Chip-firing Using M-Bases

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1 Introduction

A chip-firing game on a graph involves changing between chip configurations of the graph by "firing" its vertices, redistributing the chips at each vertex to obtain a new configuration. These games can be defined according to many different rules which can lead the chip-firing process to have certain properties, such as the existence of configurations where the game cannot continue. By studying certain types of configurations with respect to a chip-firing game, such as z-superstable configurations, we can understand more about the graph itself and its combinatorial properties.

It is therefore of interest to find a way to define a chip-firing game on the dual of a graph. Typically, only planar graphs have well-defined dual graphs but by moving to the more general setting of matroids one can consider the cographic matroid which is dual to the matroid associated to a graph. In particular, by finding a basis of the cycle space of the graph with certain properties, known as an M-basis, one can define such a chip-firing game and study its properties. To this end, multiple types of M-bases can be defined such as cycle M-bases, which consist of cycles of the graph and therefore have more combinatorial properties than an arbitrary basis. By understanding the chip-firing game on the dual of a graphic matroid, it is possible that one could answer longstanding questions such as Stanley's h-vector conjecture for graphic matroids. In this thesis, we give appropriate background information to understand these M-bases, describe some of their known properties, and contribute new results concerning cycle M-bases and fundamental cycle M-bases.

In Section 2, we introduce many of the main concepts in chip-firing including M-matrices, the critical group, and z-superstable configurations.

Section 3 defines matroids and discusses their properties and various definitions. In particular, we describe graphic matroids, their dual matroids known as cographic matroids, and regular matroids.

We define M-bases in Section 4, both the more general flow M-bases and the more combinatorial cycle M-bases. Here we also discuss how these M-bases allow us to define chip-firing games on the duals of the matroids associated to graphs, or more generally on regular matroids.

Sections 5 and 6 are both joint work with Chi Ho Yuen and contain new results which we have obtained. Section 5 introduces fundamental cycle M-bases and provides a precise characterization of the class of graphs which admit these types of cycle M-bases. In Section 6 we discuss various computational results that we have obtained regarding cycle M-bases and briefly mention future directions this work will be continued.

2 Chip-firing on Graphs

Chip-firing is a process which can be defined on finite graphs, describing transitions between chip configurations. One can think of stacks of chips on every vertex of the graph, which can be distributed to adjacent vertices along the edges incident to the vertex according to some rule. In particular, there are conditions under which these chips are allowed to "fire," meaning distribute their chips, and specific numbers of chips they will send to each adjacent vertex when they fire. In particular, this process will be dictated by a chosen matrix. If this matrix has certain properties, one can guarantee that this process will reach a stable state where the game can no longer continue. Such a matrix is known as an M-matrix. Studying chip-firing games defined by such an M-matrix and associated concepts such as the critical group and z-superstable configurations can reveal combinatorial properties of a graph and is valuable for their study. This section is based primarily on Caroline Klivans' book on chip-firing [5].

Let $G = (V, E)$ be a finite graph with n vertices. Then we define a **chip configuration** on G to be a vector $\mathbf{c} \in \mathbb{Z}^n_{\geq 0}$. If the vertices of G are $\{v_1, \ldots, v_n\}$, then $\mathbf{c}_i$ is the number of chips at the vertex v_i in the configuration described by $\mathbf{c}$.

In the classical chip-firing game, when $\mathbf{c}_i$ is greater than or equal to the number of neighbors of v_i, meaning:

$$\mathbf{c_i} \geq \deg(v_i)$$

then we say that the vertex v_i is ready to fire, or that firing it is **legal**. When v_i fires, it sends chips to its neighbors along each incident edge. So $\mathbf{c}_i$ is reduced by $\deg(v_i)$, and for each neighbor v_j of v_i, $\mathbf{c}_j$ is increased by 1. Notice that this always gives a new chip configuration on G, due to the constraint on when v_i is ready to fire.

One can continue to fire vertices in G so long as there exist vertices which are ready to fire in the resulting chip configurations. A chip configuration $\mathbf{c}$ is called **stable** if, for every vertex $v_i \in V$:

$$\mathbf{c}_i < \deg(v_i)$$

So stable chip configurations are those in which no vertex can legally fire and thus the chip-firing game cannot continue.

Next we define a matrix associated to a simple graph G known as the **graph Laplacian**.

Definition 2.1 (Graph Laplacian). The graph Laplacian $\Delta(G)$ associated to G is the $n \times n$

matrix defined by:

$$\Delta(G)_{i,j} = \begin{cases} -1 & \text{if } i \neq j \text{ and there exists } e \in E \text{ between } v_i \text{ and } v_j, \\ \deg(v_i) & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Equivalently, if A is the adjacency matrix associated to G and D is the diagonal matrix such that $D_{i,i} = \deg(v_i)$, then $\Delta(G) = D - A$. Finally, if ∂_G is the signed incidence matrix for some chosen orientation of G, we see that $\Delta(G) = \partial_G \partial_G^T$.

The alternate definition $\Delta(G) = \partial_G \partial_G^T$ implies that $\Delta(G)$ is also symmetric. Notice that we can also define the graph Laplacian for graphs which are not simple- the off-diagonal terms $L_{i,j}$ will simply be $-$(the number of edges between v_i and v_j).

We can describe the process of chip-firing using this graph Laplacian. Say we have a chip configuration $\mathbf{c}$ and we fire the vertex v_i to obtain a new chip configuration $\mathbf{c}'$. Then if $\mathbf{e}_i$ is the ith standard basis vector of $\mathbb{R}^n$:

$$\mathbf{c}' = \mathbf{c} - \Delta(G)^T \mathbf{e}_i$$

This corresponds to the firing of v_i since the vector $\Delta(G)^T \mathbf{e}_i$ is the ith column of $\Delta(G)$. Note that since $\Delta(G)$ is symmetric, $\Delta(G)^T = \Delta(G)$ so we may use them interchangeably. We see by the definition of the graph Laplacian that subtracting this vector, subtracting $\deg(v_i)$ chips from $\mathbf{c}_i$ and adding one to the number of chips at all of v_i's neighbors, gives the chip-configuration vector $\mathbf{c}'$ corresponding to the firing of this vertex.

So far, we have only considered firing one vertex at a time. If we consider a subset of the vertices $V' = \{v_{i_1}, \ldots, v_{i_m}\} \subseteq V$ then we can define a characteristic vector for X:

$$\chi_{V'} = \sum_{j=1}^{m} \mathbf{e}_{i_j}$$

The **cluster-fire** of V' from configuration $\mathbf{c}$ gives a new configuration $\mathbf{c}'$ defined by:

$$\mathbf{c}' = \mathbf{c} - \Delta(G)\chi_{V'}$$

Similarly to before, cluster-firing is called legal when $\mathbf{c}' \in \mathbb{Z}^n_{\geq 0}$. By its definition, we can see that it is equivalent to firing the vertices in V' in succession. However, the intermediate steps in a legal cluster fire need not be legal firing moves when performed sequentially if the end result is a configuration where all $\mathbf{c}_i \geq 0$.

We remark that the choices made in a chip-firing game of which vertices to fire and when are not particularly important.

Theorem 2.1. [5, Theorem 2.2.2]

1. If $\mathbf{c}_1$ and $\mathbf{c}_2$ are two configurations which can both be reached from $\mathbf{c}$ after firing a single vertex, there is some configuration $\mathbf{c}'$ which can be reached from both $\mathbf{c}_1$ and $\mathbf{c}_2$ after a single firing.

2. If $\mathbf{c}'$ is a stable configuration which can be reached from a configuration $\mathbf{c}$ after a finite number of legal fires, then $\mathbf{c}'$ is the unique such stable configuration.

Therefore, the choices made in a chip-firing game do not affect the end result, assuming the game reaches a stable state.

Next, consider a graph $G = (V, E)$ which has $n + 1$ vertices, where the 0th vertex, denoted q, is called the **sink** vertex. Then we can also define a chip-firing game in this context, but for a chip-configuration $\mathbf{c}$ on G, we let $\mathbf{c}_q$ be any integer, including negative integers. Furthermore, we do not allow q to ever fire, hence denoting it a sink: chips which enter q never leave. Finally, in this context, a stable configuration is one in which no vertex other than q can legally fire.

Since we do not care about the number of chips in the sink, we can instead consider configurations $\mathbf{c}$ as vectors of length n, as before. Then we can consider the **reduced graph Laplacian** $\Delta(G)_q$, which is $\Delta(G)$ with the 0th row and column deleted (those corresponding to q). Then as before, this matrix dictates the chip-firing action. Firing a vertex v_i yields a new configuration:

$$\mathbf{c}' = \mathbf{c} - \Delta(G)_q^T \mathbf{e}_i$$

Notably, in this setting we know that all chip-firing games must eventually result in a stable configuration.

Proposition 2.1. [5, Proposition 2.5.2] If G is a finite graph with sink q, then any chip configuration $\mathbf{c}$ will reach a stable configuration after a finite number of vertices other than q are fired.

This property of $\Delta(G)_q$ is also known as **avalanche-finiteness**.

Notice that for a finite graph $G = (V, E)$ with $|V| = n$, we could potentially define a chip-firing action using an arbitrary $n \times n$ matrix L by letting the configuration obtained from $\mathbf{c}$ by firing v_i be:

$$\mathbf{c}' = \mathbf{c} - L^T \mathbf{e}_i$$

In this general setting, a stable configuration will be one where $\mathbf{c}_i < L_{i,i}$. One could also define a chip-firing action with sink by taking an $(n-1) \times (n-1)$ matrix L instead. A natural question is which matrices L define a chip-firing game which always reaches a stable configuration, like the classical chip-firing with sink. To this end, we formally define **avalanche-finiteness** of a matrix:

Definition 2.2. A matrix L is called avalanche-finite if the chip-firing action it defines has the property that any chip configuration $\mathbf{c}$ will reach a stable configuration after a finite number of vertices are fired.

So more precisely, we want to know which matrices are avalanche-finite.

If L is a matrix such that $L_{i,j} \leq 0$ for $i \neq j$ and $L_{i,j} \geq 0$ for $i = j$, then L is called a

Z-matrix. For a Z-matrix, we have the following characterization of avalanche-finiteness:

Proposition 2.2. [3, Definition 2.2] Let L be an $n \times n$ Z-matrix. Then we call L a non-singular M-matrix when any of the following equivalent statements are true:

1. L is avalanche-finite.

2. The real parts of the eigenvalues of L are positive.

3. The inverse matrix L^{-1} exists and $L^{-1}_{i,j} \geq 0$ for all i, j.

4. There exists a vector $\mathbf{x}$ such that $\mathbf{x} \geq 0$ and $L\mathbf{x}$ has all positive entries.

These equivalences are summarized in a paper by Johnny Guzmán and Caroline Klivans and their proofs lie in the references therein.

For simplicity, we will simply call these matrices M-matrices, dropping the non-singular descriptor. When one has avalanche-finiteness, every chip configuration $\mathbf{c}$ has a unique stable configuration which is reachable from it by the definition of avalanche-finiteness and from the second part of Theorem 2.1. This unique configuration will be denoted $\text{stab}(\mathbf{c})$.

We define a **critical** configuration to be a chip configuration which is stable and is reachable from some configuration where every non-sink vertex is ready to fire. Notice that for any two critical configurations $\mathbf{c}$ and $\mathbf{d}$, $\text{stab}(\mathbf{c}+\mathbf{d})$ is a critical configuration as well. Therefore, we have a sum defined on critical configurations which yields a group structure on the set of critical configurations of a given finite graph G with sink. This group is called the **critical group** or **sandpile group** of G, and is denoted $\kappa(G)$. It is not immediately clear that this sum actually gives a group structure, but there is an alternate definition as a quotient group:

Definition 2.3. [5, Definition 4.2.1] The critical group for a graph G with n vertices including a sink q is the integer cokernel of the reduced Laplacian:

$$\kappa(G) \cong \mathbb{Z}^{n-1}/\text{im}(\Delta(G)_q) \cong \text{coker}(\Delta(G)_q)$$

Similarly, one can define the critical group for the chip-firing action given by any M-matrix L, which consists of critical configurations with respect to this action.

Definition 2.4. [5, Definition 6.2.1] The critical group of an avalanche-finite matrix L is the integer cokernel of its transpose:

$$\kappa(L) \cong \mathbb{Z}^n/\text{im}(L^T) \cong \text{coker}(L^T)$$

Here, n is the number of vertices in a graph G on which we define a chip-firing action. Then a configuration $\mathbf{c}$ is **z-superstable** if for every $\mathbf{z} \in \mathbb{Z}^n$ where $\mathbf{z} \geq 0$ and $\mathbf{z} \neq 0$ there is some i such that:

$$\mathbf{c}_i - (L^T\mathbf{z})_i < 0.$$

In other words, a configuration is called z-superstable if there are no legal cluster-fires (with multiplicity).

There is a duality between the notions of critical and z-superstable configurations. Let L be an M-matrix and define $\mathbf{c}_{\max}$ to be the chip configuration such that:

$$\mathbf{c}_{\max_i} = L_{i,i} - 1$$

Then we have that a configuration $\mathbf{c}$ is critical if and only if the configuration $\mathbf{c}_{\max} - \mathbf{c}$ is z-superstable [5, Theorem 6.3.8].

3 Matroids

Matroids serve as a combinatorial generalization of the idea of independence and have many equivalent characterizations. They are useful for generalizing properties of different combinatorial structures. In particular, matroids can be used to define a dual of a graph which is not planar by considering the dual matroid of a matroid which can be represented by the graph. This section is based primarily on the exposition in [1].

We define matroids on some finite ground set, which we will denote by E.

Definition 3.1 (Independent Sets)**.** A matroid is a pair $M = (E, \mathcal{I})$, consisting of a finite set E and a collection of subsets of E denoted by $\mathcal{I}$, satisfying the following axioms:

(I1) $\varnothing \in \mathcal{I}$.

(I2) If $J \in \mathcal{I}$ then $I \subseteq J$ implies $I \in \mathcal{I}$.

(I3) If $I, J \in \mathcal{I}$ then $|I| < |J|$ implies there exists some $j \in J - I$ such that $I \cup \{j\} \in \mathcal{I}$.

The elements of $\mathcal{I}$ are called **independent sets**. Furthermore, we call any maximal independent set a **basis** of the matroid. Using the axiomatic definition of independence, we immediately make the following observation about bases:

Proposition 3.1. All bases of a matroid have the same size.

Proof. Let B_1, B_2 be two bases of a matroid $M = (E, \mathcal{I})$. For contradiction, suppose that $|B_1| < |B_2|$. Then since $B_1, B_2 \in \mathcal{I}$, by axiom (I3) we see that there exists $b \in B_2 - B_1$ such that $B_1 \cup \{b\} \in \mathcal{I}$. But this suggests that there exists an independent set which strictly contains B_1, meaning B_1 is not maximal. This contradicts the fact that B_1 is a basis. Thus, $|B_1| = |B_2|$. □

Let $\mathcal{B}$ be the collection of all bases of a matroid M. We can alternatively define M as follows:

Definition 3.2 (Bases)**.** A matroid is a pair $M = (E, \mathcal{B})$, consisting of a finite set E and a collection of subsets of E denoted by $\mathcal{B}$, satisfying the following axioms:

(B1) $\mathcal{B} \neq \varnothing$.

(B2) If $B_1, B_2 \in \mathcal{B}$ and $b_1 \in B_1 - B_2$ then there exists some $b_2 \in B_2 - B_1$ such that $(B_1 - b_1) \cup \{b_2\} \in \mathcal{B}$.

In this way, we can define a matroid by its bases, describing the same combinatorial object in a different way. One can easily pivot between these definitions, obtaining the bases by taking maximal independent sets and obtaining the independent sets by taking all subsets of the bases.

Any subset of E which is not an independent set is called a **dependent set**. In particular, the **circuits** of a matroid M are its minimal dependent sets. Like with bases, one can directly define a matroid using a set of axioms for circuits:

Definition 3.3 (Circuits)**.** A matroid is a pair $M = (E, \mathcal{C})$, consisting of a finite set E and a collection of subsets of E denoted by $\mathcal{C}$, satisfying the following axioms:

(C1) $\varnothing \notin \mathcal{C}$.

(C2) If $C_1 \in \mathcal{C}$ then $C_1 \subsetneq C_2$ implies $C_2 \notin \mathcal{C}$.

(C3) If $C_1, C_2 \in \mathcal{C}$ and $e \in C_1 \cap C_2$ then there exists $C \in \mathcal{C}$ such that $C \subseteq (C_1 \cup C_2) - \{e\}$.

So far, we have characterized matroids by different collections of subsets of E. However, we can also describe a matroid using a function. We define a function on the power set of E called the **rank function** which sends a subset $A \subseteq E$ to the size of the largest independent subset of A. Therefore, the independent subsets of E will be exactly the subsets which are sent to their own cardinality by the rank function. As one might expect, such a function must satisfy certain axioms and can be used to define a matroid without explicitly giving the independent sets, bases, or circuits.

Definition 3.4 (Rank Function)**.** A matroid is a pair $M = (E, r)$, consisting of a finite set E and a function $r : 2^E \to \mathbb{N}$ satisfying the following axioms:

(R1) $0 \leq r(A) \leq |A|$ for every $A \subseteq E$.

(R2) If $A \subseteq B \subseteq E$ then $r(A) \leq r(B)$.

(R3) For every $A, B \subseteq E$, $r(A) + r(B) \geq r(A \cup B) + r(A \cap B)$.

We will refer to the value the rank function takes on a subset of E (given some matroid M on E) as the **rank** of that set. Furthermore, we denote the size of a basis (which we have shown is always the same) by r.

A **flat** is a subset A of E such that for any $e \notin A$, $A \cup e$ contains an independent set which is larger than any independent subset of A, i.e., $r(A \cup \{e\}) > r(A)$. In this way, a flat can be thought of as a maximal set of a given rank.

One can obtain a lattice L of these flats by constructing their poset, ordered by inclusion. Then we can define a matroid via its flats in the following way:

Definition 3.5 (Flats)**.** A matroid is a pair $M = (E, \mathcal{F})$, consisting of a finite set E and a collection of subsets of E denoted by $\mathcal{F}$ such that the lattice L of these subsets ordered by inclusion is a geometric lattice.

For more information on the criteria for a lattice to be geometric, see [1, p.66].

Perhaps the most explicitly geometric view of a matroid is via the matroid polytope. Given the bases $\mathcal{B}$ of the matroid M, we can obtain a polytope $P_M = \mathrm{conv}\{\mathbf{e}_{b_1} + \cdots + \mathbf{e}_{b_r} : \{b_1, \ldots, b_r\} \in \mathcal{B}\}$ in $\mathbb{R}^E$. We can similarly define a matroid using this construction of polytope:

Definition 3.6 (Matroid Polytope)**.** A collection of subsets $\mathcal{B}$ of E is a collection of bases of a matroid on E if and only if every edge of the polytope $P = \mathrm{conv}\{\mathbf{e}_{b_1} + \cdots + \mathbf{e}_{b_r} : \{b_1, \ldots, b_r\} \in \mathcal{B}\}$ is a translate of $\mathbf{e}_i - \mathbf{e}_j$ for some $i, j \in E$.

A useful class of matroids for understanding these abstract definitions are **linear matroids**. One can take the set E to be some finite subset of a vector space V over a field K. Then a linear matroid is obtained by letting a subset of E be independent if and only if the vectors in the subset are all linearly independent. Therefore, when we take E such that the span of E is V, the bases of the matroid will be the bases of V as a vector space.

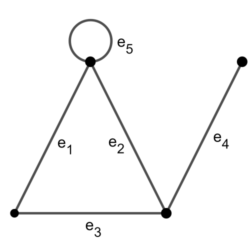
Now let E be the edges of some connected graph G. Then one can define a matroid from this G, called a **graphic matroid**, by letting the circuits of the matroid be the cycles of the graph. Then the independent sets of this matroid will be the forests of the graph and the bases will be the spanning forests.

While there are matroids which are neither linear nor graphic, some matroids are actually both linear and graphic. Take for example, the matroid $M = (E, \mathcal{B})$ where $E = \{e_1, e_2, e_3, e_4, e_5\}$ and $\mathcal{B} = \{\{e_1, e_2, e_4\}, \{e_1, e_3, e_4\}, \{e_2, e_3, e_4\}\}$. Then we can take these elements to be vectors in $\mathbb{R}^3$:

$$e_1 = (1, 0, 0),\ e_2 = (0, 1, 0),\ e_3 = (1, 1, 0),\ e_4 = (0, 0, 1),\ e_5 = (0, 0, 0)$$

It is simple to check that the maximal linearly independent subsets of these vectors are indeed the set of bases we have given. Notice that this representation of the matroid is not unique, as for example, one could take $e_3 = (x, y, 0)$ for any nonzero $x, y \in \mathbb{R}$.

Now we can also let these elements be the edges in a graph.



This is the graph associated to our matroid since first we notice that e_1, e_2, e_3 must be a dependent set of elements, meaning that their associated edges must form a cycle, giving the copy of K_3 we see. On the other hand, e_4 is not dependent on any of these edges, so it must be an edge between K_3 and a new vertex. Since e_5 is always dependent, even by itself, it must be a cycle on its own, and thus a loop on the graph. Notice that e_4 could be taken to be adjacent to any of the vertices in the copy of K_3 and e_5 could be a loop at any vertex of the graph and the matroid structure would be the same.

This illustrates the power of matroids as abstract combinatorial objects, generalizing the study of graph theory and linear algebra, not to mention other subjects.

This example also exhibits some special types of elements that occur in matroids. The element e_5 from the example is appropriately called a **loop**. In general, a loop is an element $e \in E$ such that $r(\{e\}) = 0$, meaning e is dependent by itself. On the other hand, a **coloop** is an element $e \in E$ such that $r(E - \{e\}) = r - 1$. Such an element is therefore independent of all other elements of E and contained in every basis of the matroid. In our example, e_4 is a coloop.

Given a matroid M, there are various operations on M which will yield more matroids. First, one can take the **dual matroid** of M:

Definition 3.7 (Dual Matroid)**.** Given a matroid $M = (E, \mathcal{B})$, the dual matroid of M is $M^* = (E, \mathcal{B}^*)$ where $\mathcal{B}^* = \{E - B : B \in \mathcal{B}\}$.

It is clear from the definition that the dual matroid of M^* is M once again. We call the circuits of M^* the **cocircuits** of M. Also, note that the duals of graphic matroids are known as cographic matroids.

The familiar operations of deletion and contraction on graphs extend to operations on general matroids:

Definition 3.8 (Deletion)**.** Let $M = (E, \mathcal{I})$ be a matroid. Given $e \in E$, let $\mathcal{I}' = \{I \in \mathcal{I} : e \notin I\}$. Then $M \backslash e = (E - e, \mathcal{I}')$ is a matroid known as the **deletion** of e from M.

Definition 3.9 (Contraction)**.** Let $M = (E, \mathcal{I})$ be a matroid. Given $e \in E$, let

$$\mathcal{I}' = \begin{cases} \mathcal{I} & \text{when } e \text{ is a loop,} \\ \{I \subset E - \{e\} : I \cup \{e\} \in \mathcal{I}\} & \text{otherwise.} \end{cases}$$

Then $M/e = (E - e, \mathcal{I}')$ is a matroid known as the **contraction** of e from M.

These two operations commute so the matroid obtained successive deletion and contraction can be written simply as $M \backslash A/B$ for some disjoint subsets $A, B \subset E$ (where the elements in A have been deleted and the elements in B have been contracted). Matroids of this form are known as the **minors** of M. We also note that the dual matroid of the deletion of e from M is the contraction of e from M^*:

$$(M \backslash e)^* = M^*/e$$

We introduce an interesting class of matroids known as **regular matroids**:

Definition 3.10. A regular matroid is a matroid which, for any field K, can be represented as a linear matroid over K. An equivalent characterization is that they are precisely the matroids which can be defined over $\mathbb{R}$ from a totally unimodular matrix, a matrix whose minors are all $0, 1$, or -1.

We note that every graphic or cographic matroid is a regular matroid. On the other hand, any regular matroid can be constructed from a graphic matroid, cographic matroid, and a regular matroid with 10 elements known as R_{10} by operations known as 1-sums, 2-sums, and 3-sums [7].

This particular regular matroid R_{10} is therefore significant in the general theory of regular matroids. It is neither graphic nor cographic, and it has the following matrix representation:

$$R_{10} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & -1 \end{pmatrix}$$

An incredibly useful tool in the study of matroids is the **Tutte polynomial** of a matroid $M = (E, r)$. It is a two-variable polynomial which is uniquely associated to the matroid M, depending explicitly on the finite ground set E and the rank function r. It is given by the expression:

$$T_M(x, y) = \sum_{A \subseteq E} (x-1)^{r-r(A)} (y-1)^{|A|-r(A)}$$

Many invariants of matroids result as evaluations of this polynomial. For example, the number of independent sets of the matroid is $T_M(2, 1)$ and the number of bases is $T_M(1, 1)$. A more specific use is its relation to the chromatic polynomial of a graph. The chromatic polynomial of G, denoted $\chi_G(q)$, yields the number of proper colorings of G using q colors. It is related to the Tutte polynomial of G as a graphic matroid M by:

$$\chi_G(q) = (-1)^{v-c} q^c T_M(1-q, 0)$$

where v is the number of vertices in G and c is the number of connected components in G. The Tutte polynomial appears in many other parts of graph theory, including chip-firing on graphs.

The Tutte polynomial also satisfies a recursive relation which can be useful for its computation.

Theorem 3.1. [1, Proposition 7.6.1] For any matroid M and $e \in E$, we have

$$T_M(x, y) = \begin{cases} yT_{M \setminus e}(x, y) & \text{if } e \text{ is a loop,} \\ xT_{M/e}(x, y) & \text{if } e \text{ is a coloop,} \\ T_{M \setminus e}(x, y) + T_{M/e}(x, y) & \text{otherwise.} \end{cases}$$

A final useful property of the Tutte polynomial is the relationship between the Tutte polynomials of a matroid and its dual:

Theorem 3.2. [1, Proposition 7.6.2] Given a matroid M,

$$T_{M^*}(x, y) = T_M(y, x)$$

4 Integral Flow and Cycle Chip-firing

Now that we have an understanding of chip-firing and matroids, we can define chip-firing on the duals of graphic matroids. Specifically, we consider consider bases of the cycle spaces of graphs which can be used to yield an M-matrix and define an avalanche-finite chip-firing game on the dual matroid. In general, these bases are called flow M-bases, but we also consider cycle M-bases which have more combinatorial significance. We also discuss the motivation for defining this chip-firing game, particularly in how chip-firing has been used in the past to prove Stanley's h-conjecture for cographic matroids. This section is based on the work of Anton Dochtermann, Eli Meyers, Raghav Samavedan, and Alex Yi [2].

Let $G = (V, E)$ be a finite connected simple graph, where V is the set of vertices and E is the set of edges. Then we define the **lattice of integral flows** $\mathcal{F}(G)$ by

$$\mathcal{F}(G) = \mathbb{Z}^{|E|} \cap \ker(\partial_G)$$

and the **lattice of integral cuts** $\mathcal{C}(G)$ by

$$\mathcal{C}(G) = \mathbb{Z}^{|E|} \cap \operatorname{im}(\partial_G^T).$$

Removing a row from ∂_G, say one corresponding to a chosen sink vertex, gives a matrix $\tilde{\partial}_G$ whose columns form a $\mathbb{Z}$-basis of $\mathcal{C}(G)$. Notice that $\Delta(G)_q = \tilde{\partial}_G \tilde{\partial}_G^T$, so we can use such a basis to define an avalanche-finite chip-firing action on G.

Similarly, we can consider such a basis of $\mathcal{F}(G)$, which will allow us to obtain a chip-firing action on the dual of G. For planar graphs, one can embed G in the plane and then take the vertices of its dual graph G^* to be the regions of this embedding, with edges between them if they are bordered by the same edge in the embedding of G. In general, we can always consider the dual of G by looking at the dual of the graphic matroid associated to G. In the case of planar G, the dual matroid will be the matroid associated to this dual graph that we have described. However, when G is not planar the cographic dual matroid will not be a graphic matroid.

Returning to $\mathcal{F}(G)$, the dimension of the lattice of integral flows will be the **genus** g of G, a non-negative integer defined by:

$$g = |E| - |V| + 1$$

In the case of planar graphs, the genus can be thought of as the number of holes in the graph, or one less than the number of regions in an embedding.

So suppose $\{\mathbf{f}_1, \ldots, \mathbf{f}_g\}$ is a $\mathbb{Z}$-basis for $\mathcal{F}(G)$. Then letting ι be the matrix whose columns are these basis vectors, we can obtain a matrix $\mathcal{L}^* = \iota^T \iota$ with dimension $g \times g$ which we call the **dual Laplacian**. Thus, we can use $\mathcal{L}^*$ to define a chip-firing action on the dual graph G^*. This has a natural interpretation in the case of planar graphs as a chip-firing action on the dual graph, but in general gives a matrix which describes an abstract chip-firing action on the dual of a graphic matroid.

Ideally, this chip-firing action will be avalanche-finite, i.e., $\mathcal{L}^*$ will be an M-matrix. When $B = \{\mathbf{f}_1, \ldots, \mathbf{f}_g\}$ yields a dual Laplacian $\mathcal{L}^*$ which is an M-matrix, then we call B an **integral flow M-basis** or just a **flow M-basis**. Notice that by its definition, $\mathcal{L}^*$ is a symmetric positive definite matrix, meaning it always has positive real eigenvalues. Therefore, by our previous characterizations of avalanche-finiteness, $\mathcal{L}^*$ will be avalanche-finite whenever it is a Z-matrix. Regardless of G, one can always choose a basis B such that this is true:

Theorem 4.1. [2, Corollary 3.13] Any graph G admits an integral flow M-basis.

Therefore, one can always define this type of avalanche-finite chip-firing action on the dual matroid of the graph.

It is interesting to note that the dual Laplacian can be used to calculate the critical group of the chip-firing action on G. In particular:

Proposition 4.1. [2, Proposition 3.3] Let G be a graph and let $\mathcal{L}^*$ be its dual Laplacian. Then there is a group isomorphism:

$$\kappa(G) \cong \mathbb{Z}^g / \mathrm{im}(\mathcal{L}^*)$$

Given a flow M-basis B, we can define a chip configuration on the dual of G to be a **z-superstable flow configuration** when it is z-superstable with respect to the chip-firing action given by the dual Laplacian $\mathcal{L}^*$ associated to B. Then we remark that the bijection between z-superstable configurations on G and its critical configurations, along with the previous proposition, will yield that the number of z-superstable flow configurations is the same as the number of z-superstable configuration on G.

This hints at potential applications and benefits of this process of defining a chip-firing game on the dual of a graphic matroid. Stanley's h-vector conjecture is a long-standing problem in matroid theory concerning the nature of h-vector corresponding to a matroid M. The entries of this vector occur as coefficients of an evaluation of the Tutte polynomial, specifically:

$$T_M(x, 1) = \sum_i h_i x^{r-i}$$

Stanley's conjecture is that this h-vector is a pure O-sequence [4]. More specifically, this means that the h-vector has the following properties:

Theorem 4.2 (Stanley's h-vector conjecture)**.** For any matroid M, there exists a set X of monomials such that the following hold:

1. If m and m' are monomials with $m \in X$ and $m' \mid m$, then $m' \in M$.
2. All maximal monomials in X have the same degree.
3. The number of monomials of degree i in X is precisely h_i.

Criel Merino proved that cographic matroids satisfy the h-vector conjecture [6, Theorem 4.2]. In fact, in their proof they showed that the collection of critical configurations of the dual of the matroid corresponds to this set of monomials. This can be determined using the relationship between duality and the Tutte polynomial seen in Theorem 3.2:

$$T_{M^*}(x, 1) = T_M(1, y)$$

Merino's proof motivates defining and understanding chip-firing games on graphs and their duals. In particular, this conjecture has yet to be proven on graphic matroids. The z-superstable flow configurations are dual to the critical configurations of the chip firing action on the dual, so Merino's work suggests that understanding them and flow M-bases of graphs may aid in proving the h-vector conjecture for graphic matroids using similar methods. Another potential application includes studying the critical group for chip-firing games via the duality of matroids.

Next, we consider a more restrictive type of basis of the lattice of integral flows, known as a **cycle M-basis**.

Definition 4.1. A basis $B = \{\mathbf{f}_1, \ldots, \mathbf{f}_g\}$ is a cycle M-basis if B is a flow M-basis such that each $\mathbf{f}_i$ consists only of entries in $\{0, 1, -1\}$, with the non-zero entries corresponding to an oriented cycle of G.

We note that such a basis will be a cycle M-basis whenever the associated dual Laplacian is a Z-matrix. Thus by the definition of the dual Laplacian, we have the following observation:

Observation 4.1. A basis $B = \{\mathbf{f}_1, \ldots, \mathbf{f}_g\}$ is a cycle M-basis if the standard inner product of every pair of distinct basis elements is non-positive:

$$\langle f_i, f_j \rangle \leq 0 \text{ for } i \neq j$$

This observation makes it much simpler to check which cycle bases are cycle M-bases.

The flow M-basis is primarily a linear algebraic construction, but because this type of M-basis consists of cycles of the graph, it has a more direct combinatorial interpretation. As an example, Mac Lane's planarity criterion says that a finite undirected graph is planar if and only if its cycle space (taken as a vector space over $\mathbb{Z}_2$) has a basis of cycles such that each edge is included in at most two of the basis cycles. This relates a property of the graph, planarity, to a cycle-basis. Therefore, studying cycle M-bases could be useful in establishing combinatorial properties of graphs which cannot be described using the more general flow M-bases, hence our motivation for studying them.

One class of graphs which always admit a cycle M-basis are planar graphs.

Proposition 4.2. [2, Proposition 4.2] Any planar graph admits a cycle M-basis.

Proof. Any planar graph G has g bounded faces, so take the cycles along the boundary of each of these faces. Choose some orientation for one of the bounded faces. Then one can always choose a compatible orientation for the cycles which share edges with it by choosing them to have opposite orientation along these edges. Continuing this process to orient all the cycles bounding faces of G, these oriented cycles form a cycle basis. Clearly, this will be an M-basis since the inner product of any two cycles which intersect will be negative by the choice of opposite orientations on shared edges. If they do not intersect then the inner product will be zero. Therefore, any planar graph G will admit a cycle M-basis. $\square$

Some non-planar graphs also have cycle M-bases, namely K_5 and $K_{3,3}$- this is proved in [2, Proposition 4.3].

Some other results regarding the nature of cycle M-bases involve limitations on the form which these bases can take for certain graphs, such as complete graphs. For example, we have the following propositions:

Proposition 4.3. [2, Proposition 4.4] Let $n \geq 5$ and B be a cycle M-basis for K_n. Then at least one element of B has more than 3 nonzero entries, i.e., there is a cycle in the basis which is not a 3-cycle.

Proposition 4.4. [2, Proposition 4.5] Let $3 \leq m \leq n$ and B be a cycle M-basis for $K_{m,n}$. Then at least one element of B has more than 4 nonzero entries, i.e., there is a cycle in the basis which is not a 4-cycle.

We can also consider cycle M-bases of regular matroids. In this case, we simply replace ∂_G with a totally unimodular real matrix representing the regular matroid. These matroids naturally extend the definition of M-bases for graphs. Indeed, the directed circuits of a regular matroid can be viewed as vectors with entries in $\{0, 1, -1\}$ and will be elements of the lattice of integral flows due to total unimodularity of the matrix. Therefore, a cycle M-basis of a regular matroid will be an M-basis consisting of circuits of the matroid. In this context, one may call them circuit M-bases for this reason.

5 Fundamental Cycle M-Bases

Proving theoretical results about cycle M-bases turns out to be difficult due to the potentially complicated intersections the cycles in the M-basis can have with each other. Recall that by Observation 4.1, one only needs to check that the inner product of any two cycles in the basis is non-positive. In general, these cycles can have intersections consisting of multiple disjoint paths, along some of which the inner product may be positive and along others negative so long as the total inner product is non-positive. This makes it harder to check whether this Z-matrix condition can be ensured on some new basis after performing operations on a graph with an M-basis such as taking minors of the graph. In fact, we have shown computationally that this cannot be the case. If simpler intersections between the cycles in a basis could be guaranteed, then proving more general properties of M-bases would be easier.

To this end, it is interesting to study a subset of cycle M-bases which will have nicer properties. In this section, we prove multiple theoretical results concerning fundamental cycle M-bases, a specific type of cycle M-basis also introduced in [2]. In particular, we give a complete characterization of graphs which admit a fundamental cycle M-basis as planar graphs whose 2-connected components have a planar dual consisting of an outerplanar graph and an additional apex vertex.

Recall that given a graph $G = (V, E)$, a spanning tree T of G is a connected acyclic subgraph of G which contains all of V. Note that T will contain exactly $|V| - 1$ edges. Furthermore, for any edge e in G but not in T, one can obtain a cycle by considering the unique cycle in $T + e$, which we will denote $C(T, e)$. This type of cycle is known as a **fundamental cycle** with respect to T. The number of these cycles will be the number of edges of G not included in T, so there are $g = |E| - |V| + 1$. In fact, these fundamental cycles form a basis of the cycle space.

We also introduce the concept of a **cut** of a graph G. A cut is a partition of the vertices of G into two disjoint subsets. The edges in the cut are the edges incident to a vertex in each of these subsets. Notice that removing an edge from a spanning tree T of G partitions the vertices into two subsets- those in each connected component of the tree. We define the **fundamental cut** of an edge e with respect to a spanning tree T (where $e \in T$) to be the cut obtained by removing e from T in this way.

We now consider **fundamental cycle M-bases**. They are defined as follows:

Definition 5.1. Let G be a finite connected graph with genus g. Then a fundamental cycle M-basis (FCMB) of G is a cycle M-basis which consists exactly of the g fundamental cycles with respect to a fixed spanning tree of G with some choice of orientations.

The cycles in an FCMB have nicer intersections like we desired.

Proposition 5.1. Let T be a fixed spanning tree of the graph G. Then two fundamental cycles with respect to T intersect along a single (possibly empty) path. In particular, if two such cycles both contain an edge e, then a choice of orientations on these cycles have non-positive inner product if and only if they have the opposite orientation on e.

Proof. Say we have two distinct fundamental cycles with respect to T, C_1 and C_2. Then their intersection must be contained entirely in T. In particular, it is the intersection of two paths contained in a tree, which must be a path (assuming it is nonempty): If the intersection contained multiple disjoint paths, then one could take a vertex v in one and vertex u in another. Then one has two paths between u and v in T, one along C_1 and one along C_2 which must be distinct, meaning there is a cycle contained in T. This is a contradiction so the intersection is indeed a path.

Now we let C_1 and C_2 be two fundamental cycles which have an edge e in their intersection.

Since C_1 and C_2 must intersect along a path, when they are oriented, they must either have the same orientation on every edge in the intersection or opposite orientation on every edge in the intersection. Notice that each edge in the intersection where they have opposite orientation contributes a summand of -1 to the inner product, while edges in the intersection where they have the same orientation contribute $+1$. The inner product is therefore strictly negative when the edges all have opposite orientations and strictly positive when they all have the same orientation, as it is exactly $\pm|C_1 \cap C_2|$. It is never zero since the intersection is non-empty by the assumption of e's existence. Since the orientations of the cycles on e determine the orientations on the entire path, we conclude that these cycles have non-positive inner product exactly when they have opposite orientations on e. $\square$

This nice intersection property gives rise to a property of FCMBs which will be extremely useful.

Proposition 5.2. Suppose that T is a spanning tree of a graph G which induces an FCMB. Then each edge of T is included in at most two fundamental cycles. In other words, the fundamental cut of an edge $e \in T$ is of size at most three.

Proof. If T induces an FCMB, then any two cycles of the basis must have non-positive inner product. Say $e \in T$ is an edge contained in more than two fundamental cycles. By 5.1, these cycles must all have pairwise opposite orientations on e. But by the pigeonhole principle, a third fundamental cycle will have the same orientation as one of the first two since there are only two choices of orientation. Therefore, it cannot have non-positive inner product with both of them, contradicting the assumption that these cycles form an FCMB. So each edge in T must only be included in at most two of the fundamental cycles. This implies that the fundamental cut of e with respect to T is at most size three, containing e and the at most two edges whose fundamental cycles with respect to T contain e. $\square$

This property immediately implies that some graphs do not admit an FCMB.

Proposition 5.3. K_5 and $K_{3,3}$ (and their subdivisions) do not admit an FCMB.

Proof. Fix a spanning tree T of K_5. Every cycle of K_5 contains at least 3 edges total, so every fundamental cycle with respect to T must contain at least 2 edges from T. K_5 has 10 edges and 5 vertices, so T contains a total of 4 edges and the genus of K_5 is 6. So each of the 6 fundamental cycles contain at least 2 of the 4 edges of T. By the pigeonhole principle, at least one edge of T must therefore be contained in more than 2 fundamental cycles with respect to T. Then by Proposition 5.2, K_5 does not admit an FCMB.

Similarly, fix a spanning tree T of $K_{3,3}$. Every cycle of $K_{3,3}$ contains at least 4 edges total, so every fundamental cycle with respect to T must contain at least 3 edges from T. $K_{3,3}$ has 9 edges and 6 vertices, so T contains a total of 5 edges and the genus of $K_{3,3}$ is 4. So each of the fundamental cycles contain at least 3 of the 5 edges of T. By the pigeonhole principle, at least one edge of T must therefore be contained in more than 2 fundamental cycles with respect to T. Then again by Proposition 5.2, $K_{3,3}$ does not admit an FCMB.

Subdivisions of K_5 and $K_{3,3}$ replace edges with paths which must always be entirely contained in any fundamental cycle they are included in. Therefore, the same arguments apply and subdivisions of these graphs cannot admit FCMBs either. $\square$

Unlike with cycle M-bases, subgraphs of graphs which admit FCMBs will also admit FCMBs.

Theorem 5.1. The class of graphs which admit an FCMB is closed under taking subgraphs.

Proof. Let G be a graph which admits an FCMB and let T be a spanning tree of G such that some orientation on the fundamental cycles with respect to T is an FCMB. Let e be an edge of G. We claim that $G - e$ has an FCMB.

If $e \notin T$, then the fundamental cycles of $G - e$ with respect to T are clearly an FCMB, as their pairwise inner products are unaffected. Similarly, if e is a bridge then no cycle of G includes e, so the fundamental cycles with respect to T become the fundamental cycles with respect to the subgraphs of T in the respective connected component of $G - e$ and we see that we still have an FCMB of each connected component.

Given this, we can assume that $e \in T$ and $G - e$ is a connected graph. Then the cut of e with respect to T contains edges other than e, so there is at least one edge f in the cut. Note that by Proposition 5.2 it contains at most two edges other than e, so there may be an additional edge f' in the cut. Then we see that $T - e + f$ is a spanning tree of $G - e$. Its fundamental cycles with respect to edges not in the cut will be the same as those of T. If there is no f', then we are done. It remains to check the fundamental cycle of f' with respect to $T - e + f$ is compatible with the other fundamental cycles as an M-basis when f' exists in the cut. Note that since we had an FCMB, the fundamental cycles $C(T, f)$ and $C(T, f')$ must have opposite orientation along their intersection, including e. Then one can take the symmetric difference of these cycles to obtain a new cycle $C(T, f)\Delta C(T, f')$, the cycle which follows $C(T, f)$ until it reaches its intersection with $C(T, f')$ and then follows that path, opposite orientations along the intersection guaranteeing that all the edges in the symmetric difference have the same orientation as in the original fundamental cycles. We notice that this cycle only has two edges not in T, f and f'. Thus, it is in fact the fundamental cycle of f' with respect to the tree $T - e + f$. But since only two fundamental cycles can use any given edge in the tree, the fundamental cycles of T do not have edges in the intersection of $C(T, f)$ and $C(T, f')$. So the inner product of a fundamental cycle with respect to T which does not include e with $C(T - e + f, f')$ is the sum of its inner product with $C(T, f)$ and $C(T, f')$. Since these inner products are non-positive by assumption, their sum will also be non-positive, meaning that this tree does in fact give an FCMB of $G - e$. So $G - e$ has a fundamental cycle M-basis.

By induction, we conclude that any subgraph of G has a fundamental cycle M-basis, so this property is closed under taking subgraphs. $\square$

Combining this theorem with 5.1, we can greatly narrow down the class of graphs which admit FCMBs.

Corollary 5.1. Every graph that admits an FCMB is planar.

Proof. By 5.3, K_5, $K_{3,3}$, and their subdivisions do not admit FCMBs. Furthermore, Theorem 5.1 tells us that no graph which admits an FCMB can contain K_5, $K_{3,3}$, or any of their subdivisions as a subgraph. Therefore, by Kuratowski's theorem any graph which admits an FCMB is planar. □

Since every graph which admits an FCMB is planar, we can discuss the nature of planar embeddings of this class of graphs. Specifically, first we look at 2-connected graphs, i.e., graphs which remain connected when any one of their vertices is deleted.

Theorem 5.2. Let G be a planar 2-connected graph. Then G admits a fundamental cycle M-basis if and only if there exists a plane embedding of G such that some FCMB consists exactly of the boundaries of bounded faces (with orientations induced by the plane orientation).

Proof. Pick an arbitrary planar embedding of G and fix some fundamental cycle M-basis B with respect to a spanning tree T. Every cycle in B bounds some number of the bounded faces of the embedding. We define the size of a basis B to be the sum of the number of faces bounded by each cycle in the basis. Since these cycles are non-trivial, they all bound at least one face, so if the size of B is g, each cycle must bound exactly one face, in which case we have the desired FCMB. Note that because G is 2-connected, the boundary of every bounded face will indeed be a cycle.

Now assume that some fundamental cycle $C(T, f)$ bounds more than one face. We want to find another fundamental cycle $C(T, f')$ which bounds exactly one of the faces F which $C(T, f)$ does and shares an edge e with $C(T, f)$.

Say such a fundamental cycle does not exist. Then the boundary of every bounded face with a boundary edge in $C(T, f)$ must contain at least two edges not in the spanning tree T. Every fundamental cycle of one of these edges will be contained in the region bounded by $C(T, f)$ since otherwise there would be a cycle contained in the tree composed of the part of the fundamental cycle outside of the region and the path between the vertices where it leaves the region along $C(T, f)$. This path is guaranteed to be contained in T since if it contained f, this new cycle would also be the fundamental cycle of f with respect to T which is a contradiction.

Notice that at least one of these non-$C(T, f)$ fundamental cycles must bound more than one face. If not, then for every non-tree edge f' inside the face bounded by $C(T, f)$, $C(T, f')$ must be the boundary of some internal face, i.e. a face whose boundary cycle does not intersect $C(T, f)$. So fix a face F which shares at least one edge with $C(T, f) - f$, i.e., one which is not an internal face, and whose boundary cycle does not contain f. One can take a maximal path P in the boundary cycle which consists of edges not contained in $C(T, f)$. Denote the endpoints of P by u and v. These vertices are both in $C(T, f)$ or else P could be extended and would not be maximal. So there is a path between u and v only using edges in $C(T, f) - f$. On the other hand, one can travel from u to v along

P, but whenever there is an edge $f' \notin T$, instead go along the path $C(T, f') - f'$ before continuing along P. By assumption, $C(T, f') - f'$ is disjoint from $C(T, f)$ so this yields a path P' between u and v which does not contain any edges in $C(T, f)$. Then P' together with the path along $C(T, f)$ will give a cycle contained in T. This gives a contradiction, so there must be some fundamental cycle contained in this region which bounds more than one face. Take this cycle to be the new $C(T, f)$ since it bounds more than one face. We note that it contains strictly fewer faces than the previous choice of $C(T, f)$. As long as the boundaries of the faces intersecting our choice of $C(T, f)$ contain at least two edges not in T, we can continue choosing smaller fundamental cycles to be $C(T, f)$. This process must terminate so eventually we must obtain a fundamental cycle $C(T, f)$ such that the boundary of a face intersecting $C(T, f)$ contains only one edge not in T. Then one can find the desired $C(T, f')$ which bounds exactly one of the faces F which $C(T, f)$ does and shares an edge e with $C(T, f)$.

Now recall that by Proposition 5.2, no cycles other than $C(T, f)$ and $C(T, f')$ in the FCMB can include e. So consider the spanning tree $T - e + f'$. The fundamental cycles with respect to T which do not contain e will still be fundamental cycles of this spanning tree. Furthermore, notice that $C(T-e+f', e) = C(T, f')$, since you are simply adding in whichever edge is not included in the tree. The last fundamental cycle of $T - e + f'$ to be considered is $C(T - e + f', f)$. By definition, this cycle does not contain e, so it cannot be the same as $C(T, f)$. Let F be a face with e in its boundary. Then we see that $C(T - e + f', f)$ cannot contain the face F anymore but cannot bound faces which $C(T, f)$ did not bound. So the size of the fundamental cycle basis with respect to $T - e + f'$ is strictly smaller than the previous basis with respect to T. Therefore, by repeating this process, one may obtain an FCMB of size g.

Clearly, once we have the desired FCMB, an orientation of the cycles is induced by the plane, like with the cycle M-basis of a planar graph. Furthermore, it is obvious that if we have a fundamental cycle M-basis with this property, we have one in general, so this statement is both necessary and sufficient. $\square$

This theorem leads to another criterion for a graph to admit a fundamental cycle M-basis. While the previous criterion further is concerned with the properties of FCMBs, we can instead obtain a criterion which links existence of an FCMB to combinatorial properties of the graph.

First, we must define an **outerplanar graph**.

Definition 5.2. An outerplanar graph is a planar graph which has a planar embedding such that all vertices of the graph are on the unbounded face of the embedding. In other words, no vertex is completely enclosed by edges.

We also must define an **apex graph** and **apex vertices**:

Definition 5.3. An apex graph is a graph which becomes planar when some vertex is removed. This includes graphs which are already planar. An apex vertex is any vertex of an apex graph whose removal results in a planar graph.

Now we can state and prove our combinatorial criterion for a 2-connected planar graph to have an FCMB:

Theorem 5.3. Let G be a 2-connected planar graph. Then G admits a fundamental cycle M-basis if and only if the planar dual of G is an outerplanar graph plus an apex vertex which is adjacent to every other vertex in the dual.

Proof. Let G^* be the planar dual of G. Then Theorem 5.2 implies that there is a spanning tree T of G^* such that the fundamental cuts with respect to T contain a single edge. Indeed, since the bounded faces in G are fundamental cycles, their duality to cuts implies we have a spanning tree of the dual planar graph such that each dual vertex has only one incident edge in the tree (corresponding to the edge missing from the tree of G giving rise to this FCMB). The only exceptional vertex is the one corresponding to the unbounded region of G. But a tree with all degree one vertices except for one vertex must be a star. Since every vertex must be incident to this vertex, the subgraph obtained by deleting it is outerplanar and it is therefore an apex vertex.

For the converse, one can see the fundamental cuts of such an outerplanar dual graph similarly give rise to fundamental cycles of G which are exactly the bounded faces. Then by the previous theorem again, we obtain that the graph admits an FCMB. □

We note that any graph can be decomposed into 2-connected components which are connected by bridges or glued together at vertices. In fact, the cycle space of an arbitrary graph can be written as the direct sum of the cycle spaces of its 2-connected components since no cycles can contain edges from different 2-connected components, as the existence of such a cycle would imply that they are in fact the same 2-connected component. Therefore, one can find an FCMB by finding FCMBs of a graph's 2-connected components, leading us to the following classification theorem for graphs with FCMBs in general:

Theorem 5.4. A graph G has a fundamental cycle M-basis if and only if it is a planar graph such that each of its 2-connected components has a planar dual which consists of an outerplanar graph plus an apex vertex which is adjacent to every other vertex in the dual.

6 New Results on Cycle M-Bases

Despite it being more difficult to obtain general theoretical results for cycle M-bases than fundamental cycle M-bases, we are still able to understand them better through computations. By writing code which searches for cycle M-bases given the incidence matrix of a graph, we have been able to answer various questions about them by finding cycle M-bases for graphs or showing that they do not admit any cycle M-basis by checking that no possible cycle basis is an M-basis.

The first natural question to ask about cycle M-bases is whether, like flow M-bases, any graph G admits a cycle M-basis. This is Question 5.1 in [2]. We have the following counterexample:

Proposition 6.1. The graph $K_{3,5}$ does not admit a cycle M-basis.

Furthermore, it is the smallest complete bipartite graph of the form $K_{3,n}$ which does not admit a cycle M-basis. This shows that having a cycle M-basis is indeed a more precise property than having a flow M-basis.

Next, we consider similar characterizations of a cycle M-basis to Propositions 4.3 and 4.4. When searching for a cycle M-basis, it could be helpful to have restrictions on the size of cycles in an M-basis. These earlier propositions established a minimum size for one of the cycles, but must a cycle M-basis contain cycles of a sufficiently large size?

In particular, we would like to know whether any cycle M-basis for K_n must contain an n-cycle. Similarly, must any cycle M-basis for $K_{m,n}$ $(m \leq n)$ contain a $2m$-cycle? This is Question 5.4 in [2]. We have answered these questions by finding the following two counterexamples:

Proposition 6.2. The graph K_6 admits a cycle M-basis which does not contain a 6-cycle.

Proof. Consider the following basis of cycles, where the columns are the basis elements:

$$\iota = \begin{pmatrix}
1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 1 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0
\end{pmatrix}$$

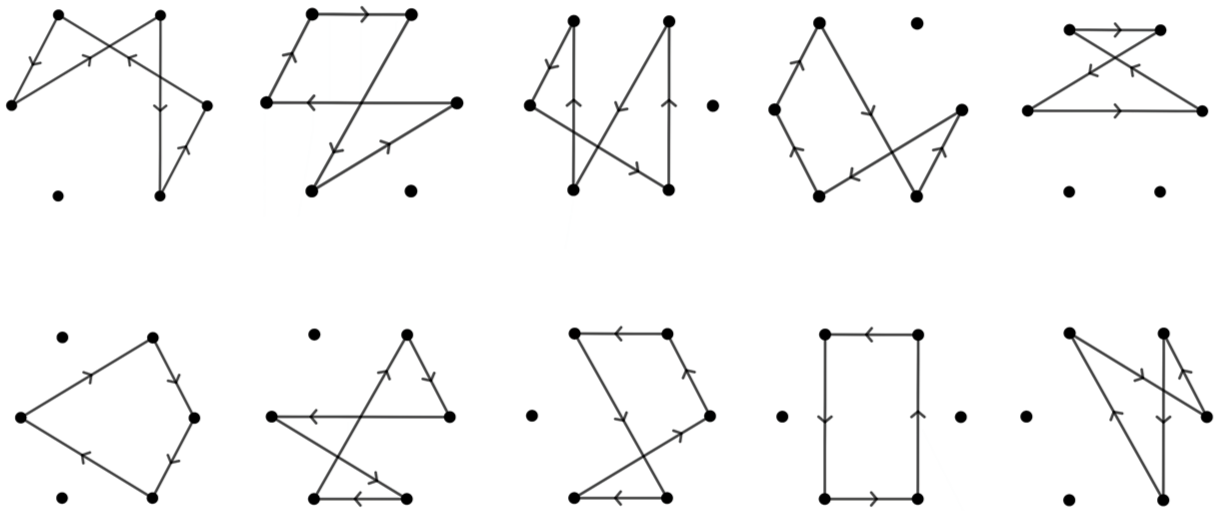


Figure 1: Cycles in a cycle M-basis of K_6

Then we can calculate the dual Laplacian:

$$\mathcal{L}^* = \iota^T \iota = \begin{pmatrix} 5 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 5 & -1 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 5 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 4 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 & 4 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 5 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 5 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 4 & -1 \\ 0 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & -1 & 4 \end{pmatrix}$$

This is a Z-matrix, so we see that this is indeed an M-basis. Therefore, we have found a cycle M-basis for K_6. In particular, we notice that none of the cycles are 6-cycles. $\square$

Proposition 6.3. The graph $K_{4,4}$ admits a cycle M-basis which does not contain an 8-cycle.

Proof. Consider the following basis of cycles, where the columns are the basis elements:

$$\iota = \begin{pmatrix} 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

Then we can calculate the dual Laplacian:

$$\mathcal{L}^* = \iota^T \iota = \begin{pmatrix} 4 & -3 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -3 & 6 & -3 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & -3 & 6 & 0 & 0 & 0 & 0 & -2 & 0 \\ -1 & 0 & 0 & 6 & -1 & -1 & -1 & -2 & -1 \\ 0 & 0 & 0 & -1 & 6 & -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & -1 & -2 & 6 & -2 & 0 & 0 \\ 0 & 0 & 0 & -1 & -2 & -2 & 6 & 0 & 0 \\ 0 & -1 & -2 & -2 & 0 & 0 & 0 & 6 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

This is a Z-matrix, so we see that this is indeed an M-basis. Therefore, we have found a cycle M-basis for $K_{4,4}$. In particular, we notice that none of the cycles are 8-cycles. $\square$

Note that the existence of a cycle M-basis for K_6 and $K_{4,4}$ has not previously been shown, so these results are significant on their own, not just as counterexamples.

Next, we have a few more interesting computational results:

Proposition 6.4. The Petersen graph admits a cycle M-basis.

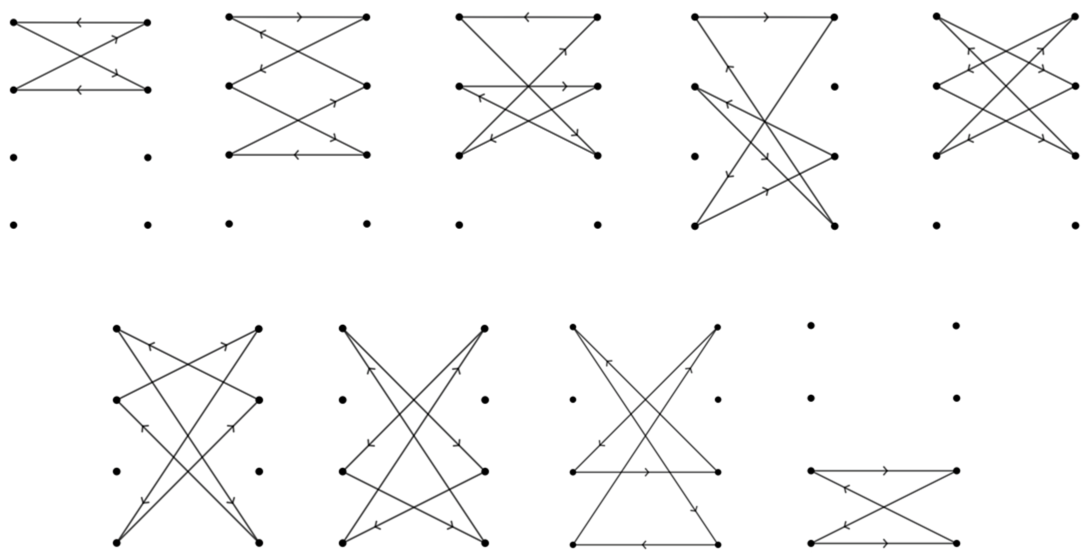


Figure 2: Cycles in a cycle M-basis of $K_{4,4}$

Proof. Consider the following basis of cycles, where the columns are the basis elements:

$$\iota = \begin{pmatrix} 1 & -1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 \end{pmatrix}$$

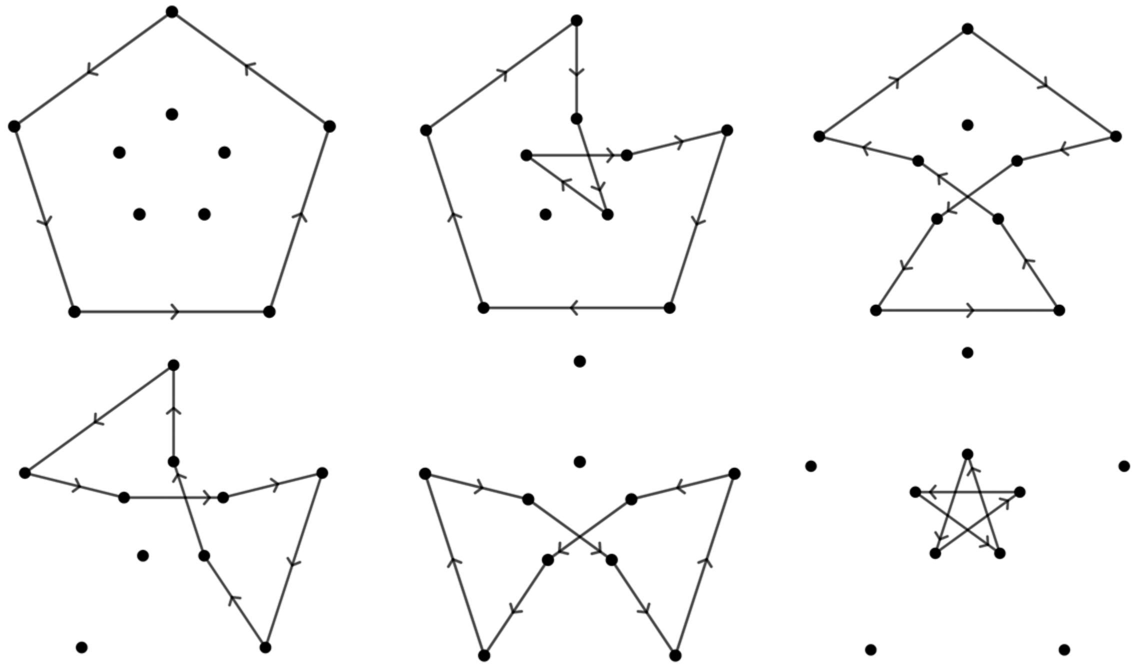


Figure 3: Cycles in a cycle M-basis of the Petersen graph

Then we can calculate the dual Laplacian:

$$\mathcal{L}^* = \iota^T \iota = \begin{pmatrix} 5 & -4 & -1 & 0 & 0 & 0 \\ -4 & 9 & 0 & 0 & -2 & -3 \\ -1 & 0 & 9 & -2 & 0 & -2 \\ 0 & 0 & -2 & 8 & -2 & 0 \\ 0 & -2 & 0 & -2 & 8 & 0 \\ 0 & -3 & -2 & 0 & 0 & 5 \end{pmatrix}$$

This is a Z-matrix, so we see that this is indeed an M-basis. Therefore, we have found a cycle M-basis for the Petersen graph. □

Proposition 6.5. The regular matroid R_{10} admits a circuit M-basis.

Proof. Consider the following basis of circuits, where the columns are the basis elements:

$$\iota = \begin{pmatrix} 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 1 & -1 \\ 0 & 1 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 \\ -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

Then we can calculate the dual Laplacian:

$$\mathcal{L}^* = \iota^T \iota = \begin{pmatrix} 6 & -3 & 0 & 0 & 0 \\ -3 & 6 & -3 & 0 & 0 \\ 0 & -3 & 6 & 0 & -3 \\ 0 & 0 & 0 & 6 & -3 \\ 0 & 0 & -3 & -3 & 6 \end{pmatrix}$$

This is a Z-matrix, so we see that this is indeed an M-basis. Therefore, we have found a circuit M-basis for R_{10}. $\square$

Finally, we examine whether the property of having a cycle M-basis is minor-closed. That is, if a graph G has a cycle M-basis, do the operations of edge deletion and contraction preserve this property? This question arises naturally when trying to find operations on graphs under which the existence of cycle M-basis is maintained in order to construct more examples of graphs which have cycle M-bases. In general, the property is not minor-closed, as we have the following counterexample:

Proposition 6.6. Let G be the graph obtained by adding an edge e to $K_{3,5}$ between two vertices in the bipartite set of size 5. Then G admits a cycle M-basis.

Proof. Consider the following basis of cycles, where the columns are the basis elements:

$$\iota = \begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Then we can calculate the dual Laplacian:

$$\mathcal{L}^* = \iota^T \iota = \begin{pmatrix} 4 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 6 & -4 & 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & -4 & 6 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 5 & -1 & 0 & -2 & 0 & 0 \\ -1 & -1 & 0 & -1 & 4 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 5 & -1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 0 & -1 & 4 & -1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 5 & 0 \\ 0 & -1 & -1 & 0 & -1 & 0 & -1 & 0 & 4 \end{pmatrix}$$

This is a Z-matrix, so we see that this is indeed an M-basis. Therefore, we have found a cycle M-basis for G. $\square$

Since $K_{3,5}$, a graph which does not admit a cycle M-basis, is a minor of this graph G, we see that having a cycle M-basis is not a minor-closed property. This further emphasizes the difficulty of determining which graphs have a cycle M-basis without explicit computation. It also calls into question whether one can construct large classes of non-planar graphs which admit a cycle M-basis. In particular, is every graph a subgraph of some graph which admits a cycle M-basis? This idea will be explored further in the future.

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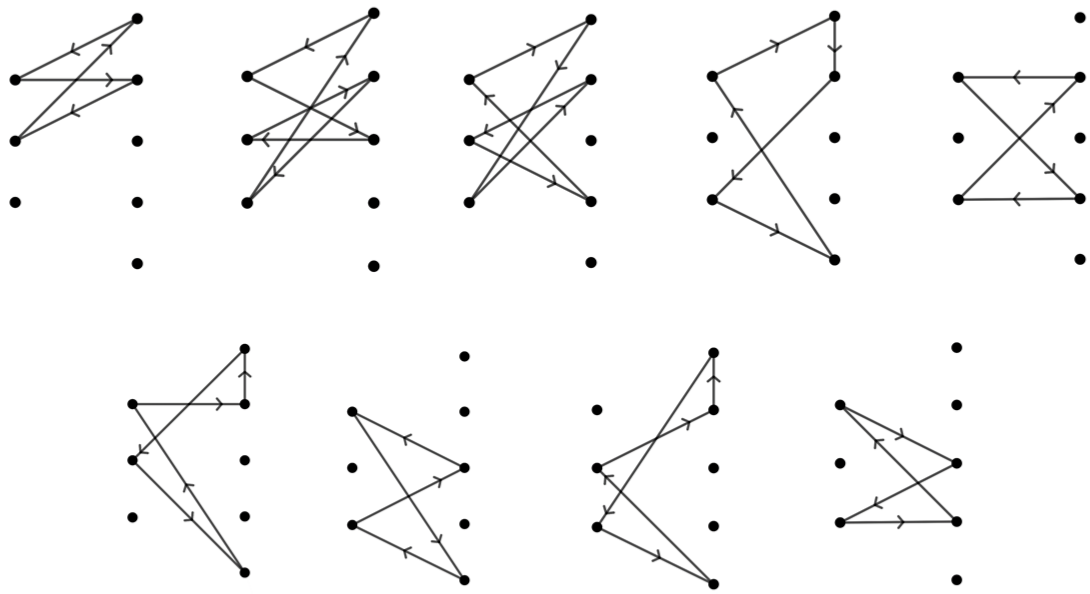

Figure 4: Cycles in a cycle M-basis of G

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