DESSIN D'ENFANTS AND EQUIVALENT SETS

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Abstract. In this paper, we explore Grothendieck's *dessins d'enfants*, explaining the bijection between this set of bicolored maps and equivalent sets from combinatorics and algebraic geometry. In particular, we also calculate explicit examples of these correspondences.

1. Introduction

Arguably, one of the most fascinating phenomena in mathematics is when classes of objects arise from seemingly vastly different subfields, but reveal themselves to be in bijection. Such is the case with Grothendieck's *dessins d'enfants*, bicolored graphs embedded on surfaces, the data of which can be expressed purely combinatorially, as triples of permutations; topologically, as coverings of the complex sphere ramified at only three points; algebro-geometrically, as Riemann surfaces defined over $\overline{\mathbb{Q}}$; or number theoretically, as equations in complex projective space. In particular, if we choose to look at only *dessins* of genus one, which can be embedded on the torus, we can follow through this complicated chain of equivalences to represent these *dessins* as specific elliptic curves.

In this paper, we explore these different equivalences and how one gets back and forth between each set.

Theorem 1.1. *There are natural bijections between the following sets, each up to isomorphism:*

- $\{$*dessins d'enfants*$\}$
- $\{$*combinatorial maps, i.e.* 3*-constellations* $[g_1, g_2, g_3]$ *such that* g_2 *is an involution without fixed points*$\}$
- $\{$*Belyĭ maps, i.e. a compact, connected Riemann surface* X *together with a function* $f : X \to \overline{C}$ *ramified at at most 3 points*$\}$.

Remark: the precise notions of isomorphisms for each set will be defined in the corresponding sections.

Theorem 1.2. *(Belyĭ). The Riemann surfaces which admit Belyĭ maps are precisely the ones that may be defined over* $\overline{\mathbb{Q}}$.

The paper is organized as follows: we start, in Section 2, with the graph theory version: graphs embedded on surfaces, building up to the definition of a *dessin*. In Section 3, we prove the bijection between *dessins* and combinatorial maps, which are specific sets of permutations called *constellations*. We then prove in Section 4 an intermediate step in the bijection between *dessins* and Belyĭ maps, proving that our set of *dessins* is bijective to the set of coverings for the complex sphere ramified at three points. Section 5 takes these coverings and completes the second bijection by identifying them with algebraic geometric constructions called *Riemann surfaces*, which can be expressed in terms of equations, along with functions to $\overline{\mathbb{C}}$ called *Belyĭ functinos*. In particular, genus one Riemann surfaces are also known as *elliptic curves*, which are defined in Section 7. In Section 6, we prove Belyĭ's Theorem (Theorem 1.2), to give a better description of which Riemann surfaces participate in the Belyĭ maps of our third set. Finally, Sections 8 and 9 are concrete examples: taking a particularly nice genus one *dessin* and a not-so-nice genus one *dessin*, respectively, through each set on the way to an elliptic curve and its j-invariant.

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2. The Graph Theory Part

To understand the first category, *dessins d'enfants*, we first need some definitions from graph theory.

2.1. **Some Basic Graph Theory.**

Definition 2.1. A *graph* $\Gamma = (V, E, I)$ is a triple of a set of vertices V, a set of edges E, and an incidence relation I between edges and vertices such that we say an edge $e \in E$ is incident to two vertices v_i, v_j which may or may not be distinct. If $v_i = v_j$, we call e a *loop*.

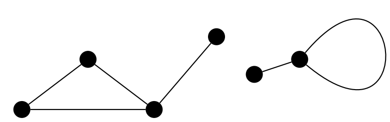

Figure 1. An example of a graph.

If we have two edges e_1, e_2 between the same pair of vertices, we call them *multiple edges*. We call that the number of edges incident to a vertex v the *degree* of v and denote it $\deg(v)$, noting that loops from v to v count twice. Note that each edge is incident to 2 vertices, so

$$\sum_{v \in V} \deg(v) = 2|E|.$$

Definition 2.2. A *path* is a sequence $v_0, e_1, v_1, e_2, \ldots, v_n$ in which e_i is incident to v_{i-1} and v_i. If $v_0 = v_n$, we call the path a *cycle*.

Definition 2.3. A graph Γ is *connected* if any two vertices may be connected by a path.

However, it is impossible to embed certain graphs in the plane or the sphere without edge crossings: we think of these maps as belonging on a different kind of surface, namely one with holes around which we can detour the extra edges to eliminate crossings.

2.2. **Graphs Embedded on Surfaces.**

Definition 2.4. A *surface* is a two-dimensional topological manifold. We will consider our surfaces to be not only orientable but to already have a fixed orientation.

Furthermore, every surface has an invariant called the genus, g, which intuitively, we think of as being the number of holes through the surface. Note that the genus cannot be negative.

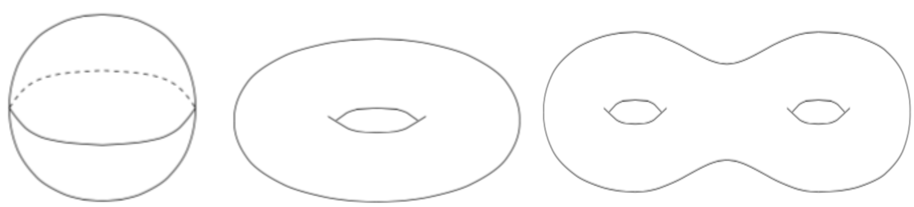

FIGURE 2. Surfaces of different genus: the sphere (genus 0), the torus (genus 1), and the two-holed torus (genus 2).

Definition 2.5. A *map* M is a graph G_M together with an embedding of G_M onto a surface X (so we can view $\Gamma \subset X$) such that:

- the vertices are distinct points on the surface;
- the edges are curves on the surface that intersect only at the appropriate vertices;
- if we cut the surface along the graph, the remainder $X \setminus \Gamma$ is a disjoint union of connected components homeomorphic to disks, which we call *faces*.

To distinguish between maps that are essentially the same, we introduce a notion of equivalence.

Definition 2.6. Two maps $M_1 \subset X_1$ and $M_2 \subset X_2$ are *isomorphic* if there exists an orientation preserving homeomorphism $u : X_1 \to X_2$ such that $u|_{M_1}$ is a graph isomorphism between G_{M_1} and G_{M_2}.

Note that we need to be careful here: there are homeomorphisms for surfaces of genus $g \geq 1$ which realize non-trivial elements of what is called the *mapping class group* [3, Ch. 2,3]. This group can be generated by constructions known as *Dehn twists*: cutting the surface along a closed curve which doesn't bound a topological disk, we twist one of the borders around by 2π and glue it back together with the other (see Figure 3).

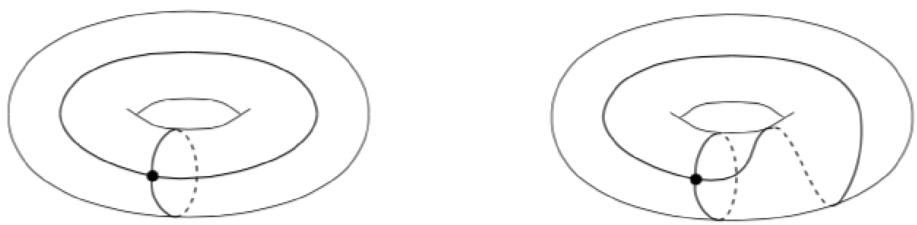

FIGURE 3. The Dehn twist of a map is in the same isomorphism class.

Note that while we consider these two maps as isomorphic, we cannot get the second map by slowly deforming the first, i.e. by applying an isotopy of the surface.

Furthermore, we comment that the graph G_M associated to a map M *must* be connected. If we try to draw a disconnected graph on a surface, we necessarily get a face violating the third condition for maps, as we can see in Figure 4.

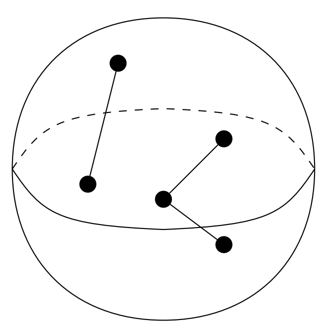

FIGURE 4. A disconnected map can't give us faces homeomorphic to disks.

The last condition then gives rise to the question: which graphs can be embedded as maps on which surfaces? For instance, we can embed the graph G below on the torus as shown in Figure 5, but it fails the third condition necessary to be a map.

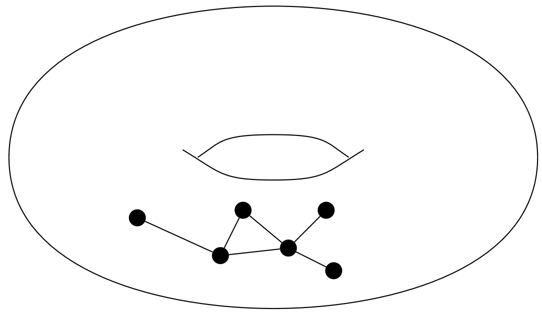

FIGURE 5. A graph that isn't a map because the back face has a hole in it.

This conundrum gives rise to the notion of the *genus* of a map, which we can calculate using the standard Euler characteristic.

Theorem 2.7. *(Euler, Lhuilier). We associate to a map M the* Euler characteristic

$$\chi(M) = |V| - |E| + |F|.$$

Furthermore, we have that

$$\chi(M) = 2 - 2g,$$

where we call g the genus *of the map. Note that the genus of a map M will be the same as the genus of its surface X.*

Proof. See [7, §1.3]. □

However, note that the graph G does not determine the map M, since we can embed a graph in many different ways. For instance, the two maps in Figure 6, shown embedded on the sphere $\mathbb{S}^2$, have the same associated graph G.

FIGURE 6. Two different maps, embedded on the sphere, from the same graph.

Surprisingly, G_M does not even determine the genus of M. For example, take the tetrahedron, which we would describe as:

$$V = \{A, B, C, D\},$$
$$E = \{AB, AC, AD, BC, BD, CD\}.$$

Note that we are folding the incidence relation into the edge set by labelling the edges with their incident vertices.

There is a planar embedding of the tetrahedron, i.e. an embedding with genus 0, as shown in Figure 7.

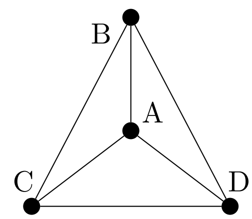

FIGURE 7. A planar embedding of the tetrahedron.

However, we can embed the same graph on the torus as in Figure 8, in which case the resulting map has genus 1.

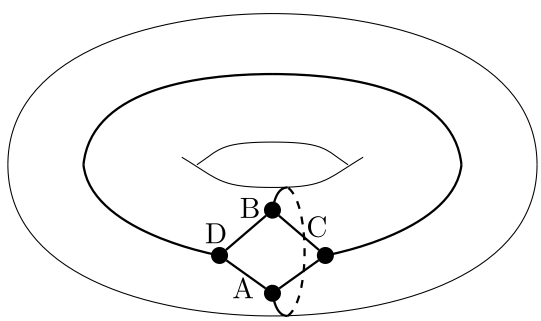

FIGURE 8. A genus 1 embedding of the tetrahedron on a torus.

Remark. In some cases, we look at a map M of genus 1 or greater, but for simplicity of drawing we want to present M as drawn in the plane. To do this, consider each edge of M as two pieces, each incident to one vertex of M. Given that M's surface is oriented, we can provide a cyclic order on the half-edges incident to a vertex by considering them in the counter-clockwise direction, as shown in Figure 9.

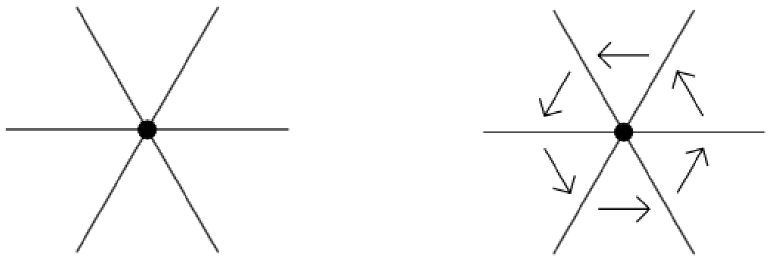

FIGURE 9. Cyclic ordering around a vertex.

We then label the half-edges in order in the counter-clockwise direction, putting the label on the left as we exit the vertex. Then, if we force the map into the plane, the labelling of these half-edges allows us to reconstruct the original map: we just redraw the half-edges around a particular vertex in numerical order and connect half-edges of the same edge together. For an example, see Figure 10, which reconstructs to Figure 12. Another example is the *dessin* in Section 8.

2.3. **Defining *Dessins D'enfants*.**

Grothendieck's *dessins d'enfants* are a specific type of map along with an extra piece of information; to define them, we first need one more important definition.

Definition 2.8. We say that a graph is *bipartite* if we can color its vertices using only two colors, such that no edge connects vertices of the same color. We say a *bicoloring* of a connected bipartite graph is a choice of one of the two possible colorings.

Note that given any map, we can turn it into a bicolored map by coloring its existing vertices black and placing a white vertex at the midpoint of every edge. Then, the original half-edges have now become edges connecting a black vertex to a white one. For instance, take the map on the left in Figure 10, which has genus one. We add extra vertices at the midpoints of the edges, and we get the bicolored map on the right.

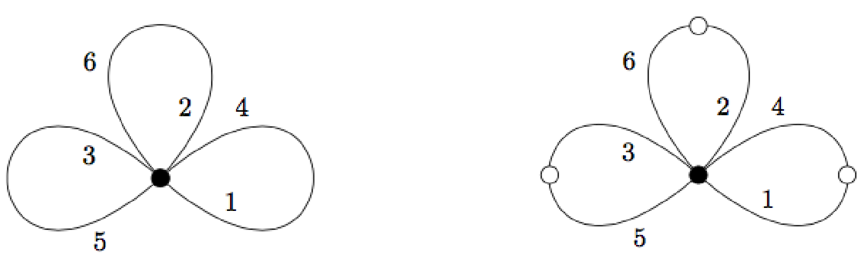

FIGURE 10. A map becomes a dessin.

To maintain the same numbering convention as before, we note that the label of an edge still goes on the left-hand side as we travel from the black vertex to the white. We then say that an edge is incident to a face if its label is within that face.

Definition 2.9. We'll call a bicolored map a *dessin d'enfant*, or a *dessin* for short. Furthermore, we call the segment $[0, 1]$ on the sphere, where 0 is a black vertex and 1 a white vertex, the *elementary dessin* (see Figure 11).

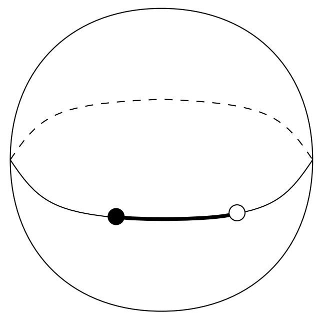

FIGURE 11. The elementary *dessin*.

Definition 2.10. Two *dessins* M_1, M_2 are *isomorphic* if they are isomorphic as maps under an isomorphism which preserves the bicoloring, i.e. if there exists an orientation preserving homeomorphism $u : X_1 \to X_2$ such that $u|_{M_1}$ is a graph isomorphism between G_{M_1} and G_{M_2} and u sends black vertices of M_1 to black vertices of M_2 and white vertices of M_1 to white vertices M_2.

3. The Combinatorics Part

3.1. *Dessins* as Combinatorial Data.

To prove the bijection between *dessins* and combinatorial maps, we start with the question: given a *dessin* M, can we encode it into purely combinatorial data? In fact, all the data of M can be provided by a set of two permutations. To form these, take M and consider the half-edges incident to each vertex, recalling the notion of cyclic order defined in Figure 9.

Given a dessin M, let σ be the permutation $\sigma = c_1 c_2 \cdots c_n$, where c_i is the cyclic order of edges around black vertices v_i. Let α be the permutation $\alpha = t_1 t_2 \cdots t_m$, where t_i is the cyclic order of edges around white vertices w_i.

Note: in calculating these permutations, we must view the *dessin* as embedded on its surface, rather than as forcibly drawn in the plane.

Example. The *dessin* M in Figure 10 has genus one and thus embeds onto the torus as in Figure 12. Note that we visualize the torus as a square with opposite sides identified. Then, we can easily see that $\sigma = (1, 2, 3, 4, 5, 6)$ and $\alpha = (1, 4)(2, 6)(3, 5)$.

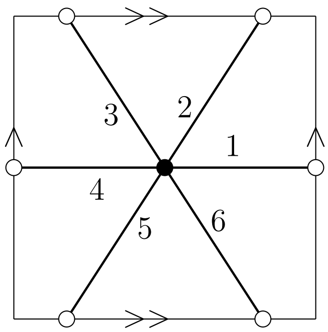

FIGURE 12. The *dessin* in Figure 10 embedded on the torus.

Furthermore, we can construct a permutation ϕ which illuminates which edges are incident to each face.

Lemma 3.1. *The permutation σ represents the action around black vertices, and the permutation α represents the action around white vertices, so*

$$\phi = \alpha^{-1}\sigma^{-1}$$

represents the faces of the dessin associated to α and σ.

Proof. Immediately visualizable from the *dessin*. Else, see [7, §1.3]. $\square$

Example. For instance, using the same *dessin* as in Figure 12, we calculate

$$\begin{aligned}\phi &= \alpha^{-1}\sigma^{-1} \\ &= (1,4)(2,6)(3,5)(1,6,5,4,3,2) \\ &= (1,3,4,6)(2,5).\end{aligned}$$

Remark. Following the convention of [7], we multiply permutations from left to right here.

Note: in the outer face on a planar map embedded on the sphere, we read the labels from the other side of the sphere, so they appear to go in a clockwise direction from our planar drawing but actually follow the same counterclockwise rule as the other faces.

3.2. Combinatorial Data as a *Dessin*.

If, on the other hand, we have a triple of permutations $[\sigma, \alpha, \phi]$ which we know corresponds to a dessin M, we can reconstruct M embedded on its surface by the following construction:

- Create a m-gon for every cycle of length m in ϕ, giving it a counter-clockwise-oriented border and labeling the edges inside by the cycle in that direction.

- Glue polygons together along their edges according to α so that the orientation of sides glued together is opposite This ensures that our surface will be oriented properly.
- Place black vertices at jointures of polygons following cycles of σ, and white vertices for cycles of α.

Since $\sigma = \phi^{-1}\alpha^{-1}$, the polygons will already give the correct cyclic order around each vertex, and their orientations will match around the vertices. The only thing we have left to do is check that what we have actually is a surface: that is, that a neighborhood around any point is homeomorphic to a disk in $\mathbb{R}^2$. Obviously for points on the interior of a polygon, we are fine. For points on an edge, we take half the disk from each polygon. For points on a vertex with multiple polygons coming together, we take a slice from each.

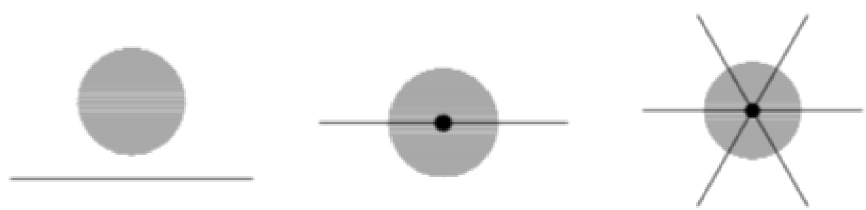

FIGURE 13. The three different cases for disks.

Note that if we take a map illegally embedded on a surface–that is to say, with not all faces being homeomorphic to a disk, and we encode it as permutations, then reconstruct it, the resulting map will be legally embedded on a different surface and will show us the true genus of the map.

However, this construction raises the question: how do we know which sets of permutations correspond to a map? To classify these sets, we introduce the notion of constellations.

3.3. **Constellations.**

Definition 3.2. A *constellation* or *k-constellation* is a sequence $[g_1, g_2, \ldots, g_k]$ of elements $g_i \in S_n$ such that:

- the group $G = \langle g_1, g_2, \ldots, g_k \rangle$ acts transitively on the set $\{1, \ldots, n\}$;
- the product $g_1 g_2 \cdots g_k = id$, the identity permutation.

We then say that n is the *degree* of the constellation, and k is its *length.*

Definition 3.3. The group $G = \langle g_1, g_2, \ldots, g_k \rangle$ is called the *monodromy group* or the *cartographic group.* Note that G is a permutation group.

However, we are interested in a specific set of constellations, namely those equivalent to dessins.

Definition 3.4. A *combinatorial map* is a 3-constellation $[\sigma, \alpha, \phi]$.

Remark. By a result of Dixon [2], the monodromy group of a randomly chosen set of permutations is almost always either S_n, the full permutation group on n elements, or A_n, the alternating group on n elements.

Definition 3.5. Two combinatorial maps $C = [g_1, g_2, g_3]$ and $C' = [g_1', g_2', g_3']$ acting on sets E and E', respectively, are *isomorphic* if there exists a bijection $f : E \to E'$ such that $g_i' = f^{-1} g_i f$ for $i = 1, 2, 3$. Furthermore, C and C' are *conjugate* if $E = E'$ and $h \in G = \langle g_1, g_2, g_3 \rangle$.

Now that we have fully defined our combinatorial maps and their isomorphisms, we prove the bijection between the first and second sets of Theorem 1.1.

The constructions in Section 3.1 give us functions:

$$f : \{dessins\} \to \{\text{combinatorial maps}\},$$

and

$$g : \{\text{combinatorial maps}\} \to \{dessins\}.$$

We thus need to check that these functions give us a bijection between the two sets, up to isomorphism on each.

First we tackle the obvious problem: there are multiple ways to label a *dessin*. However, the way we have defined isomorphisms on combinatorial maps gives us that two labelings of a *dessin* result in isomorphic combinatorial maps. Furthermore, isomorphisms on *dessins* carry through to combinatorial maps, creating a subgroup in the monodromy group G.

Definition 3.6. The *automorphism group* of a combinatorial map M, called $Aut(M)$, is the centralizer of the monodromy group G_M. That is to say,

$$Aut(M) = \{h : h^{-1} g h = g, \forall g \in G\}.$$

Remark: we can consider elements of the automorphism group as isomorphisms of the underlying *dessins*.

To see this, think about the action of $Aut(M)$ on a *dessin*. In particular, $h \in G_M$ is in $Aut(M)$ if h commutes with both σ and α, which means that h respects incidence and ordering of edges at both black and white vertices, as well as incidences of vertices and edges to faces. Thus, we can think of the automorphism group of a combinatorial map as the automorphism group of the embedded *dessin* underlying the constellation. In fact, most combinatorial maps have trivial automorphism group $Aut(M) = \{id\}$.

Since $Aut(M) < G$, we have that f is injective, since two combinatorial maps in the same isomorphism group must come from either different labelings of the same dessin or from labelings of isomorphic *dessins*. Furthermore, f is surjective, since we can take any combinatorial map and use the second construction of Section 3.1 to find an appropriate *dessin*. Thus, we have that the set of *dessins d'enfants* is bijective to the set of combinatorial maps via the natural bijection f.

4. An Intermediate Topological Part

Now that we have turned our *dessin* into a constellation, we move to the third equivalent set: isomorphism classes of Riemann surfaces over $\overline{\mathbb{Q}}$. On the way, we move through an equivalent set: isomorphism classes of coverings of the complex sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ ramified at three points. To explore this equivalence, we first must define ramified and unramified coverings.

4.1. **Covering the Sphere.**

Definition 4.1. Let X and Y be two path-connected topological spaces, with a continuous mapping $f : X \to Y$ between them. We say that the triple (X, Y, f) is an *unramified covering*, which we'll call just a *covering*, of Y by X if for any $y \in Y$ there exists a neighborhood V of y such that the preimage $f^{-1}(V) \subset X$ is homeomorphic under f to $V \times S$, where S is a discrete set. In this case, we call the function f the *projection* from X to Y.

Note that often, due to laziness or abuse of language, the function f is called a covering when X and Y are clear from context.

Definition 4.2. Given a covering (X, Y, f), we call the connected components of $f^{-1}(V)$ the *sheets of the covering over* V, the preimage $f^{-1}(y)$ the *fiber over* y, and the cardinality of S—i.e., the number of sheets in the covering—the *degree*, which we denote $deg(f)$.

Example. For instance, in Figure 14, the circle $X = S^1$ equipped with the map $f(q) = q^8$ is an unramified covering of itself, where $q = e^{2\pi i z}$.

Figure 14. An unramified covering of the circle.

However, there are times when we have a covering that is unramified except for a few points.

Definition 4.3. Suppose we have a triple (X, Y, f) which satisfies Definition 4.1 at all but a finite set of points $\{y_1, \ldots, y_n\}$ in Y, at which f is continuous, and the fiber above y_i has $deg(f)$ points x_{ij} when counting with *multiplicity* for all y_i and for a neighborhood V_i of y_i, the preimage $f^{-1}(V_i \setminus \{y_i\}) = \bigsqcup (U_j \setminus \{x_{ij}\})$ where $U_j \setminus \{x_{ij}\}$ is isomorphic to a punctured disk. We call such an object a *ramified covering* of Y by X.

Note that by *multiplicity*, we mean the number of times the same point appears on different sheets of the covering. For instance, if some x_{ij} over y_i is contained in 3 sheets of the covering, then x_{ij} has multiplicity 3.

We then say that the points $\{y_i\}$ are *critical values* or *ramification points*, their preimages $\{x_i\}$ are *critical points*.

Definition 4.4. Two ramified coverings $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ are *isomorphic* if there exists an orientation preserving homeomorphism $u : X_1 \to X_2$ such that the following diagram is commutative:

$$\begin{tikzcd} X_1 \arrow[rr, "u"] \arrow[dr, "f_1"'] & & X_2 \arrow[dl, "f_2"] \\ & Y & \end{tikzcd}$$

Now, we prove that the sets of *dessins* and coverings ramified at 3 points are in bijection.

4.2. **Dessins as Coverings.**

We construct a map $f : \{dessins\} \to \{\text{coverings of } \overline{\mathbb{C}} \text{ ramified at 3 points}\}$. To do this, we first triangulate the *dessin*.

Place a vertex $*$ in the center of every face and connect it to the black vertices adjacent to its face with dashed lines and to the white vertices with dotted lines (see Figure 15)

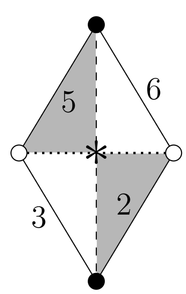

FIGURE 15. Triangulation of one of the faces in Figure 12.

We'll color these triangles by their orientation: we note that each triangle has three vertices: $\bullet$, $\circ$, and $*$, and three sides: plain, dotted, and dashed. Starting at the black vertex, travel counterclockwise around the boundary of the triangle: if the next vertex is white, we call the triangle *positive*. If the next vertex is the star, we call the triangle *negative*. By convention, we shade the positive triangles.

Note that the positive triangles are in bijection with the edges of the *dessin*. This is fairly easy to see: each edge belongs to two triangles, a positive one and a negative one, where the positive one will be on the left as we exit the black vertex. Thus, we can think of the label for a edge as being inside the positive triangle to which it belongs. Then, the action of ϕ become immediately visible: it sends the positive triangles of a face around in the counterclockwise direction.

We can triangulate the elementary *dessin* as well. We have only one edge, so there is only one face: placing a star in the middle of this face (i.e. at the point at infinity) and connecting it to the black and white vertices, we get two triangles: one negative, one positive, which correspond to the two hemispheres, as we see in Figure 16.

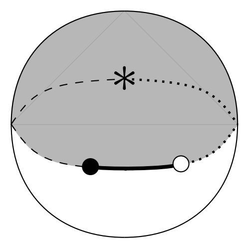

FIGURE 16. Triangulation of the elementary *dessin*.

Given a triangulation of a *dessin* M, we thus construct a covering of the sphere $\overline{\mathbb{C}}$ by sending the black vertices of M to 0, the white vertices to 1, and the star vertices to ∞. Since three points determine a triangle on the triangulated *dessin*, the positive triangles of the *dessin* are sent to the upper hemisphere of $\overline{\mathbb{C}}$ and the negative triangles to the lower hemisphere, with all edges sent to the real line, $\mathbb{R}$. Note that this construction gives us a continuous mapping to $\overline{\mathbb{C}}$. Thus, we have a covering of the sphere with only three ramification points: 0,1, and ∞.

4.3. **Coverings as Dessins.**

To create the inverse map

$$f^{-1} : \{ \text{ coverings of } \overline{\mathbb{C}} \text{ ramified at three points}\} \to \{\textit{dessins}\},$$

we start with a covering of the elementary *dessin*, ramified only at 0, 1, and ∞. Place a black vertex for each point in the preimage of 0 and a white vertex for each in the preimage of 1. We then add edges between these vertices, noting that the ordering of edges at a vertex must respect the ordering of sheets in the cover, and we have a map, which will be embedded on the surface that is the covering. Note that since the covering can be triangulated as a covering of the elementary *dessin*, we are guaranteed that the faces of the resulting *dessin* will be homeomorphic to disks, so this *dessin* will be legally embedded on the covering surface.

Furthermore, we can see that f is a bijection: any two isomorphic covering spaces must come from isomorphic *dessins*, so f is injective. Also, under the construction of Section 4.3, we can find a *dessin* M which maps to any covering in the set, so f is surjective, and f^{-1} is the inverse function.

5. The Algebraic Geometry Part

To get all the way to the third set of Theorem 1.1, we need to relate these ramified coverings to Belyĭ maps. To start, we define Riemann surfaces.

Definition 5.1. A *Riemann surface* is a complex analytic manifold of complex dimension one. Here, we only care about *compact*, *connected* Riemann surfaces, which we will just call Riemann surfaces.

Definition 5.2. We say that two Riemann surfaces are *isomorphic* if there exists a biholomorphic bijection, i.e. a complex isomorphism, between them.

Note: by convention, we call the one-point compactification of the complex plane, $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the *Riemann complex sphere*. This is the same surface as the complex projective line $\mathbb{CP}^1$.

Note that complex isomorphisms are *not* the same thing as topological isomorphisms (homeomorphisms). Two surfaces which are complex isomorphic are also topologically isomorphic, but the reverse is false. For example, all Riemann surfaces of genus one with a marked point, called *elliptic curves* (see section 7) are topologically isomorphic, but not complex isomorphic.

This dilemma raises the question: how do we specify a particular Riemann surface? In fact, there are two natural options: we can either define it by a system of polynomial equations in a complex projective space, or we can define it by a particular ramified covering of $\overline{\mathbb{C}}$.

To explain either of these, we first need one more definition.

Definition 5.3. Let X be a Riemann surface. Then we say a holomorphic function $f : X \to \overline{\mathbb{C}}$ is a *meromorphic function* on X. We say that points $x \in X$ such that $f(x) = 0$ are *zeros* and points $x \in X$ such that $f(x) = \infty$ are *poles*.

For example, meromorphic functions on $\overline{\mathbb{C}}$ look like rational functions with obvious zeros and poles. If we have a function on $\overline{\mathbb{C}}$ with only one pole at ∞, we have a polynomial.

Theorem 5.4. *A Riemann surface can be described as an algebraic curve in a complex projective space, i.e. by a system of polynomial equations over that space.*

We omit the proof here, noting that it is in [5, Appendix B, §3].

Definition 5.5. If we can express a Riemann surface X with a system of equations with coefficients in a subfield $K \subset \mathbb{C}$, then we say that X *is defined over* K.

Proposition 5.6. *A nonconstant meromorphic function $f : X \to \overline{\mathbb{C}}$ considered as a mapping of X as a topological space, gives us a ramified covering of $\overline{\mathbb{C}}$ by X.*

Proof. Follows directly from the fact that f is meromorphic. □

With the fact that there exists such a meromorphic function on every Riemann surface [7, 1.6], we have that every Riemann surface may be represented as a ramified covering of $\overline{\mathbb{C}}$. So, we can create a mapping:

$$g : \{\text{Riemann surfaces with mero. functions to } \overline{\mathbb{C}} \text{ ramified at three points}\}$$
$$\downarrow$$
$$\{\text{coverings of } \overline{\mathbb{C}} \text{ ramified at three points}\}. \tag{5.1}$$

Thus, we can extend our notions from ramified coverings up to Riemann surfaces together with particular meromorphic functions, so we can denote a ramified covering of $\overline{\mathbb{C}}$ by (X, f). Choose local coordinates around $x \in X$ and $y = f(x) \in \overline{\mathbb{C}}$ such that $x \neq \infty$ and $y \neq \infty$. Then x is a *critical point* and y a *critical value* if and only if $f'(x) = 0$. If x is critical, we can choose local coordinates such that $x = 0$, $y = 0$, and f looks like $f(s) = s^d$ in the local coordinate. We then call d the *degree, multiplicity,* or *order* of the critical point. We furthermore call the set of critical values in $\overline{\mathbb{C}}$ the *ramification locus.*

Definition 5.7. Two complex ramified coverings (X_1, f_1) and (X_2, f_2) over $\overline{\mathbb{C}}$ are *isomorphic* if there exists a complex isomorphism $u : X_1 \to X_2$ such that the following diagram is commutative.

$$\begin{tikzcd} X_1 \arrow[rr, "u"] \arrow[dr, "f_1"'] & & X_2 \arrow[dl, "f_2"] \\ & \overline{\mathbb{C}} & \end{tikzcd}$$

Note that while isomorphic complex ramified coverings must have isomorphic Riemann surfaces, the converse is not true, since the same Riemann surface can give distinct non-isomorphic coverings using different meromorphic functions $f_1, f_2 : X \to \overline{\mathbb{C}}$, so we do need to include the data of the meromorphic function.

To determine whether two meromorphic functions give the same covering, we note that the only isomorphisms of $\overline{\mathbb{C}}$ are the linear fractional transformations

$$z \mapsto \frac{az+b}{cz+d}.$$

So, two meromorphic functions f_1 and f_2 give isomorphic coverings if and only if

$$f_1(z) = f_2\left(\frac{az+b}{cz+d}\right)$$

for some a, b, c, d such that $ad - bc \neq 0$.

We know already that we can take a Riemann surface and represent it as a ramified covering of the complex sphere, so g in (5.1) is injective; this raises the question of whether the converse is true. Can we take a ramified covering of the complex sphere and represent it as a Riemann surface with an appropriate map? In fact, we can.

Theorem 5.8. *(Riemann's existence theorem) Given a constellation* $[g_1, \dots, g_k]$*,* $g_i \in S_n$*, there exists a compact Riemann surface* X *and a meromorphic function* $f : X \to \overline{\mathbb{C}}$ *such chosen points* $y_1, \dots, y_k \in \overline{\mathbb{C}}$ *are the critical values of* f *and* $g_1, \dots, g_k$ *are the corresponding generators of the monodromy group. Furthermore, this construction is unique up to isomorphism on both sides.*

Proof. See [7, §1.8] □

Then, the covering of $\overline{\mathbb{C}}$ related to that constellation is equivalent to the resulting Riemann surface, so, if we set $k = 3$, we get that g in (5.1) is also surjective, up to isomorphism. In fact, we have just proven the rest of the second bijection in Theorem 1.1: given a pair (X, f) in the first set of 5.1 with three ramification points, we can assume, without loss of generality, that these points are $0, 1, \infty$. (See the construction in Theorem 6.1). We then note that our pair (X, f) is actually a Belyĭ map.

Definition 5.9. A *Belyĭ function* is a meromorphic function $f : X \to \overline{\mathbb{C}}$ that is unramified outside of $\{0, 1, \infty\}$. and say that (X, f) is a *Belyĭ pair.*

Noting that the notion of isomorphism on Belyĭ pairs is equivalent to that on complex ramified coverings, we rephrase it as follows.

Definition 5.10. We say that two Belyĭ pairs (X_1, f_1) and (X_2, f_2) are *equivalent* if there exists an isomorphism $\phi : X_1 \to X_2$ such that $f_2 \circ \phi = f_1$.

So the first set in (5.1) is the same as the third set in Theorem 1.1, so then, our bijection g is actually

$$g : \{\text{Belyĭ pairs}\} \to \{\text{coverings of } \overline{\mathbb{C}} \text{ ramified at three points}\},$$

up to isomorphism on both sets.

However, just as we saw that only some coverings can be related to our *dessins*, the same is true for Riemann surfaces, so we need a way to determine if a Riemann surface is a suitable candidate for creating a map. Our solution was discovered by Belyĭ, and is the subject of the next section.

6. Belyĭ Theorem

We spend this section proving Theorem 1.2, which we state more precisely below.

Theorem 6.1. *Let X be a Riemann surface. Then X is defined over $\overline{\mathbb{Q}}$ if and only if there exists a meromorphic function $f : X \to \overline{\mathbb{C}}$ which is unramified outside of $\{0, 1, \infty\}$. Furthermore, if such a function f exists, it can be chosen to also be defined over $\overline{\mathbb{Q}}$.*

Proof. We here prove the more illuminating "Only If" part of the proof: suppose we have a Riemann surface X defined over $\overline{\mathbb{Q}}$. Following the proof provided in [7, §2.6], we proceed in three steps to construct an appropriate f.

Step 1: take any meromorphic function $h : X \to \overline{\mathbb{C}}$ defined over $\overline{\mathbb{Q}}$. Look at the critical values of h: some will be rational, others irrational. Taking only the irrational (algebraic) points, consider also their conjugates and call this set S_0. Let $|S_0|= n$.

Step 2: let P_0 be the polynomial which annihilates S_0, noting that P_0 is defined over $\mathbb{Q}$ and that $degP_0 = n$. Considering the critical values (the roots of the derivative) of P_0, we note that there can be at most $n-1$ of them: let this set be called S_1, noting that it already contains the conjugates of its elements, with annihilating polynomial P_1. We then proceed similarly, until we have a set $\{P_0, \ldots, P_{n-1}\}$. Taking the composition of all of these,

$$P_{n-1} \circ \cdots \circ P_1 \circ P_0,$$

we see that it sends all the critical values of the original function h to rational points. Note that we don't have to do anything about the rational values, because all the coefficients of each polynomial are rational, so they automatically go to rational points.

Step 3: Now, we just need to push all our critical points into the set $\{0, 1, \infty\}$, which we can do by applying affine transformations. First, apply an affine transformation A sending all the points into the segment $[0, 1]$. Let $p_{m,n}(x)$ be the polynomial

$$p_{m,n}(x) = \frac{(m+n)^{m+n}}{m^m n^n} x^m (1-x)^n.$$

Then, $p_{m,n}(0) = 0, p_{m,n}(1) = 0, p_{m,n}(\infty) = \infty$, and $p_{m,n}\left(\frac{m}{m+n}\right) = 1$, while preserving the segment $[0, 1]$. So, applying a sufficient number of such functions, since we have a finite number of critical values, will move all the critical values to either $0, 1$, or

∞, and the resulting composition

$$p_{m_i n_i} \circ \cdots p_{m_k n_k} \circ A \circ P_{n-1} \circ \cdots \circ P_1 \circ P_0 \circ h$$

is a Belyĭ function for X.

Note that this is not the most efficient way to find a Belyĭ function, but it does always work.

The other direction of the proof requires heavy machinery which we choose not to cover here; interested readers may consult [7, §2.6]. □

7. The Number Theoretic Part

In particular, genus one *dessins d'enfants* can be transformed into Riemann surfaces of genus one, which are equivalent to a number theoretic object called *elliptic curves*. We only touch on the bare basics of this subject here; see [8] for a more thorough treatment.

Definition 7.1. Riemann surfaces of genus one with a specified base point are *elliptic curves*. By [8, Ch. 3], we know that elliptic curves are defined by equations that look like

$$E : \{y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6\} \cup \{\infty\}.$$

Any such equation is called a *Weierstrass equation*, but note that there are many Weierstrass equations which describe a single elliptic curve.

Provided E is defined over a field K where $char(K) \neq 2$, we can easily rearrange this equation with the substitution

$$y \mapsto \frac{1}{2}(y - a_1x - a_3)$$

to get an equation of the form

$$E : y^2 = 4x^3 + b_2x^2 + 2b_4x + b_6.$$

Also, if $char(K) \neq 2, 3$, we can do a second substitution

$$(x, y) \mapsto \left(\frac{x - 3b_2}{36}, \frac{y}{108}\right)$$

that gives us the simpler form

$$E : y^2 = x^3 - 27c_4x - 54c_6,$$

where c_4, c_6 are defined below.

Elliptic curves have all sorts of identifying characteristics, among them several interesting invariants.

Definition 7.2. The *j-invariant* is an invariant of an elliptic curve that classifies them up to complex isomorphism.

We calculate the j-invariant as follows:
Define the quantities:

$$\begin{aligned} c_4 &= b_2^2 - 24b_4 \\ c_6 &= -b_2^3 + 36b_2b_4 - 216b_6 \\ \Delta &= \frac{c_4^3 - c_6^2}{1728} \end{aligned}$$

Note that Δ is called the *discriminant* for that particular Weierstrass equation. Then we calculate the j-invariant by

$$j = \frac{c_4^3}{\Delta}.$$

Note that the j-invariant does not depend on the choice of equation to define the elliptic curve.

8. A Cool Example

One of the typical problems, however, in this area is the lack of computability: we know that we can take a genus one *dessin d'enfant* and produce an elliptic curves, but which elliptic curve? Can we calculate its j-invariant? To explore these issues, we follow the entire process of computation for a particularly nice *dessin*.

Let M be the *dessin* in Figure 17.

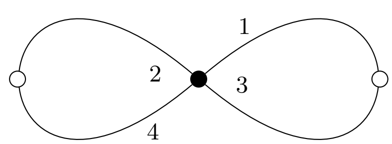

FIGURE 17. The *dessin* M.

Since we need to make the half-edges travel around the black vertex in counter-clockwise order, we wind up with a map of genus one.. So, M embeds onto the torus as described in Figure 18.

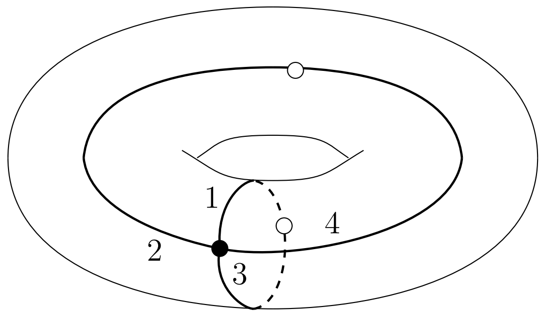

FIGURE 18. The *dessin* M on the torus.

To make the rest of the calculations more intuitive, we choose to represent the torus in a different standard way: as a square with opposite sides identified in the same direction, in which case our map looks like that in Figure 19.

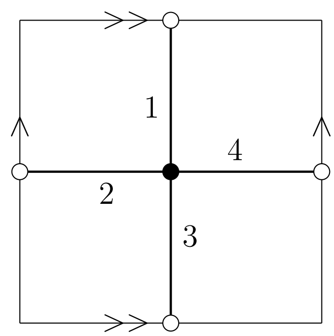

FIGURE 19. Another way to view the torus with M embedded.

From here, we calculate our permutations to get the combinatorial map associated to M. Reading off half-edges around the two white vertices,

$$\alpha = (1,3)(2,4).$$

Then, reading off the half-edges around our black vertex,

$$\sigma = (1,2,3,4).$$

So, by our definition,

$$\begin{aligned}\phi &= \alpha^{-1}\sigma^{-1} \\ &= (1,3)(2,4)(1,4,3,2) \\ &= (1,2,3,4)\end{aligned}$$

Thus, our combinatorial map is $[(1,2,3,4),(1,3)(2,4),(1,2,3,4)]$.

If we take Figure 19 and triangulate it as in Section 4, adding in the star vertex in the middle of the single face, then connecting black and white vertices to the star, we can see the meaning of ϕ for the face a bit more clearly, as in Figure 20.

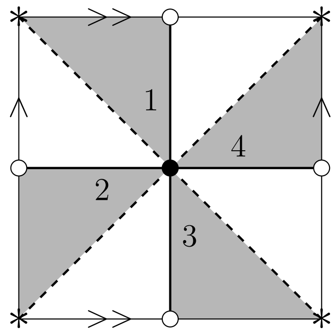

FIGURE 20. M triangulated.

Notice that the positive triangles are already labelled, since the labels for their associated black-to-white edges were on the left bank and are thus already inside the triangle.

Furthermore, we can think of the identification of the sides as the action of two translations: one moving vertically, one horizontally. This allows us to view this expression of the torus as the fundamental domain of the action of the group Λ generated by these translations over the complex plane. That is to say, we can tile the plane with copies of this fundamental domain by these translations, as shown in Figure 21.

We then note that our original square, now centered at the black vertex at $\left(\frac{1}{2}, \frac{1}{2}\right)$ in Figure 21, is a fundamental domain for this tiling, since it contains a single copy of every labelled positive triangle. This domain, which we will call D, makes clear that the two translations generating Λ are $z \mapsto z + i$ and $z \mapsto z + 1$. That is to say, $\Lambda = \mathbb{Z} \oplus i\mathbb{Z}$.

Note that $\mathbb{Z} \oplus i\mathbb{Z}$ has an extra automorphism in the plane: we can map $[z] \mapsto [iz]$ and retain the same lattice, .i.e. $\Lambda = i\Lambda$, so our lattice is special.

Now that we have identified what the appropriate covering space should be, we need a Belyǐ function that will take our fundamental domain, and thus the entire tiling determined by our lattice, to the elementary hypermap on the one-point compactification of the sphere. That is to say, we need to find a meromorphic function f on $\mathbb{C}/\Lambda$ such that $D = f^{-1}([0, 1])$ where the black vertices map to 0 and the white to 1. If we can find such an f, it will follow that the Riemann surface associated to the map as a covering space is $S_D = \mathbb{C}/\Lambda$ and that $f_D = f$.

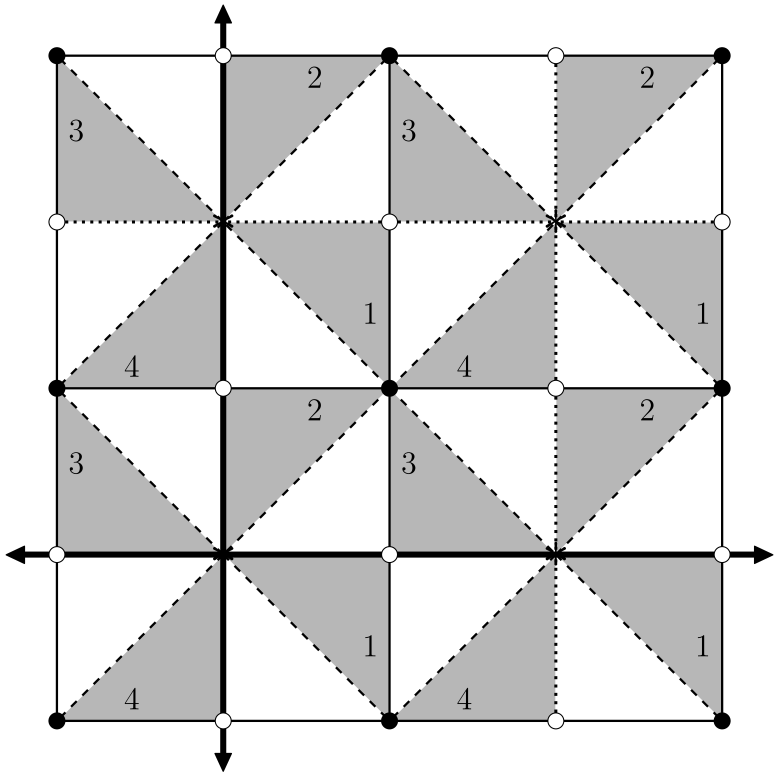

FIGURE 21. Tiling the plane with this fundamental domain.

Based on the previous exploration of Belyĭ maps on lattices, as explored in [6], we guess that our function is going to involve the Weierstrass $\wp$ function, so we check its action on the lattice. Recall that the Weierstrass $\wp$ function is defined as

$$\wp(z) = \frac{1}{z^2} + \sum_{0 \neq \omega \in \Lambda} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

Substituting iz for z produces the identity

$$\wp(iz) = -\wp(z), \tag{8.1}$$

as we can see by calculating:

$$\begin{aligned} \wp(iz) &= \frac{1}{(iz)^2} + \sum_{0 \neq \omega \in \Lambda} \left(\frac{1}{(iz-\omega)^2} - \frac{1}{\omega^2} \right) \\ &= -\frac{1}{z^2} + \sum_{0 \neq \omega \in \Lambda} \left(\frac{1}{(iz-i\omega)^2} - \frac{1}{i\omega^2} \right) \end{aligned}$$

since we can already noticed that $i\Lambda = \Lambda$, so we can shift the indexing of the lattice points

$$\begin{aligned} &= -\frac{1}{z^2} - \sum_{0\neq\omega\in\Lambda}\left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right) \\ &= -\wp(z) \end{aligned}$$

We can do the same thing with the derivative to get an additional identity. Clearly,

$$\wp'(z) = -\frac{1}{z^3} - \sum_{0\neq\omega\in\Lambda}\frac{1}{(z-\omega)^3}.$$

Then, we substitute in iz as before and get the identity

$$\wp'(iz) = i\wp'(z), \tag{8.2}$$

since

$$\begin{aligned} \wp'(iz) &= -\frac{1}{(iz)^3} - \sum_{0\neq\omega\in\Lambda}\frac{1}{(iz-\omega)^3} \\ &= \frac{1}{i^3}\cdot\frac{-1}{z^3} - \sum_{0\neq\omega\in\Lambda}\frac{1}{(iz-i\omega)^3} \end{aligned}$$

by the same shifting argument as above, so we get

$$\begin{aligned} &= i\frac{1}{z^3} - \frac{1}{i^3}\sum_{0\neq\omega\in\Lambda}\frac{1}{(z-\omega)^3} \\ &= i\wp'(z) \end{aligned}$$

We know that f has to respect the single automorphism on the lattice, i.e. $f(iz) = f(z)$, so we claim that the function we are looking for is $f = c\wp^2$, where $c \in \mathbb{R}$ is a constant that we will determine later.

We first check to see that this f respects the automorphism $z \mapsto iz$:

$$c\wp^2(iz) = c(-\wp(z))^2 = c\wp^2(z)$$

by the identity (8.1).

In order to check that this function works as a Belyĭ function, we need to verify that it sends every edge in the graph to the segment $[0, 1]$, that it sends black vertices to 0, that it sends white vertices to 1, and that it sends the star vertex to the pole.

Remark. Note first that by definition of a Weierstrass function,

$$\wp(z) = \wp(z+1) = \wp(z+i), \tag{8.3}$$

by [8, Ch. 3].

Furthermore, it suffices to check that most of these properties hold for $\wp^2$, ignoring the constant, since $c \in \mathbb{R}$.

We start with the edges:

- $\wp^2(\mathbb{R}) \subset \mathbb{R}$ and $\wp^2(i\mathbb{R}) \subset \mathbb{R}$. These are the horizontal and vertical boundaries of D, respectively. By the above Remark, it suffices to check the lower horizontal and the leftmost vertical, i.e. the real and imaginary axes.

  To see this, we recall that if $z = \overline{z}$ for $z \in \mathbb{C}$, then $z \in \mathbb{R}$. Then, given $x \in \mathbb{R}$, we have that

$$\begin{aligned}\overline{\wp(x)} &= \frac{1}{x^2} + \sum_{0 \neq \omega \in \Lambda} \left(\frac{1}{(x - \overline{\omega})^2} - \frac{1}{\overline{\omega}^2} \right) \\ &= \frac{1}{x^2} + \sum_{0 \neq \omega \in \Lambda} \left(\frac{1}{(x - \omega)^2} - \frac{1}{\omega^2} \right) \\ &= \wp(x)\end{aligned}$$

  where the second statement follows from the fact that $\overline{\Lambda} = \Lambda$ by reindexing. Thus, we're done, since

$$\overline{\wp^2(x)} = \left(\overline{\wp(x)}\right)^2 = \wp^2(x).$$

  Similarly, using identity (8.1)

$$\overline{\wp^2(ix)} = \left(\overline{\wp(ix)}\right)^2 = \left(-\overline{\wp(x)}\right)^2 = (-\wp(x))^2 = \wp^2(ix).$$

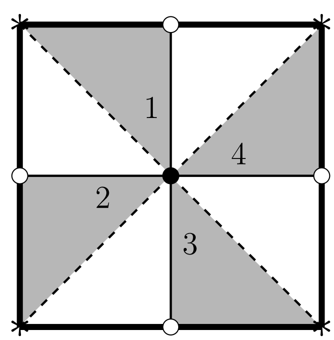

FIGURE 22. Covering the sides.

- $\wp^2(\sqrt{i}\mathbb{R}) \subset \mathbb{R}$ and $\wp^2(-\sqrt{i}\mathbb{R}) \subset \mathbb{R}$. These are the diagonals corresponding to $Re(z) = Im(z)$ and $Re(z) = -Im(z)$, respectively, where we translate the second one to match the second diagonal in D.

$$\begin{aligned}
\overline{\wp(\sqrt{i}x)} &= \overline{\frac{1}{(\sqrt{i}x)^2}} + \sum_{0\neq\omega\in\Lambda}\left(\frac{1}{\left(\overline{\sqrt{i}x} - \overline{\omega}\right)^2} - \frac{1}{\overline{\omega}^2}\right) \\
&= \frac{1}{ix^2} + \sum_{0\neq\omega\in\Lambda}\left(\frac{1}{\left(-\sqrt{i}x + \omega\right)^2} - \frac{1}{\omega^2}\right)
\end{aligned}$$

by reindexing, since $\Lambda = -\overline{\Lambda}$. Then, we have

$$\begin{aligned}
&= \frac{1}{(\sqrt{i}x)^2} + \sum_{0\neq\omega\in\Lambda}\left(\frac{1}{\left(\sqrt{i}x - \omega\right)^2} - \frac{1}{\omega^2}\right) \\
&= \wp(\sqrt{i}x).
\end{aligned}$$

So as before,

$$\overline{\wp^2(\sqrt{i}x)} = \wp^2(\sqrt{i}x).$$

Similarly, we can use the identity (8.1) to show that

$$\overline{\wp^2(-\sqrt{i}x)} = \left(\overline{\wp(-\sqrt{i}x)}\right)^2 = \left(-\overline{\wp(\sqrt{i}x)}\right)^2 = \left(-\wp(\sqrt{i}x)\right)^2 = \wp^2(-\sqrt{i}x).$$

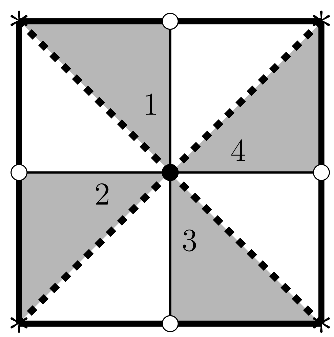

FIGURE 23. Covering the diagonals.

- $\wp^2(\{Re(z) = \frac{1}{2}\}) \subset \mathbb{R}$ and $\wp^2(\{Im(z) = \frac{1}{2}\}) \subset \mathbb{R}$. These are the cross-bars of D, vertical and horizontal, respectively. Consider $z = \frac{1}{2} + ix$, for some $x \in \mathbb{R}$. A series of elementary calculations similar to those above give us

$$\overline{\wp^2\left(\frac{1}{2} + ix\right)} = \wp^2\left(-\frac{1}{2} + ix\right) = \wp^2\left(\frac{1}{2} + ix\right)$$

where the second equality comes from applying the identity (8.3). Similarly, we calculate

$$\overline{\wp^2\left(x+\frac{1}{2}i\right)} = \left(-\overline{\wp\left(\frac{1}{2}-ix\right)}\right)^2 = \overline{\wp^2\left(\frac{1}{2}-ix\right)}$$
$$= (-1)^2\wp^2\left(\frac{1}{2}-ix\right) = \wp^2\left(x+\frac{1}{2}i\right)$$

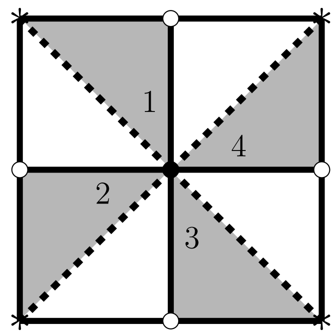

FIGURE 24. Covering the middle edges.

Now that we've shown that all edges are sent to the real line, it remains to show that the vertices behave as desired.

- The Weierstrass function $\wp$ has only one pole, located at $z = [0]$, so $\wp^2$ only has one pole, again located at $z = [0] = [\star]$, so $\wp(\star) = \infty$ as desired.
- Our black vertex should be the sole zero of $\wp^2$. We know that $[\bullet] = \left[\frac{1}{2}+\frac{1}{2}i\right]$, so we check:
$$\wp\left(\frac{1}{2}+\frac{1}{2}i\right) = \wp\left(i\left(\frac{1}{2}-\frac{1}{2}i\right)\right) = -\wp\left(\frac{1}{2}-\frac{1}{2}i\right) = -\wp\left(\frac{1}{2}+\frac{1}{2}i\right),$$
where the middle equivalence is due to identity (8.1) and the right one to identity (8.3). Thus, we have that $\wp\left(\frac{1}{2}+\frac{1}{2}i\right) = 0$, and thus that
$$\wp^2\left(\frac{1}{2}+\frac{1}{2}i\right) = 0.$$
- The last set of vertices are the two white ones, $[\circ_1] = \left[\frac{1}{2}\right]$ and $[\circ_2] = \left[\frac{1}{2}i\right]$, which we claim are the ramification points of $\wp$. By [4, §2.2.1], $\wp$ satisfies the algebraic relation
$$\wp'^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3.$$

Using identities (8.1) and (8.2), we get also that

$$\wp'^2(z) = 4\wp^3(z) - g_2\wp(z) + g_3,$$

so we have that $g_3 = 0$ and therefore, $g_2 \neq 0$. Furthermore, for any ramification point z, we then have that

$$\wp^2(z) = \frac{g_2}{4}.$$

Since we want our ramification points to go to 1 under f, we know that our constant $c = \frac{4}{g_2}$. We know that our curve can only have two ramification points, since

$$0 = \wp(z)(4\wp^2(z) - g_2)$$

gives us three solutions: the two points sent to $\frac{g_2}{4}$ and the single zero. Thus, we check our white vertices; since neither is 0, we can refigure the equation for the $\wp'$ into

$$\wp'(z) = -\sum_{\omega\in\Lambda} \frac{1}{(z-\omega)^3}.$$

Then, if we take $z = \frac{1}{2}$, we get

$$\begin{aligned}
\wp'\left(\frac{1}{2}\right) &= -\sum_{\omega\in\Lambda} \frac{1}{\left(\frac{1}{2}-\omega\right)^3} \\
&= -\sum_{\substack{\omega\in\Lambda \\ Re(\omega)<\frac{1}{2}}} \frac{1}{\left(\frac{1}{2}-\omega\right)^3} - \sum_{\substack{\omega\in\Lambda \\ Re(\omega)>\frac{1}{2}}} \frac{1}{\left(\frac{1}{2}-\omega\right)^3}.
\end{aligned}$$

We then can rephrase the first sum, since for ω such that $Re(\omega) < \frac{1}{2}$, $\omega = 1 - \omega'$ for some unique ω' such that $Re(\omega') > \frac{1}{2}$.

$$\begin{aligned}
&= -\sum_{\substack{\omega\in\Lambda \\ Re(\omega)>\frac{1}{2}}} \frac{1}{\left(-\frac{1}{2}+\omega\right)^3} - \sum_{\substack{\omega\in\Lambda \\ Re(\omega)>\frac{1}{2}}} \frac{1}{\left(\frac{1}{2}-\omega\right)^3}. \\
&= \sum_{\substack{\omega\in\Lambda \\ Re(\omega)>\frac{1}{2}}} \frac{1}{\left(\frac{1}{2}-\omega\right)^3} - \frac{1}{\left(\frac{1}{2}-\omega\right)^3} = 0.
\end{aligned}$$

Similarly, for the other white vertex, we split along the line $z = \frac{1}{2}i$:

$$\begin{aligned}\wp'\left(\frac{1}{2}i\right) &= -\sum_{\omega\in\Lambda} \frac{1}{\left(\frac{1}{2}i-\omega\right)^3} \\ &= -\sum_{\substack{\omega\in\Lambda \\ Im(\omega)<\frac{1}{2}}} \frac{1}{\left(\frac{1}{2}-\omega\right)^3} - \sum_{\substack{\omega\in\Lambda \\ Im(\omega)>\frac{1}{2}}} \frac{1}{\left(\frac{1}{2}-\omega\right)^3}.\end{aligned}$$

Rephrasing the first sum, since for ω such that $Im(\omega) < \frac{1}{2}$, $\omega = i - \omega'$ for some unique ω' such that $Im(\omega') > \frac{1}{2}$.

$$\begin{aligned}&= -\sum_{\substack{\omega\in\Lambda \\ Im(\omega)>\frac{1}{2}}} \frac{1}{\left(-\frac{1}{2}i+\omega\right)^3} - \sum_{\substack{\omega\in\Lambda \\ Im(\omega)>\frac{1}{2}}} \frac{1}{\left(\frac{1}{2}i-\omega\right)^3}. \\ &= \sum_{\substack{\omega\in\Lambda \\ Im(\omega)>\frac{1}{2}}} \frac{1}{\left(\frac{1}{2}i-\omega\right)^3} - \frac{1}{\left(\frac{1}{2}i-\omega\right)^3} = 0.\end{aligned}$$

Thus, $[\circ_1]$ and $[\circ_2]$ are ramification points for $\wp^2$, and since we only have two ramification points, we have all of them.

Therefore, we have shown that the fundamental domain D is mapped to the elementary hypermap as desired, with the black triangles mapping to $\mathbb{H}^+$ and the white to $\mathbb{H}^-$.

So, our Belyĭ pair is

$$(S_D, f_D) = \left(\frac{\mathbb{C}}{\mathbb{Z}+i\mathbb{Z}}, \frac{4}{g_2}\wp^2\right).$$

Furthermore, we can identify the elliptic curve corresponding to this surface by sending $\wp \mapsto X$ and $\wp' \mapsto Y$, so by [8], this surface is equivalent to the Belyĭ pair

$$\left(\{y^2 = 4x^3 - g_2 x\}, \frac{4}{g_2}x^2\right),$$

which is isomorphic to the Belyĭ pair

$$\left(\{y^2 = x^3 - x\}, x^2\right),$$

with j-invariant $j = 1728$.

9. A Not So Nice Example

Just to demonstrate that these equivalences work for more than just the really nice cases, an example of a slightly more complicated *dessin* is M shown in Figure 25.

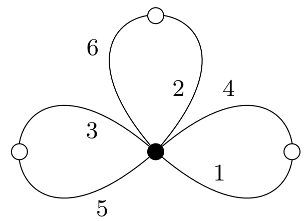

FIGURE 25. A slightly more complicated dessin.

We already calculated the combinatorial map for this *dessin* in Section 3: it is

$$[\sigma, \alpha, \phi] = [(1, 2, 3, 4, 5, 6),\ (1, 4)(2, 6)(3, 5),\ (1, 3, 4, 6)(2, 5)].$$

Furthermore, the corresponding Belyĭ pair, as computed by matrix model computations in [1], is

$$\left(\{y^2 = x(x + 3)(x - 1)\}, \frac{4x^3}{27(x - 1)}\right),$$

which has j-invariant $j = \frac{35152}{9}$.

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