffe [4] and the Theorem 11.2.2 in this book.

Theorem 4.5.6 (Poincaré's Fundamental Polyhedron Theorem on Hyperbolic plane)**.** *Let Φ be a subset of the group $SL(2,\mathbb{R})$, and let it be a proper $SL(2,\mathbb{R})$-side-pairing for a convex polygon P. Suppose when we glue the edges of P according to the pairing Φ, the resulting hyperbolic surface is complete. Then P is the fundamental domain of the group in $SL(2,\mathbb{R})$ generated by Φ.*

Proposition 4.5.7. *Let G be the subgroup of $SL(2,\mathbb{Z})$ generated by $A=\begin{pmatrix}1 & 4\\ 0 & 1\end{pmatrix}, B=\begin{pmatrix}-2 & 3\\ -3 & 4\end{pmatrix}, R=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$. G acts on $L=(\mathbb{Z}^2-\{(0,0)\})/\sim$, where $(a,b)\sim(c,d)$ iff there exist $m,n\in\mathbb{Z}-\{0\}$ such that $m(a,b)=n(c,d)$. (i.e. L is the set of lines with rational slope.) Then this action has three orbits, one containing direction $(1,0)$, one containing direction $(1,1)$, and one containing direction $(2,1)$.*

Proof. We start by looking at the action of G on the hyperbolic plane $\mathbb{H}^2$. We adopt the upper half plane model for now. Then $A(-2)=2, A(\infty)=\infty$, so A would send the geodesic through $-2,\infty$ to the geodesic through $2,\infty$. $R(i)=i, R(1)=-1$, so R would send the geodesic segment from 1 to i to the geodesic segment from -1 to i. Finally, $RB(2)=-2, RB(1)=-1$, so RB would send the geodesic through $1,2$ to the geodesic through $-1,-2$. Then we obtain a convex hyperbolic pentagon $P_{\mathbb{H}}$ as shown in Figure 4.5.10, and $\Phi=\{A,A^{-1},R,R^{-1},RB,(RB)^{-1}\}$ is a proper $SL(2,\mathbb{R})$-side-pairing. Furthermore, if we glue the edges according to the pairing by Φ, then the resulting hyperbolic surface M is clearly complete. So by the Poincaré's fundamental polyhedron theorem, $P_{\mathbb{H}}$ is a fundamental domain of G.

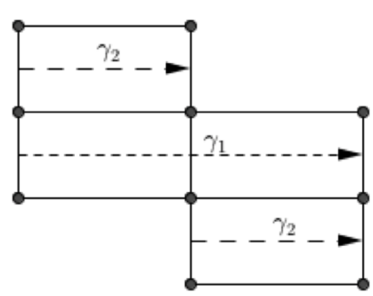
Figure 4.5.9: γ_1 and γ_2

Now because the fundamental domain $P_{\mathbb{H}}$ have five vertices, all rational points on x-axis together with the point ∞ can be mapped by to one of the five vertices by elements of G. Further more by the G-side-pairing on $P_{\mathbb{H}}$, we see that there are exactly three orbits, containing $1, 2, \infty$ respectively. Now G acts on the rational points on x-axis together with the point ∞ in the same as G acts on the set L. So the action of G on L has three orbits, containing $(1, 1), (2, 1)$ and $(1, 0)$ respectively. □

Proof of the Enumeration Theorem 4.5.5. We know that γ must have rational slope. Let $(a, b) \in \mathbb{Z}^2$ be the direction vector of the slope of γ. Clearly $(a, b) \not\sim (0, 0)$. Then there exists $M \in G$ such that $M(a, b) \sim (1, 0), (1, 1)$ or $(2, 1)$. As the derivatives of τ_0, τ_1, r are exactly the generators of G, we can find $\psi \in \langle \tau_0, \tau_1, r \rangle$ with derivative M. Then $\psi(\gamma)$ must have slope $(1, 0), (1, 1), (2, 1)$, so it must be one of the desired geodesics. □

Now we can analyze the action of the affine automorphism on homology. By the enumeration theorem, we will only use the action of τ_0, τ_1, r, so we omit the others.

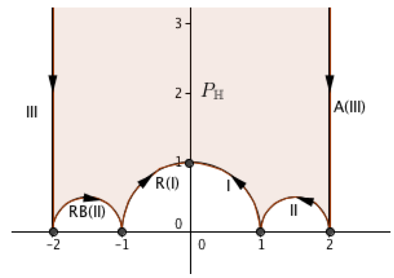

Figure 4.5.10: The hyperbolic polygon $P_{\mathbb{H}}$. Here Roman numerals I,II,III are labels of edges, and A(III),R(I),RB(II) are how the matrices A,R,RB identify pairs of edges of $P_{\mathbb{H}}$.

$$
\begin{aligned}
r(\omega) &= \omega \\
r(\alpha_i) &= \alpha_{i-1} \\
r(\beta_i) &= \beta_{i-1} \\
\tau_0(\omega) &= \omega \\
\tau_0(\beta_1) &= \beta_1 \\
\tau_0(\beta_3) &= \beta_3 \\
\tau_0(\alpha_1) &= \alpha_1 + (\alpha_1 + \alpha_2 + \beta_1) \\
\tau_0(\alpha_2) &= \alpha_2 - (\alpha_1 + \alpha_2 + \beta_1) \\
\tau_0(\alpha_3) &= \alpha_3 + (\alpha_3 + \alpha_4 + \beta_3) \\
\tau_0(\alpha_4) &= \alpha_4 - (\alpha_3 + \alpha_4 + \beta_3) \\
\tau_0(\beta_2) &= \beta_2 - (\alpha_1 + \alpha_2 + \beta_1) + (\alpha_3 + \alpha_4 + \beta_3) \\
\tau_0(\beta_4) &= \beta_4 + (\alpha_1 + \alpha_2 + \beta_1) - (\alpha_3 + \alpha_4 + \beta_3) \\
\tau_1(\omega) &= \omega \\
\tau_1(\alpha_1) &= \alpha_1 \\
\tau_1(\alpha_3) &= \alpha_3 \\
\tau_1(\beta_1) &= \beta_1 - 3\alpha_1 - (\beta_4 + \beta_1 + \alpha_1) \\
\tau_1(\beta_2) &= \beta_2 + 3\alpha_5 + (\beta_2 + \beta_3 + \alpha_3) \\
\tau_1(\beta_3) &= \beta_3 - 3\alpha_5 - (\beta_2 + \beta_3 + \alpha_3) \\
\tau_1(\beta_4) &= \beta_4 + 3\alpha_1 + (\beta_4 + \beta_1 + \alpha_1) \\
\tau_1(\alpha_2) &= \alpha_2 - (\beta_4 + \beta_1 + \alpha_1) + (\beta_2 + \beta_3 + \alpha_3) \\
\tau_1(\alpha_4) &= \alpha_4 + (\beta_4 + \beta_1 + \alpha_1) - (\beta_2 + \beta_3 + \alpha_3)
\end{aligned}
$$

These computations yield the following insight:

Proposition 4.5.8. *For any closed curve γ on $MT(P_1)$ and any $g \in \langle \tau_0, \tau_1, r \rangle$, we have $\alpha^*(g(\gamma) - \gamma) = 2\beta^*(g(\gamma) - \gamma)$. In particular, $\alpha^*(g(\gamma) - \gamma)$ is always even.*

Proof. From the computation above we see that the statement is true when g is τ_0, τ_1 or r, and γ is α_i, β_i or ω. So the statement must be true when g is τ_0, τ_1 and γ is any closed curve.

The rest is induction. Suppose the statement is true for an element $g \in \langle \tau_0, \tau_1, r \rangle$, then by assumption, we have:

$$\alpha^*(g(\gamma) - \gamma) = 2\beta^*(g(\gamma) - \gamma).$$

Then we can deduce that:

$$
\begin{aligned}
\alpha^*(\tau_0 g(\gamma)-\gamma) =& \alpha^*(\tau_0 g(\gamma)-g(\gamma))+\alpha^*(g(\gamma)-\gamma)\\
=& 2\beta^*(\tau_0 g(\gamma)-g(\gamma))+2\beta^*(g(\gamma)-\gamma)\\
=& 2\beta^*(\tau_0 g(\gamma)-\gamma)\\
\alpha^*(\tau_1 g(\gamma)-\gamma) =& \alpha^*(\tau_1 g(\gamma)-g(\gamma))+\alpha^*(g(\gamma)-\gamma)\\
=& 2\beta^*(\tau_1 g(\gamma)-g(\gamma))+2\beta^*(g(\gamma)-\gamma)\\
=& 2\beta^*(\tau_1 g(\gamma)-\gamma)\\
\alpha^*(rg(\gamma)-\gamma) =& \alpha^*(rg(\gamma)-g(\gamma))+\alpha^*(g(\gamma)-\gamma)\\
=& 2\beta^*(rg(\gamma)-g(\gamma))+2\beta^*(g(\gamma)-\gamma)\\
=& 2\beta^*(rg(\gamma)-\gamma)
\end{aligned}
$$

□

Proposition 4.5.9. *Let $\phi : MT(P_1) \to \mathcal{D}P_1$ be the natural covering map. For a closed curve γ on $MT(P_1)$, if $\phi(g(\gamma))$ is null-homologous for some $g \in \langle \tau_0, \tau_1, r \rangle$, then $\beta^*(\gamma)$ is even, and $\alpha^*(\gamma) = 2\beta^*(\gamma)$.*

Proof. Note that $\beta^*(\tau_1(\beta_i)-\beta_i), \beta^*(\tau_1(\alpha_i)-\alpha_i), \beta^*(\tau_1(\omega)-\omega)$ are all even. So $\beta^*(\tau_1(\gamma')-(\gamma'))$ is always even for any loop γ'. In particular, $\beta^*(\tau_1(\gamma'))$ is even iff $\beta^*(\gamma')$ is even.

Now note that $b^*(\tau_0(\beta_i)-\beta_i), \beta^*(\tau_0(\omega)-\omega)$ are even, while $\beta^*(\tau_0(\alpha_i)-\alpha_i)$ is odd. As a result, we have $\beta^*(\tau_1(\gamma')-(\gamma'))-\alpha^*(\gamma')$ always even. Now because $\phi(g(\gamma))$ is null-homologous, $\alpha^*(g(\gamma))=0$, and because $\alpha^*(g(\gamma)-\gamma)$ is even by the above proposition, we have $\alpha^*(\gamma)$ even. Finally for any $h \in \langle \tau_0, \tau_1, r \rangle$, we apply the above proposition again, and we have $\alpha^*(h(\gamma))$ even. So for $\gamma' = h\gamma$ for some $h \in \langle \tau_0, \tau_1, r \rangle$, $\beta^*(\tau_0(\gamma'))$ is even iff $\beta^*(\gamma')$ is even.

Finally $\beta^*(r(\gamma')) = \beta^*(r(\gamma'))$ for all γ'. So by induction, we can conclude that $\beta^*(\gamma)$ is even iff $\beta^*(g(\gamma))$ is even. But $\phi(g(\gamma))$ is null-homologous, so $\beta^*(g(\gamma))=0$. So $\beta^*(\gamma)$ is even. Finally, by the above proposition again, $\alpha^*(g(\gamma)-\gamma) = 2\beta^*(g(\gamma)-\gamma)$, but $\alpha^*(g(\gamma)) = \beta^*(g(\gamma)) = 0$, so $\alpha^*(\gamma) = 2\beta^*(\gamma)$. □

Proof of Main Theorem 4.5.1. Suppose there is a stable periodic billiard path. Then we can find a closed geodesic γ on $MT(P_1)$ such that $\phi(\gamma)$ is null-homologous. By enumeration theorem we know γ is homologous to $g(\gamma')$ for $\gamma' \in \{\pm\gamma_1, ..., \pm\gamma_{10}\}$ and some $g \in \langle \tau_0, \tau_1, r \rangle$. Then we must have $\beta^*(\gamma')$ even and $\alpha^*(\gamma) = 2\beta^*(\gamma)$. Now we compute the pair $(\alpha^*(\gamma_i), \beta^*(\gamma_i))$ for each i, none would be the desired one. So our theorem is proven. □

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