Differential Operators and Algorithmic Weighted Resolution

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1 Introduction

1.1 Some history

As Dan Abramovich likes to say, singularities of algebraic varieties are beautiful, yet we try to get rid of them. Why do we do this crime? We do so because it is easier to understand smooth varieties. An algebraic variety has a resolution of singularities if some smooth variety projects onto it in a proper and birational way, that is, if it is the "shadow" of some smooth variety. A prominent question in algebraic geometry is whether every algebraic variety has a resolution of singularities. In his monumental paper [17], Hironaka answered this question in the affirmative for varieties over fields of characteristic zero, proving that there exists a resolution by repeatedly blowing them up.

The case of prime characteristic has proven to be more challenging. Though singular curves and surfaces in arbitrary characteristic can be resolved, as well as three-folds defined over perfect fields, the question of resolution of singularities in positive characteristic and in all dimensions is still wide open. It is well known that Hironaka's techniques in characteristic zero outright fail in positive characteristic. Furthermore, Hironaka's proof is incredibly intricate and complicated. For many years, people worked hard to simplify Hironaka's proof, so much so that the proof can now be responsibly presented at the end of a first course in algebraic geometry. People have also implemented algorithmic resolution of singularities in computer algebra systems, such as Anne Frühbis-Krüger and Gerhard Pfister's implementation in Singular [8]. Yet all these further developments still rely on Hironaka's involved bookkeeping of the exceptional divisors, which are byproducts of the blowing ups.

1.2 Recent progress

Recently, Dan Abramovich, Michael Temkin, and Jarosław Włodarczyk developed a new algorithm for resolution of singularities in characteristic zero by using stack-theoretic weighted blowing ups [1]. Their "weighted resolution algorithm" is much simpler than any of the other existing algorithms because it does away with the history of the exceptional divisors. The supposed cost of achieving this simpler algorithm is having to expand one's worldview, which may only consist of algebraic varieties, to allow for Deligne-Mumford stacks. This is not unlike how the Fourier transform of integrable functions, for which there is an explicit formula, is extended to general square integrable functions in order to obtain a unitary operator on Hilbert space L^2. Extending Fourier analysis to L^2 is natural, and we take the same attitude towards stack-theoretic resolution of singularities.

1.3 My honors thesis project

One important aspect of the weighted algorithm is that it is explicitly computable! I have had the wonderful privilege to have Professor Dan Abramovich (Brown University) and Professor Anne Frühbis-Krüger (University of Oldenburg) coadvise me through developing an implementation of the weighted resolution algorithm for my honors thesis project, which we implemented in SINGULAR. Currently the implementation can compute the centers along which to weighted blowup as well as the charts covering the stack-theoretic weighted blowing up of a variety. With these charts covering the stack, it remains to obtain their gluing data.

1.4 Overview of this document

In this theoretical document of the honors thesis project, I describe the implementation of the weighted resolution algorithm. I also devote a large portion of my honors thesis document to the notion of higher order differential operators, which is fundamental to weighted resolution and the theory of resolution of singularities in general. In Section 2, I introduce the general theory of the module of differential operators as well as the module of principal parts, which gives rise to all differential operators just as the module of Kähler differentials gives rise to all derivations. One can find this general theory introduced in [12]. In characteristic zero, the module of differential $D^{\leq n}$ of order up to n has many names, going by D^n in [14] and Δ^n in computational resolution of singularities (see [6], [7], and [24]).

In Section 3, we prove the following two theorems:

Theorem 1 (Differential Operators extend uniquely under Localization)**.** *Let A be a B-algebra and M an A-module. Let $D \in D^{\leq n}_{A/B}(A, M)$ be a differential operator of order up to n. Then there is a unique $S^{-1}D \in D^{\leq n}_{S^{-1}A/B}(S^{-1}A, S^{-1}M)$ extending D.*

Theorem 2 (See [12, Proposition 16.4.14])**.** *Let A be a B-algebra and $S \subset A$ a multiplicative system. Then the canonical map $S^{-1}P^n_{A/B} \to P^n_{S^{-1}A/B}$ is an isomorphism.*

The second theorem is Proposition 16.4.14 in [12], for which Grothendieck gives an incomplete, if not incorrect, proof (see Remark 1). Unfortunately, I have found no other references on localization and the module of principal parts. To make up for it, I furnish two separate proofs of the second theorem: in Section 3.1, we prove the first theorem, showing the second as a corollary, and then vice versa in Section .

Section 4 highlights more ways that differential operators interact with commutative algebra, in particular with ideals and adic completions. In Section 4.1, we prove the following theorem that applying differential operators to ideals commutes with localization:

Theorem 3. *We have the inclusion of ideals* $S^{-1}D^{\leq n}_{A/B}I \subset D^{\leq n}_{S^{-1}A/B}(S^{-1}I)$ *in* $S^{-1}A$*. If the ring map* $B \to A$ *is of finite presentation, then this inclusion is bijective.*

In Section 4.2, we prove that differential operators extend uniquely under completion:

Proposition 1 (Differential Operators extend uniquely under Completion)**.** *Let* $D \in D^{\leq n}_{A/B}(A, M)$*,* $I \subset A$ *an ideal. Then the differential operator* D *extends uniquely to a differential operator* $\hat{D} \in D^{\leq n}_{\hat{A}/B}(\hat{A}, \hat{M})$ *on the* I*-adic completions.*

In Section 5, we develop the theory of Hasse derivatives. These are also called Hasse-Dieudonne derivatives, Hasse-Schmidt derivatives, or Hasse-Schmidt derivations (see [22], as well as [14, 3.74.6]). I learned about Hasse derivatives mainly from [21, Appendix A.3], where Cutkosky describes the notion of higher derivations, and the blog posts of Felix Fontein in [5], where Fontein proves several properties about Hasse derivatives on polynomial rings $R[x_1, \dots, x_n]$. In Section 8, we prove the following theorem:

Theorem 4 (See [21, Theorem A.19])**.** *Suppose that* k *is a perfect field,* X *a smooth* k*-scheme, and* $\mathcal{I} \subset \mathcal{O}_X$ *a coherent ideal sheaf. Then* $p \in V(\mathcal{D}^{\leq r-1}_{X/k}\mathcal{I})$ *if and only if* $\mathrm{ord}_p \mathcal{I} \geq r$*.*

The proof of this theorem relies on understanding the behavior of Hasse derivatives and their extensions to adic completions. Indeed it was in seeking the proof to this theorem that I was led to the theory of Hasse derivatives. In Section 5, I first describe the most general conditions on a B-algebra A to admit a theory of Hasse derivatives that resembles the case on polynomial rings. In order to do so, I introduce the notion of an almost-quasi-regular sequence of an ideal in A, which is weaker than the notion of a quasi-regular sequence (see [19, Theorem 27]). The notion of almost-quasi-regular allows us to consider the most general condition on A such that A admits a theory of Hasse derivatives, that is, the ring of differential operators on A is free (see Proposition 11). The theory developed in Section 5 is quite pretty in my opinion, where an almost-quasi-regular sequence induces Hasse derivatives that behave just like Hasse derivatives on polynomial rings, such that when these Hasse derivatives are extended to any adic completion, these extensions behave just like Hasse derivatives on formal power series rings.

In Section 6, we globalize the module of differential operators and principal parts to their sheaf-theoretic counterparts. We begin with the theory of conormal invariants associated to a locally closed embedding. We use this theory to prove that the sheaf of principal parts is quasicoherent. Then we show that the sheaf of prinicipal parts is affine-locally the module of principal parts from Section 2. Afterwards, we proceed with the theory of the sheaf of differential operators, proving their quasicoherence in a special but reasonably general case, then computing this sheaf affine-locally.

In Section 7, we analyze differential operators on the germs of smooth varieties with separably generated residue fields. We show that we can import the theory of Hasse derivatives described in Section 5 into Section 7 since such germs admit a theory of Hasse derivatives. This allows us to completely describe the structure of differential operators on smooth varieties stalk-locally at points with separable residue field.

The work of Section 7 culminates in Section 8 when we prove that the stratifications of the order of vanishing function of an ideal a smooth variety over a perfect field is cut out by applying differential operators to the ideal:

Theorem 5 (See [21, Theorem A.19]). *Suppose that k is a perfect field, X a smooth k-scheme, and $\mathcal{I} \subset \mathcal{O}_X$ a coherent ideal sheaf. Then $p \in V(\mathcal{D}^{\leq r-1}_{X/k}\mathcal{I})$ if and only if $\operatorname{ord}_p \mathcal{I} \geq r$.*

Cutkosky essentially gives the same proof of the above theorem in [21] Theorem A.19. Cutkosky's proof describes Hasse derivatives on affine neighborhoods which extend to Hasse derivatives on the completions of the germs of closed points, justifying these extensions with results from Grothendieck's EGA on etale maps. My proof proceeds completely stalk-locally at any point, closed or not, relying on self-contained results about Hasse derivatives in Section 5 to extend Hasse derivatives on the germs of points on a smooth variety to their adic completions.

In Section 9, we give an explicit algorithm for applying first order differential operators on an ideal of a smooth variety. In Section 10, we follow Kollár [14] in introducing the theory of maximal contact hypersurfaces and show how to algorithmically cover a smooth variety in characteristic zero with local maximal contact hypersurfaces. In Section 11, we follow [1] and define the lexicographic order invariant and associated parameters for smooth varieties in characteristic zero, providing an algorithm for explicitly computing the top locus. In Section 12, we first lay out an algorithm for computing the weighted blowing up along a reduced center. We conclude Section 12, as well as this honors thesis document, by describing the weighted resolution algorithm, the realization of a dream algorithm (see [1, Section 1.2]).

I include an appendix with two subsections. The first subsection deals with Zariski closures and the ideal-theoretic operation of saturation. I include this because of the lack of a reference that scheme-theoretically describes the relation between ideal saturation and Zariski closures. The second subsection describes an algorithm for explicitly computing the orthogonal idempotents on a smooth affine variety.

1.5 Notation

Since everybody seems to have their own definition of an algebraic variety, we make clear which definition we will use. We choose the most general definition and follow [18, Definition 2.3.47], defining a variety over a field k to be a scheme of finite type over k.

We also introduce some multi-indexing notation, in particular to describe Hasse derivatives (see Section 5). First of all, we make clear that the natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ will be the nonnegative integers. We consider the additive structure on the set $\mathbb{N}^n$ of multi-indices of length n given by component-wise addition. We will use subscripts to indicate the components of a multi-index, so that if $s \in \mathbb{N}^n$, then $s = (s_1, \dots, s_n)$. For $s \in \mathbb{N}^n$, define

$$|s| := s_1 + \cdots + s_n \qquad s! := (s_1!) \cdots (s_n!)$$

For $s, t \in \mathbb{N}^n$, define the "multi-binomial coefficient" (in particular, this is not a multinomial coefficient)

$$\binom{s}{t} := \begin{cases} \frac{s!}{t!(s-t)!} = \prod_{i=1}^n \binom{s_i}{t_i} & \text{if } s - t \in \mathbb{N}^n \\ 0 & \text{otherwise} \end{cases}$$

If R is a ring and have a sequence of n elements $x_1, \cdots, x_n \in R$, then we will denote $x^{(s)} := x_1^{s_1} \cdots x_n^{s_n} \in R$ for $s \in \mathbb{N}^n$.

2 Module of Principal Parts and of Differential Operators

For this section, let $B \to A$ be a ring map, M an A-module, and I_Δ the kernel of the multiplication map $A \otimes_B A \to A$.

Recall that a derivation over B is a B-linear map $d : A \to M$ satisfying the Leibniz rule $d(fg) = fdg + gdf$. If we give $A \otimes_B A$ the structure of an A-module through its left factor, then from the theory of Kähler differentials, the map $d : A \to I_\Delta/I_\Delta^2$ given by $df = 1 \otimes f - f \otimes 1$ is a B-derivation such that every B-derivation $A \to M$ is obtained by pulling back along a unique A-linear map $I_\Delta/I_\Delta^2 \to M$. In this sense, $d : A \to I_\Delta/I_\Delta^2$ is a universal derivation, and I_Δ/I_Δ^2 represents the functor $M \mapsto \mathrm{Der}_B(A, M)$, where $\mathrm{Der}_B(A, M)$ is the collection of B-derivations from A to M. Alternatively, $\mathrm{Der}_B(A, M)$ could be defined as the image of the map $\mathrm{Hom}_A(I_\Delta/I_\Delta^2, M) \to \mathrm{Hom}_B(A, M)$ given by pulling back along the universal derivation. We replicate this alternative approach to introduce the notion of differential operators, which generalizes the notion of derivations as we will soon see.

2.1 Definitions

Definition 1 (Module of Principal Parts, see [12, 16.3.7]). *Define the module $P^n_{A/B}$ of principal parts of order n to be*

$$P^n_{A/B} := (A \otimes_B A)/I_\Delta^{n+1}$$

We give $P^n_{A/B}$ the structure of a left A-module by $\iota : a \mapsto a \otimes 1$. Let $d^n : A \to P^n_{A/B}$ be given by $a \mapsto 1 \otimes a$.

The image of d^n generates $P^n_{A/B}$ as a left A-module. Thus we have the inclusion of A-modules

$$\mathrm{Hom}_A(P^n_{A/B}, M) \xhookrightarrow{\circ d^n} \mathrm{Hom}_B(A, M)$$
$$\phi \mapsto \phi \circ d^\infty$$

where we give $\mathrm{Hom}_B(A, M)$ its left A-module structure by post-multiplication. $\mathrm{Hom}_B(A, M)$ also has a right A-module structure by pre-multiplication. Note that when $n = 0$, the inclusion above is the restriction of scalars inclusion $\mathrm{Hom}_A(A, M) \subset \mathrm{Hom}_B(A, M)$.

Definition 2 (Module of Differential Operators, see [12, Definition 16.8.1]). *Define the A-module $D^{\leq n}_{A/B}(A, M)$ of differential operators of order up to n to be the image under the left A-linear inclusion $\mathrm{Hom}_A(P^n_{A/B}, M) \hookrightarrow \mathrm{Hom}_B(A, M)$. Observe we have natural inclusions*

$$D^{\leq 0}_{A/B}(A, M) \subset D^{\leq 1}_{A/B}(A, M) \subset D^{\leq 2}_{A/B}(A, M) \subset \cdots$$

dual to the natural surjections

$$P^0_{A/B} \twoheadleftarrow P^1_{A/B} \twoheadleftarrow P^2_{A/B} \twoheadleftarrow \cdots$$

Note that $P^n_{A/B}$ represents the functor sending $M \mapsto D^{\leq n}_{A/B}(A, M)$ by definition.

Definition 3. *If $D \in D^{\leq n}_{A/B}(A, M) \setminus D^{\leq n-1}_{A/B}(A, M)$, then we say that D is a differential operator of order n.*

Definition 4. *Define $D_{A/B}(A, M) := \bigcup_{n \geq 0} D^{\leq n}_{A/B}(A, M)$. This is the A-submodule of all differential operators in $\mathrm{Hom}_B(A, M)$.*

2.2 Zeroth and first order differential operators

We first quickly note that $D^{\leq 0}_{A/B} = \mathrm{Hom}_A(A, M) \subset \mathrm{Hom}_B(A, M)$. The next proposition handles differentials operators of order up to one and connects the theory of differential operators with the theory of derivations. Recall that $\mathrm{Der}_B(A, M) \subset \mathrm{Hom}_B(A, M)$ is a left A-submodule (but not a right A-submodule!).

Proposition 2. $D^{\leq 1}_{A/B}(A, M) = \mathrm{Hom}_A(A, M) \oplus \mathrm{Der}_B(A, M)$ *as left A-submodules of* $\mathrm{Hom}_B(A, M)$*, where the direct sum is an internal direct sum.*

Proof. The short exact sequence

$$0 \to I_\Delta/I_\Delta^2 \to (A \otimes_B A)/I_\Delta^2 \to A \to 0$$

is split by the section $\iota : A \to (A \otimes_B A)/I_\Delta^2$, so that $P^1_{A/B} = \iota(A) \oplus (I_\Delta/I_\Delta^2)$ is an internal direct sum. Observe that $d^\infty : A \to P^1_{A/B} = A \oplus (I_\Delta/I_\Delta^2)$ is given by

$$a \mapsto 1 \otimes a = a \otimes 1 + (1 \otimes a - a \otimes 1) = \iota(a) + da$$

where $d : A \to I_\Delta/I_\Delta^2$ is the universal derivation. Thus the result follows. □

Thus we see that multiplication by A is a differential operator of order zero, and nonzero derivations are precisely the differential operators of order one.

2.3 Properties of differential operators

We first highlight some of the structure on $\mathrm{Hom}_B(A, M)$.

Definition 5. *For $D \in \mathrm{Hom}_B(A, M)$, define the B-linear map $\phi_D : A \otimes_B A \to M$ by $x \otimes y \mapsto xDy$. Note that ϕ_D is a left A-module homomorphism, so $\phi_D \in \mathrm{Hom}_A(A \otimes_B A, M)$.*

Consider the A-linear inclusion $\mathrm{Hom}_A(A \otimes_B A, M) \xhookrightarrow{\circ d^\infty} \mathrm{Hom}_B(A, M)$ given by pulling back along $d^\infty : a \mapsto 1 \otimes a$, which is indeed an inclusion because the image of d^∞ generates $A \otimes_B A$ as a left A-module. Because $\phi_D \circ d^\infty = D$, we see that this inclusion is an isomorphism, which we summarize in the following lemma.

Lemma 1. *The A-linear inclusion* $\mathrm{Hom}_A(A \otimes_B A, M) \xhookrightarrow{\circ d^\infty} \mathrm{Hom}_B(A, M)$ *is an isomorphism, with inverse given by $D \mapsto \phi_D$.*

We remark that the previous lemma implies that $A \otimes_B A$ as a left A-module represents the functor $M \mapsto \mathrm{Hom}_B(A, M)$.

Definition 6 (Commutator)**.** *For $f \in A$ and $D \in \mathrm{Hom}_B(A, M)$, define the commutator $[D, f] \in \mathrm{Hom}_B(A, M)$ to be be given by $[D, f](g) = D(fg) - fD(g)$.*

Now we make a calculation connecting ϕ_D, derivations, and commutators.

Calculation 1. *Let $D \in \mathrm{Hom}_B(A, M)$ and $d : A \to I_\Delta/I_\Delta^2$ be the universal derivation (so $df = 1 \otimes f - f \otimes 1$). Then it is not hard to verify that*

$$\phi_D((g \otimes 1) df_1 \cdots df_{n+1}) = \sum_{H \subset [n+1]} (-1)^{|H|} \left(\prod_{i \in H} f_i \right) D\left(g \prod_{j \notin H} f_j \right) = [\dots[D, f_1], \dots, f_{n+1}](g)$$

Proposition 3 (Differential Operator Criterion (See [12, Proposition 16.8.8]))**.** *Let $D \in \mathrm{Hom}_B(A, M)$. Then the following are equivalent:*

(i) $D \in D^{\leq n}_{A/B}(A, M)$*;*

(ii) $\phi_D(I_\Delta^{n+1}) = 0$*;*

(iii) For all $f_1, \cdots, f_{n+1} \in A$, we have

$$\sum_{H \subset [n+1]} (-1)^{|H|} \Big(\prod_{i \in H} f_i \Big) D \Big(\prod_{j \notin H} f_j \Big) = 0$$

where $[n+1] = \{1, \cdots, n+1\}$ and the empty product of ring elements is equal to the identity;

(iv) If $n = 0$, then $[D, f] = 0$ for all $f \in A$; otherwise if $n > 0$, then $[D, f] \in D^{\leq n-1}_{A/B}$ for all $f \in A$.

Proof. We first show that (i) and (ii) are equivalent. Suppose $D \in D^{\leq n}_{A/B}(A, M)$, and let $\phi \in \mathrm{Hom}_A(P^n_{A/B}, M)$ such that $D = \phi \circ d^n$. Consider the diagram

$$\begin{tikzcd} A \arrow[rr, "D"] \arrow[d, "d^\infty"] \arrow[dr] & & M \\ A \otimes_B A \arrow[r, two heads] & (A \otimes_B A)/I_\Delta^{n+1} \arrow[ur, "\phi"'] & \end{tikzcd}$$

By the equality $\mathrm{Hom}_A(A \otimes_B A, M) = \mathrm{Hom}_B(A, M)$, it follows that the map $A \otimes_B A \to M$ factoring through ϕ must be equal to ϕ_D. Thus I_Δ^{n+1} is in the kernel of ϕ_D.

Conversely, assume that $\phi_D(I_\Delta^{n+1}) = 0$. Then factor ϕ_D through the natural surjection $A \otimes_B A \to (A \otimes_B A)/I_\Delta^{n+1}$ to obtain that D is in the image of $\mathrm{Hom}_A(P^n_{A/B}, A) \hookrightarrow \mathrm{Hom}_B(A, A)$:

$$\begin{tikzcd} A \arrow[rr, "D"] \arrow[dr, "d^\infty"'] & & M \\ & A \otimes_B A \arrow[r, two heads] \arrow[ur, "\phi_D"] & (A \otimes_B A)/I_\Delta^{n+1} \arrow[u] \end{tikzcd}$$

Now we show that (ii) and (iii) are equivalent. For $f \in A$, denote by $df := 1 \otimes f - f \otimes 1 \in A \otimes_B A$. By Calculation 1, we have that

$$\phi_D(df_1 \cdots df_{n+1}) = \sum_{H \subset [n+1]} (-1)^{|H|} \Big(\prod_{i \in H} f_i \Big) D \Big(\prod_{j \notin H} f_j \Big)$$

for all $f_1, \cdots, f_{n+1} \in A$. Because $df_1 \cdots df_{n+1}$'s generate I_Δ^{n+1} as a left A-module and ϕ_D is left A-linear, we see that (ii) and (iii) are equivalent.

Now we show that (ii) and (iv) are equivalent. By Calculation 1, we have for all $f_1, \cdots, f_{n+1}, g \in A$ that

$$\phi_D((g \otimes 1) df_1 \cdots df_{n+1}) = [\ldots[D, f_1], \ldots, f_{n+1}](g)$$

Thus $\phi_D(I_\Delta^{n+1}) = 0$ iff $[\ldots[D, f_1], \ldots, f_{n+1}] = 0$ for all $f_1, \cdots, f_{n+1} \in A$. If $n = 0$, then $D^{\leq 0}_{A/B}(A, M) = \mathrm{Hom}_A(A, M)$ are the A-linear maps, and $D \in \mathrm{Hom}_B(A, M)$ being A-linear is equivalent to requiring $[D, f] = 0$ for all $f \in A$, so (ii) and (iv) are equivalent when $n = 0$. Now suppose $n > 0$ and that (ii) and (iv) are equivalent for everything less than n. Then

$[D, f_1] \in D^{\leq n-1}_{A/B}(A, M)$ for all $f_1 \in A$ iff $[[D, f_1], f_2] \in D^{\leq n-2}_{A/B}(A, M)$ for all $f_1, f_2 \in A$ iff ... iff $[\ldots[D, f_1], \ldots, f_n] = D^{\leq 0}_{A/B}(A, M)$ for all $f_i \in A$ iff $[\ldots[D, f_1], \ldots, f_{n+1}] = 0$ for all $f_i \in A$ iff $\phi_D(I^{n+1}_\Delta) = 0$. Thus (ii) and (iv) are equivalent for all n. □

Corollary 1. *Let $D \in D^{\leq n}_{A/B}(A, M)$. If $\phi : C \to A$ is a B-algebra map, then $D \circ \phi \in D^{\leq n}_{C/B}(C, M)$. If $\psi : M \to N$ is an A-linear map, then $\psi \circ D \in D^{\leq n}_{A/B}(A, N)$.*

Proof. Use Proposition 3 (iii). □

Definition 7. *For notational convenience, denote $D^{\leq n}_{A/B} := D^{\leq n}_{A/B}(A, A)$.*

Proposition 4 (See [12, Proposition 16.8.9])**.** *If $D \in D^{\leq n}_{A/B}$ and $D' \in D^{\leq m}_{A/B}$, then $D \circ D' \in D^{\leq n+m}_{A/B}$.*

Proof. For $f \in A$, just note that

$$[D \circ D', f] = D \circ [D', f] + [D, f] \circ D'$$

and then inductively use Proposition 3. □

Definition 8. *Define the ring of differential operators $D_{A/B} := \bigcup_{n \geq 0} D^{\leq n}_{A/B}$. The ring of differential operators $D_{A/B}$ has the structure of a unital associative algebra.*

Note that $D^{\leq n}_{A/B}(A, M) \subset \mathrm{Hom}_B(A, M)$ inherits not only the left A-module structure on $\mathrm{Hom}_B(A, M)$ given by post-multiplication but also the right A-module given by premultiplication. This is because if $f \in A$ and $D \in D^{\leq n}_{A/B}(A, M)$, then for every $a \in A$, we have $D(fa) = [D, f](a) - fD(a)$, so that $D \cdot f = [D, f] - fD \in D^{\leq n}_{A/B}(A, M)$ by Proposition 3.

The following proposition shows that finiteness conditions on the B-algebra A transfers to the module of principal parts.

Proposition 5. *If the ring map $B \to A$ is of finite type, then $P^n_{A/B}$ is a finite A-module. Moreover, if the ring map $B \to A$ is of finite presentation, then $P^n_{A/B}$ is a finitely presented A-module.*

Proof. Suppose $B \to A$ is of finite type. We then have some B-algebra identification $A = B[x_1, \cdots, x_m] =: B[x]/J$. This induces the following identification:

$$A \otimes_B A = \frac{B[x, y]}{J(x) + J(y)}$$

Note that the ideal generated by $x - y := x_1 - y_1, \ldots, x_m - y_m$ corresponds to the kernel of the multiplication map $A \otimes_B A \to A$ because

$$\frac{B[x, y]/(J(x) + J(y))}{(x - y)} = \frac{B[x, x]/(J(x) + J(x))}{(x - y)} = B[x]/J(x)$$

Thus

$$\begin{aligned}
P^n_{A/B} &= \frac{B[x,y]/(J(x)+J(y))}{((x-y)/(J(x)+J(y)))^{n+1}} \\
&= \frac{B[x,y]}{(x-y)^{n+1}+J(x)+J(y)} \\
&= \frac{B[x,dx]}{(dx)^{n+1}+J(x)+J(x+dx)} \\
&= \frac{A[dx]/(dx)^{n+1}}{J(x+dx)}
\end{aligned}$$

where we changed variables $dx_i := y_i - x_i$. Because $A[dx]/(dx)^{n+1}$ is a finite free A-module, it follows that $P^n_{A/B}$ is a finite A-module. If J is finitely generated, then $P^n_{A/B}$ is a finitely presented A-module. $\square$

3 Differential Operators and Localization

For this section, let A be a B-algebra and M an A-module. Let $S \subset A$ be a multiplicative set. Our goal will be to show that we can extend differential operators uniquely under localization:

Theorem 6 (Differential Operators extend uniquely under Localization)**.** *Let $D \in D^{\leq n}_{A/B}(A, M)$. Then there is a unique $S^{-1}D \in D^{\leq n}_{S^{-1}A/B}(S^{-1}A, S^{-1}M)$ extending D:*

$$\begin{tikzcd}
A \arrow[r, "D"] \arrow[d] & M \arrow[d] \\
S^{-1}A \arrow[r, "S^{-1}D"] & S^{-1}M
\end{tikzcd}$$

We will present two distinct proofs of this theorem. The first approach will directly construct the extension $S^{-1}D$. On the other hand, the second approach will instead show there is a canonical isomorphism $S^{-1}P^n_{A/B} = P^n_{S^{-1}A/B}$, from which it will formally follow that differential operators extend under localization.

We first begin with a lemma that shows we will only need to prove that an extension to the localization exists.

Lemma 2 (At most one extension)**.** *Let $D \in D^{\leq n}_{A/B}(A, M)$. Then there is at most one $S^{-1}D \in D^{\leq n}_{S^{-1}A/B}(S^{-1}A, S^{-1}M)$ extending D.*

Proof. We proceed by induction on the order of D. If $D \in D^{\leq 0}_{A/B}(A, M)$, then if $\widetilde{D} \in D^{\leq 0}_{S^{-1}A/B}(S^{-1}A, S^{-1}M)$ extends D, then

$$\widetilde{D}(\tfrac{a}{s}) = \tfrac{s}{s}\widetilde{D}(\tfrac{a}{s}) = \tfrac{1}{s}\widetilde{D}(a) = \tfrac{1}{s}D(a)$$

so we see that there can only be one such $\widetilde{D}$.

Now suppose $n > 0$ and the result is true for differential operators of order up to $n-1$. Assume there exists $\widetilde{D} \in D^{\leq n}_{S^{-1}A/B}(S^{-1}A, S^{-1}M)$ that extends D. Let $s \in S$ and $a \in A$. Since $[D, s]$ has order less than n, we have its unique extension $S^{-1}[D, s] \in D^{\leq n-1}_{S^{-1}A/B}(S^{-1}A, S^{-1}M)$ by the inductive hypothesis. Because

$$[\widetilde{D}, s](a) = \widetilde{D}(sa) - s\widetilde{D}(a) = D(sa) - sD(a) = [D, s](a)$$

we see that $[\widetilde{D}, s] \in D^{\leq n-1}_{S^{-1}A/B}(S^{-1}A, S^{-1}M)$ also extends $[D, s]$, hence $[\widetilde{D}, s] = S^{-1}[D, s]$. Now observe that

$$S^{-1}[D, s](a/s) = [\widetilde{D}, s](a/s) = \widetilde{D}(a) - s\widetilde{D}(a/s) = D(a) - s\widetilde{D}(a/s)$$

Thus

$$\widetilde{D}(a/s) = \frac{D(a) - S^{-1}[D, s](a/s)}{s}$$

hence we see that there can only be one such $\widetilde{D}$. $\square$

3.1 First approach

Definition 9 (Repeated Commutator). *Let $D \in \mathrm{Hom}_B(A, M)$ and $s \in A$. Define $[D, s]^{(0)} = D$ and $[D, s]^{(n)} = [[D, s]^{(n-1)}, s]$ for $n > 0$.*

Let $D \in D_{A/B}(A, M)$ and $a/s \in S^{-1}A$. Let's suppose we could extend every differential operator to the localizations.

$$\begin{aligned} D(\tfrac{a}{s}) &= \frac{D(a) - [D, s](\frac{a}{s})}{s} \\ &= \frac{D(a) - \dfrac{[D, s](a) - [[D, s], s](\frac{a}{s})}{s}}{s} \\ &= \frac{D(a) - \dfrac{[D, s](a) - \dfrac{[[D, s], s](a) - [[[D, s], s], s](\frac{a}{s})}{s}}{s}}{s} \\ &= \cdots \end{aligned}$$

At some point the $[D, s]^{(n)}(\frac{a}{s})$ will terminate, so that we will be left with an expression where differential operators from $A \to M$ only act on the element $a \in A$, leaving s alone. Thus we are motivated to make the following definition:

Definition 10. *Let $D \in D_{A/B}(A, M)$ and $s \in A$. Define*

$$D(\tfrac{-}{s}) := \sum_{k=0}^{\infty} \frac{(-1)^k [D, s]^{(k)}}{s^{k+1}} \in D_{A/B}(A, S^{-1}M)$$

If $D \in D^{\leq n}_{A/B}(A, M)$, then $[D, s]^{(k)} = 0$ for every $k > n$ by Proposition 3, hence $D(\frac{-}{s}) \in D^{\leq n}_{A/B}(A, S^{-1}M)$.

The notation is written this way to remind ourselves that $D(\frac{-}{s})(a)$ is "equal" to "$D(a/s)$", which at the moment doesn't make sense. In what follows we will show that there is a unique extension of $D \in D^{\leq n}_{A/B}(A, M)$ to $S^{-1}D \in D^{\leq n}_{S^{-1}A/B}(S^{-1}A, S^{-1}M)$ given by $S^{-1}D(a/s) = D(\frac{-}{s})(a)$. So looking forward, this notation is "correct."

Observe that

$$D(\tfrac{-}{s}) = \frac{D - [D, s](\frac{-}{s})}{s}$$

Lemma 3. *Let $D \in D_{A/B}(A, M)$ and $s, t \in S$ and $a \in A$.*

(i) (Commutativity) $[D, a](\frac{-}{s}) = [D(\frac{-}{s}), a]$

(ii) (Cancellation) $D(\frac{-}{s})(sa) = D(a)$

(iii) (Well-defined) $D(\frac{-}{st})(at) = D(\frac{-}{s})(a)$

Proof. (i) Note that $[[D,a],s] = [[D,s],a]$, hence $[[D,a],s]^{(k)} = [[D,s]^{(k)},a]$ for all k. Observe that:

$$
\begin{aligned}
[D,a](\tfrac{-}{s}) &= \sum_{k=0}^{\infty} \frac{(-1)^k [[D,a],s]^{(k)}}{s^{k+1}} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k [[D,s]^{(k)},a]}{s^{k+1}} \\
&= \left[\sum_{k=0}^{\infty} \frac{(-1)^k [D,s]^{(k)}}{s^{k+1}}, a \right] \\
&= [D(\tfrac{-}{s}), a]
\end{aligned}
$$

□

(ii) To show that $D(\frac{-}{s})(sa) = D(a)$, we proceed by induction on the order of D. If $D \in D^{\leq 0}_{A/B}(A,M)$, then

$$D(\tfrac{-}{s})(sa) = \frac{D(sa)}{s} = \frac{sD(a)}{s} = D(a)$$

So suppose $n > 0$ and the result is true for differential operators of order up to $n-1$. Then

$$
\begin{aligned}
D(\tfrac{-}{s})(sa) &= \frac{D(sa) - [D,s](\tfrac{-}{s})(sa)}{s} \\
&= \frac{D(sa) - [D,s](a)}{s} \qquad \text{b.c. } [D,s] \text{ has order } n-1 \\
&= \frac{D(sa) - D(sa) + sDa}{s} \\
&= D(a)
\end{aligned}
$$

□

(iii) We induct on the order of D. If $D \in D^{\leq 0}_{A/B}(A,M)$, then

$$D(\tfrac{-}{st})(at) = \frac{D(at)}{st} = \frac{D(a)}{s} = D(\tfrac{-}{s})(a)$$

So suppose $n > 0$ and the result is true for differential operators of order up to $n-1$. Then we have

$$
\begin{aligned}
D(\tfrac{-}{st})(at) &= \frac{D(at) - [D,st](\tfrac{-}{st})(at)}{st} \\
&= \frac{D(at) - [D,st](\tfrac{-}{s})(a)}{st} \qquad \text{b.c. } [D,st] \text{ has order } n-1 \\
&= \frac{D(at) - [D(\tfrac{-}{s}),st](a)}{st} \qquad \text{by part (i)} \\
&= \frac{D(at) - D(\tfrac{-}{s})(sta) + stD(\tfrac{-}{s})(a)}{st} \\
&= \frac{D(at) - D(ta) + stD(\tfrac{-}{s})(a)}{st} \qquad \text{by part (ii)} \\
&= D(\tfrac{-}{s})(a)
\end{aligned}
$$

$\square$

Theorem 7 (Differential Operators extend uniquely under Localization)**.** *Let $D \in D^{\leq n}_{A/B}(A, M)$. Then there is a unique $S^{-1}D \in D^{\leq n}_{S^{-1}A/B}(S^{-1}A, S^{-1}M)$ extending D, and it is given by $S^{-1}D(a/s) = D(\frac{-}{s})(a)$.*

Proof. We proceed by induction on the order of D. If $D \in D^{\leq 0}_{A/B}(A, M)$, then $S^{-1}D \in D^{\leq 0}_{S^{-1}A/B}(S^{-1}A, S^{-1}M)$ given by

$$S^{-1}D(a/s) := D(\tfrac{-}{s})(a) = D(a)/s = \tfrac{a}{s}D(1)$$

extends D.

Now suppose $n > 0$ and the result is true for differential operators of order up to $n-1$. We claim that $S^{-1}D(a/s) := D(\frac{-}{s})(a)$ extends D. First, to see that $S^{-1}D$ is well-defined, let $a/s = a'/s'$. Then there exists $t \in S$ such that $t(s'a - sa') = 0$. Repeatedly applying Lemma 3 (iii), we obtain

$$D(\tfrac{-}{s})(a) = D(\tfrac{-}{ts's})(ts'a) = D(\tfrac{-}{ts's})(tsa') = D(\tfrac{-}{s'})(a')$$

So $S^{-1}D$ is well-defined. Because $[D, 1] = 0$, it follows that $S^{-1}D(a/1) = D(\frac{-}{1})(a) = D(a)$, and $S^{-1}D$ extends D. It is straightforward to see that $S^{-1}D$ is B-linear, hence we only have to verify that $S^{-1}D \in D^{\leq n}_{S^{-1}A/B}(S^{-1}A, S^{-1}M)$. Let $a_1/s_1, \cdots, a_{n+1}/s_{n+1} \in S^{-1}A$, and call $s_{[1,n+1]} := s_1 \cdots s_{n+1}$. Then we have

$$\begin{aligned}
\sum_{H \subset [n+1]} (-1)^{|H|} \Big(\prod_{i \in H} \frac{a_i}{s_i}\Big) S^{-1}D\Big(\prod_{j \notin H} \frac{a_j}{s_j}\Big) &= \sum_{H \subset [n+1]} (-1)^{|H|} \frac{\prod_{j \notin H} s_j \prod_{i \in H} a_i}{s_{[1,n+1]}} S^{-1}D\Big(\frac{\prod_{i \in H} s_i \prod_{j \notin H} a_j}{s_{[1,n+1]}}\Big) \\
&= \frac{1}{s_{[n+1]}} \sum_{H \subset [n+1]} (-1)^{|H|} \Big(\prod_{j \notin H} s_j \prod_{i \in H} a_i\Big) D\Big(\frac{1}{s_{[1,n+1]}}\Big)\Big(\prod_{i \in H} s_i \prod_{j \notin H} a_j\Big) \\
&= \frac{1}{s_{[n+1]}} \phi_{D(\ /s_{[1,n+1]})}\Big(\prod_{k \in 1}^{n+1} (s_k \otimes a_k - a_k \otimes s_k)\Big) \\
&= 0
\end{aligned}$$

where the last equality follows because $D(\ /s_{[1,n+1]}) \in D^{\leq n}_{A/B}(A, S^{-1}M)$ and each $s_k \otimes a_k - a_k \otimes s_k \in I$. Therefore $S^{-1}D \in D^{\leq n}_{S^{-1}A/B}(S^{-1}A, S^{-1}M)$ by Proposition 3, and we are done. $\square$

Pulling the universal differential operator $S^{-1}A \to P^n_{S^{-1}A/B}$ back to A, we obtain a canonical A-linear map $P^n_{A/B} \to P^n_{S^{-1}A/B}$, hence by localizing we get a natural $S^{-1}A$-linear map $S^{-1}P^n_{A/B} \to P^n_{S^{-1}A/B}$.

Corollary 2 (See [12, Proposition 16.4.14])**.** *The canonical map $S^{-1}P^n_{A/B} \to P^n_{S^{-1}A/B}$ is an isomorphism.*

Proof. Let N be an $S^{-1}A$-module. Consider the map

$$\mathrm{Hom}_{S^{-1}A}(P^n_{S^{-1}A/B}, N) \to \mathrm{Hom}_{S^{-1}A}(S^{-1}P^n_{A/B}, N)$$

obtained by pulling back along $S^{-1}P^n_{A/B} \to P^n_{S^{-1}A/B}$. We can identify this map with the map

$$D^{\leq n}_{S^{-1}A/B}(S^{-1}A, N) \to D^{\leq n}_{A/B}(A, N)$$

given by pulling back along $A \to S^{-1}A$. Eyeballing the following diagram shows why this identification is true.

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & P^n_{A/B} \arrow[d] \arrow[ddr] & \\
S^{-1}A \arrow[r] \arrow[dr] & P^n_{S^{-1}A/B} \arrow[d] \arrow[dr] & \\
 & S^{-1}P^n_{A/B} \arrow[r] & N
\end{tikzcd}$$

On the other hand, extending the universal differential operator $A \to P^n_{A/B}$ of order up to n to $S^{-1}A \to S^{-1}P^n_{A/B}$ by Theorem 6, we see that there is a unique $S^{-1}A$-linear map $P^n_{S^{-1}A/B} \to S^{-1}P^n_{A/B}$ through which $S^{-1}A \to S^{-1}P^n_{A/B}$ factors. Then the corresponding pullback map

$$\mathrm{Hom}_{S^{-1}A}(S^{-1}P^n_{A/B}, N) \to \mathrm{Hom}_{S^{-1}A}(P^n_{S^{-1}A/B}, N)$$

identifies with the map

$$D^{\leq n}_{A/B}(A, N) \to D^{\leq n}_{S^{-1}A/B}(S^{-1}A, N)$$

given by extending a differential operator $A \to N$ to the localization $S^{-1}A \to N$. By Theorem 6, these two pullback maps are inverse to each other. Thus the canonical map $S^{-1}P^n_{A/B} \to P^n_{S^{-1}A/B}$ is an isomorphism. $\square$

3.2 Second approach

Theorem 8 (See [12, Proposition 16.4.14])**.** *The canonical map $S^{-1}P^n_{A/B} \to P^n_{S^{-1}A/B}$ is an isomorphism.*

Proof. Let I_Δ be the kernel of the multiplication map $A \otimes_B A \to A$. Consider the multiplicative sets $S \otimes 1 := \{s \otimes 1 \in A \otimes_B A | s \in S\}$ and $S \otimes S := \{s \otimes t \in A \otimes_B A | s, t \in S\}$. We claim that the further localization maps of $A \otimes_B A$-modules below is an isomorphism for all n:

$$(S \otimes 1)^{-1} \frac{A \otimes_B A}{I^{n+1}_\Delta} \to (S \otimes S)^{-1} \frac{A \otimes_B A}{I^{n+1}_\Delta}$$

First let $n = 0$, the base case. Since $\frac{A \otimes_B A}{I_\Delta} = A$ and both $S \otimes 1$ and $S \otimes S$ map onto $S \subset A$, the further localization map for $n = 0$ identifies with the identity $S^{-1}A \to S^{-1}A$.

So suppose $n > 0$ and the claim true for $n - 1$. Consider the following map of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & (S \otimes 1)^{-1} \frac{I^n_\Delta}{I^{n+1}_\Delta} \arrow[r] \arrow[d] & (S \otimes 1)^{-1} \frac{A \otimes_B A}{I^{n+1}_\Delta} \arrow[r] \arrow[d] & (S \otimes 1)^{-1} \frac{A \otimes_B A}{I^n_\Delta} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (S \otimes S)^{-1} \frac{I^n_\Delta}{I^{n+1}_\Delta} \arrow[r] & (S \otimes S)^{-1} \frac{A \otimes_B A}{I^{n+1}_\Delta} \arrow[r] & (S \otimes S)^{-1} \frac{A \otimes_B A}{I^n_\Delta} \arrow[r] & 0
\end{tikzcd}$$

where the vertical arrows are further localization. The rightmost vertical arrow is an isomorphism by hypothesis. The leftmost vertical arrows is as well, because $I^n_\Delta / I^{n+1}_\Delta$ is a module over $(A \otimes_B A)/I_\Delta =$

A, and $S \otimes S$ and $S \otimes 1$ both map onto $S \subset A$. Thus the middle vertical arrow is an isomorphism, and the claim is true.

Now observe the natural isomorphism of rings $(S \otimes S)^{-1}(A \otimes_B A) = S^{-1}A \otimes_B S^{-1}A$. Considering $A = (A \otimes_B A)/I_\Delta$ as an $A \otimes_B A$-module and $S^{-1}A = (S^{-1}A \otimes_B S^{-1}A)/I_{S^{-1}\Delta}$ as an $S^{-1}A \otimes_B S^{-1}A$-module, we have a natural isomorphism of $S^{-1}A \otimes_B S^{-1}A$-modules $(S \otimes S)^{-1}A = S^{-1}A$. Thus we have exactness of

$$0 \to (S \otimes S)^{-1} I_\Delta \to S^{-1}A \otimes_B S^{-1}A \to S^{-1}A \to 0$$

Thus

$$P^n_{S^{-1}A/B} = \frac{S^{-1}A \otimes_B S^{-1}A}{(S \otimes S)^{-1} I^{n+1}_\Delta} = (S \otimes S)^{-1} \frac{A \otimes_B A}{I^{n+1}_\Delta}$$

Also, observe the isomorphism of rings $(S \otimes 1)^{-1}(A \otimes_B A) = (S^{-1}A) \otimes_B A$. We will observe a general fact. Let T be an $A \otimes_B A$-module. Then $(S \otimes 1)^{-1}T$ is an $S^{-1}A$-module by restricting its a priori $S^{-1}A \otimes_B A$-module to its left factor. On the other hand, we can localize T at S as a left A-module when restricting scalars to the left factor of $A \otimes_B A$ to obtain the $S^{-1}A$-module $S^{-1}T$. Then there is a natural isomorphism of $S^{-1}A$-modules $(S \otimes 1)^{-1}T = S^{-1}T$. Thus we have that

$$S^{-1} P^n_{A/B} = (S \otimes 1)^{-1} \frac{A \otimes_B A}{I^{n+1}_\Delta}$$

It is not hard to see the further localization map identifies with the canonical map $S^{-1}P^n_{A/B} \to P^n_{S^{-1}A/B}$. Thus we are done. $\square$

Remark 1. *The statement of Theorem 8 is the same as [12, Proposition 16.4.14] (note Grothendieck lets B be an A-algebra). However, the proof provided in [12, Proposition 16.4.14] is incomplete, if not incorrect. Grothendieck's proof is one sentence, where he says that it suffices to remark that $S^{-1}((A \otimes_B A)/I^{n+1}_\Delta) = S^{-1}(A \otimes_B A)/(S^{-1}I_\Delta)^{n+1}$, which is okay, and $S^{-1}(A \otimes_B A) = S^{-1}A \otimes_B S^{-1}A$, which is not okay. To justify $S^{-1}(A \otimes_B A) = S^{-1}A \otimes_B S^{-1}A$, he cites the fact that $S^{-1}(M \otimes_A N) = S^{-1}M \otimes_{S^{-1}A} S^{-1}N$ for A-modules M, N, which does not apply. If we consider localizing $A \otimes_B A$ at S as a left A-module, then $S^{-1}(A \otimes_B A) = S^{-1}A \otimes_B A$ and the natural map $S^{-1}A \otimes_B A \to S^{-1}A \otimes_B S^{-1}A$ is not an isomorphism in general, e.g. $B = k$, $A = k[x]$ and $S = \{1, x, x^2, \dots\}$.*

Corollary 3 (Differential Operators extend uniquely under Localization)**.** *Let $D \in D^{\leq n}_{A/B}(A, M)$. Then there is $S^{-1}D \in D^{\leq n}_{S^{-1}A/B}(S^{-1}A, S^{-1}M)$ extending D.*

Proof. By previous Theorem 8, the canonical map $S^{-1}P^n_{A/B} \to P^n_{S^{-1}A/B}$ is an isomorphism.

$$\begin{aligned}
D^{\leq n}_{A/B}(A, M) &= \mathrm{Hom}_A(P^n_{A/B}, M) \\
&\to S^{-1}\,\mathrm{Hom}_A(P^n_{A/B}, M) \\
&\to \mathrm{Hom}_{S^{-1}A}(S^{-1}P^n_{A/B}, S^{-1}M) \\
&= \mathrm{Hom}_{S^{-1}A}(P^n_{S^{-1}A/B}, S^{-1}M) \\
&= D^{\leq n}_{S^{-1}A/B}(S^{-1}A, S^{-1}M)
\end{aligned}$$

It is easy to check that the map $D^{\leq n}_{A/B}(A, M) \to D^{\leq n}_{S^{-1}A/B}(S^{-1}A, S^{-1}M)$ sends a differential operator to its extension. $\square$

3.3 Consequences

Proposition 6. *If $T \subset B$ is a multiplicative set that maps into $S \subset A$, then we have a canonical isomorphism $P^n_{S^{-1}A/B} \to P^n_{S^{-1}A/T^{-1}B}$, hence the canonical map $S^{-1}P^n_{A/B} = P^n_{S^{-1}A/T^{-1}B}$ is an isomorphism.*

Proof. This follows by noting that the natural map $S^{-1}A \otimes_B S^{-1}A \to S^{-1}A \otimes_{T^{-1}B} S^{-1}A$ is an isomorphism. One way to see this is that these two rings represent the functor sending an $S^{-1}A$-module N to $\mathrm{Hom}_B(S^{-1}A, N)$ and $\mathrm{Hom}_{T^{-1}B}(S^{-1}A, N)$, respectively by Lemma 1. Then just observe that the restriction of scalars inclusion $\mathrm{Hom}_{T^{-1}B}(S^{-1}A, N) \hookrightarrow \mathrm{Hom}_B(S^{-1}A, N)$ is bijective because if $D : S^{-1}A \to S^{-1}M$ was B-linear and $b/t \in T^{-1}B$ and $f \in S^{-1}A$, then

$$D(\tfrac{b}{t}f) = \tfrac{t}{t}D(\tfrac{b}{t}f) = \tfrac{b}{t}D(\tfrac{t}{t}f) = \tfrac{b}{t}D(f)$$

so that D is $T^{-1}B$-linear. □

Since differential operators extend uniquely to the localization, we have an A-linear inclusion

$$D^{\le n}_{A/B}(A, M) \hookrightarrow D^{\le n}_{S^{-1}A/B}(S^{-1}A, S^{-1}M)$$

sending a differential operator to its extension. Thus we also have an $S^{-1}A$-linear inclusion

$$S^{-1}D^{\le n}_{A/B}(A, M) \hookrightarrow D^{\le n}_{S^{-1}A/B}(S^{-1}A, S^{-1}M)$$

The following theorem addresses when this inclusion is bijective.

Theorem 9. *Suppose the ring map $B \to A$ is of finite presentation. Then the inclusion*

$$S^{-1}D^{\le n}_{A/B}(A, M) \hookrightarrow D^{\le n}_{S^{-1}A/B}(S^{-1}A, S^{-1}M)$$

is an isomorphism. Furthermore, if $T \subset B$ is a multiplicative system that maps into $S \subset A$, then

$$S^{-1}D^{\le n}_{A/B}(A, M) = D^{\le n}_{S^{-1}A/T^{-1}B}(S^{-1}A, S^{-1}M)$$

because any B-linear map between $S^{-1}A$-modules is also $T^{-1}B$-linear.

Proof. Consider the inclusion map given by extending to the localization

$$D^{\le n}_{A/B}(A, B) \hookrightarrow D^{\le n}_{S^{-1}A/B}(S^{-1}A, S^{-1}M)$$

It is easy to check that this is A-linear. Localizing, we obtain another inclusion

$$S^{-1}D^{\le n}_{A/B}(A, B) \hookrightarrow D^{\le n}_{S^{-1}A/B}(S^{-1}A, S^{-1}M)$$

This identifies with the canonical map

$$S^{-1}\,\mathrm{Hom}_A(P^n_{A/B}, M) \to \mathrm{Hom}_{S^{-1}A}(P^n_{S^{-1}A/B}, S^{-1}M)$$

which can be seen by eyeballing the following diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & P^n_{A/B} \arrow[r] \arrow[d] & M \arrow[d] \\
S^{-1}A \arrow[r] \arrow[dr] & P^n_{S^{-1}A/B} \arrow[r] \arrow[d, "="] & S^{-1}M \\
& S^{-1}P^n_{A/B} \arrow[ur] &
\end{tikzcd}$$

By Proposition 5, $P^n_{A/B}$ is a finitely-presented A-module, hence the map $S^{-1}\operatorname{Hom}_A(P^n_{A/B}, M) \to \operatorname{Hom}_{S^{-1}A}(P^n_{S^{-1}A/B}, S^{-1}M)$ is an isomorphism. So we are done. $\square$

There is also another instance the inclusion is bijective.

Proposition 7. *Let N be an $S^{-1}A$-module. Then the inclusion*

$$D^{\leq n}_{A/B}(A, N) \hookrightarrow D^{\leq n}_{S^{-1}A/B}(S^{-1}A, N)$$

is bijective.

Proof.

$$\begin{aligned} D^{\leq n}_{A/B}(A, N) &= \operatorname{Hom}_A(P^n_{A/B}, N) \\ &= \operatorname{Hom}_{S^{-1}A}(S^{-1}P^n_{A/B}, N) \\ &= \operatorname{Hom}_{S^{-1}A}(P^n_{S^{-1}A/B}, N) \\ &= D^{\leq n}_{S^{-1}A/B}(S^{-1}A, N) \end{aligned}$$

$\square$

4 More on Differential Operators

In this section we discuss applying differential operators on ideals and extending differential operators to adic completions.

4.1 Differentiating ideals

Let A be a B-algebra. We will now only consider the case when $M = A$, though everything that follows also applies for general M.

Definition 11. *Let $I \subset A$ be an ideal. Define $D^{\leq n}_{A/B}I$ to be the ideal of A generated by the image of the evaluation map $D^{\leq n}_{A/B} \times I \to A$ sending $(D, f) \mapsto D(f)$.*

Proposition 8. *Let $I \subset A$ be an ideal generated by f_i's and $D^{\leq 1}_{A/B}$ generated by D_j's as an A-module. Then the ideal $D^{\leq n}_{A/B}I \subset A$ is generated by $D_j(f_i)$'s.*

Proof. Use Proposition 3 (iii). $\square$

Theorem 10. *We have the inclusion of ideals $S^{-1}D^{\leq n}_{A/B}I \subset D^{\leq n}_{S^{-1}A/B}(S^{-1}I)$ in $S^{-1}A$. If the ring map $B \to A$ is of finite presentation, then this inclusion is bijective.*

Proof. Note that $S^{-1}D^{\leq n}_{A/B}I \subset S^{-1}A$ is an ideal generated by $D(f)$ for $D \in D^{\leq n}_{A/B}$ and $f \in I$. Because $D(f) = S^{-1}D(f/1) \in D^{\leq n}_{S^{-1}A/B}(S^{-1}I)$, hence we have an inclusion of ideals $S^{-1}D^{\leq n}_{A/B}I \subset D^{\leq n}_{S^{-1}A/B}(S^{-1}I)$.

Now suppose that the ring map $B \to A$ is of finite presentation. In particular, by Theorem 9, we then have $S^{-1}D^{\leq n}_{A/B} = D^{\leq n}_{S^{-1}A/B}$. Let $D \in D^{\leq n}_{S^{-1}A/B}$ and $a/s \in S^{-1}A$. Then

$$D(a/s) = D(\tfrac{-}{s})(a)$$

Note that $D(\frac{-}{s}) \in D^{\leq n}_{A/B}(A, S^{-1}A)$ (see Definition 10 and Theorem 7). By Theorem 7 and Theorem 9, we have that

$$D^{\leq n}_{A/B}(A, S^{-1}A) = D^{\leq n}_{S^{-1}A/B}(S^{-1}A, S^{-1}A) = S^{-1}D^{\leq n}_{A/B}$$

If we let $S^{-1}D(\frac{-}{s}) \in D^{\leq n}_{S^{-1}A/B}(S^{-1}A, S^{-1}A)$ be the extension of $D(\frac{-}{s})$, then there exists $t \in S$ such that $t \cdot S^{-1}D(\frac{-}{s}) \in D^{\leq n}_{A/B}$, thus

$$D(a/s) = \frac{1}{t} \cdot t \cdot S^{-1}D(\tfrac{-}{s})(a) \in \tfrac{1}{t}D^{\leq n}_{A/B}I \subset S^{-1}D^{\leq n}_{A/B}I$$

Hence we have the equality $S^{-1}D^{\leq n}_{A/B}I = D^{\leq n}_{S^{-1}A/B}(S^{-1}I)$, as desired. □

4.2 Differential operators and adic completions

Let A be a B-algebra and M an A-module.

Proposition 9 (Differential Operators are Continuous under any Adic Topologies)**.** *Let $D \in D^{\leq n}_{A/B}(A, M)$ and $I \subset A$ an ideal. Then for all $m \geq n$, we have $D(I^m) \subset I^{m-n}M$, hence D is continuous in the I-adic topology of A and of M.*

Proof. We first induct on n. When $n = 0$, we have D is A-linear, so that for all $f_1, \cdots, f_m \in I$, we have

$$D(f_1 \cdots f_m) = f_1 \cdots f_m D(1) \in I^m M$$

So suppose $n > 0$, and that the result is true for $n - 1$. Now we induct on m. When $m = n$, it is automatic that $D(I^m) \subset I^{m-n}M = M$. So suppose $m > n$, and assume the result is true for $m - 1$. Then for $g \in A$ and $f_1, \cdots, f_m \in I$, we have

$$\begin{aligned} D(gf_1 \cdots f_m) &= [D, f_m](gf_1 \cdots f_{m-1}) + f_m D(gf_1 \cdots f_{m-1}) \\ &\in [D, f_m](I^{m-1}) + ID(I^{m-1}) \\ &\subset I^{m-1-(n-1)}M + II^{m-1-n}M \quad \text{by Proposition 3 and inductive hypothesis} \\ &= I^{m-n}M \end{aligned}$$

□

Proposition 10 (Differential Operators extend uniquely under Completion)**.** *Let $D \in D^{\leq n}_{A/B}(A, M)$, $I \subset A$ an ideal, and $\hat{A} = \varprojlim A/I^k$ and $\hat{M} = \varprojlim M/I^k M$ the I-adic completion of A and of M. Then the differential operator D extends uniquely to a differential operator $\hat{D} \in D_{\hat{A}/B}(\hat{A}, \hat{M})$, where by extension we mean the following commutes:*

$$\begin{array}{ccc} A & \xrightarrow{D} & M \\ \downarrow & & \downarrow \\ \hat{A} & \xrightarrow{\hat{D}} & \hat{M} \end{array}$$

and $\hat{D} \in D^{\leq n}_{\hat{A}/B}(\hat{A}, \hat{M})$.

Proof. By Proposition 9, we have that $D(I^m) \subset I^{m-n}M$, hence D induces a B-module map $A/I^m \to A/I^{m-n}$. This induces a B-module map

$$\hat{D} : \varprojlim A/I^k \to \varprojlim M/I^k M$$

given by $\hat{D}((a_k)_k) = (D(a_k))_{k-n} = (D(a_{k+n}))_k$, where $(a_k)_k \in \varprojlim A/I^k \subset \prod_k A/I^k$ and $a_k \in A/I^k$.

It is easy to see that $\hat{D}$ extends D because $\hat{D}((a)_k) = (D(a))_k$ for all $a \in A$. To see that $\hat{D} \in D^{\leq n}_{\hat{A}/B}(\hat{A}, \hat{M})$, we use Proposition 3. Let $(f^1_k)_k, (f^2_k)_k, \cdots, (f^{n+1}_k)_k \in \varprojlim A/I^k$. Note that if $(a_k)_k \in \varprojlim A/I^k$, then $(a_{k+r})_k = (a_k)_k$ for all $r > 0$ in $\varprojlim A/I^k$. So we have

$$\begin{aligned}
\sum_{H\subset[n+1]} (-1)^{|H|}\Big(\prod_{i\in H}(f^i_k)_k\Big)\hat{D}\Big(\prod_{j\notin H}(f^j_k)_k\Big) &= \sum_{H\subset[n+1]} (-1)^{|H|}\Big(\prod_{i\in H}(f^i_k)_k\Big)\hat{D}\Big(\Big(\prod_{j\notin H} f^j_k\Big)_k\Big) \\
&= \sum_{H\subset[n+1]} (-1)^{|H|}\Big(\prod_{i\in H}(f^i_{k+n})_k\Big)\Big(D\Big(\prod_{j\notin H} f^j_{k+n}\Big)\Big)_k \\
&= \Big(\sum_{H\subset[n+1]} (-1)^{|H|}\Big(\prod_{i\in H}(f^i_{k+n})_k\Big)D\Big(\prod_{j\notin H} f^j_{k+n}\Big)\Big)_k \\
&= (0)_k \quad \text{by Proposition 3}
\end{aligned}$$

So by Proposition 3, we have that $\hat{D} \in D^{\leq n}_{\hat{A}/B}(\hat{A}, \hat{M})$. Because differential operators are continuous under any adic topologies by Proposition 9, and since the image of $A \to \hat{A}$ is dense and $\hat{M}$ is Hausdorff in the $\hat{I}$-adic topology, it follows that $\hat{D}$ is the unique differential operator extending D. $\square$

Thus we have an A-linear inclusion $D^{\leq n}_{A/B}(A, M) \hookrightarrow D_{\hat{A}/B}(\hat{A}, \hat{M})$ sending a differential operator to its extension.

We will use the following technical lemma in the next section on Hasse derivatives.

Lemma 4. *Let $I \subset A$ be an ideal. Let $\hat{A}$ and $\hat{M}$ be the I-adic completions of A and M respectively. Then the following map that pulls back along $A \to \hat{A}$ is bijective*

$$D^{\leq n}_{\hat{A}/B}(\hat{A}, \hat{M}) = D^{\leq n}_{A/B}(A, \hat{M})$$

Proof. We first make a general observation on completions. Let $I^k\hat{M}$ be the image under the map $I^k \otimes_A \hat{M} \to \hat{M}$. Thus $\varprojlim \hat{M}/I^k\hat{M}$ is the I-adic completion of $\hat{M}$ as an A-module. Also consider $\widehat{I^kM}$, the I-adic completion of I^kM. Identifying $\hat{M} \subset \prod_k M/I^kM$ as a subset of the product, we can see that $I^m\hat{M} \subset \widehat{I^kM}$ in $\hat{M}$. This induces an A-linear map

$$\varprojlim \hat{M}/I^k\hat{M} \to \varprojlim \hat{M}/\widehat{I^kM} = \hat{M}$$

Observe that the composition

$$M \to \varprojlim \hat{M}/I^k\hat{M} \to \hat{M}$$

identifies with the I-adic completion map $M \to \hat{M}$.

Now observe that the following composition is the identity on $D^{\leq n}_{A/B}(A, \hat{M})$

$$D^{\leq n}_{A/B}(A, \hat{M}) \hookrightarrow D^{\leq n}_{\hat{A}/B}(\hat{A}, \varprojlim \hat{M}/I^k\hat{M}) \to D^{\leq n}_{\hat{A}/B}(\hat{A}, \hat{M}) \to D^{\leq n}_{A/B}(A, \hat{M})$$

where the first map extends a differential operator to the I-adic completion (10), the second map post-composes with the map $\varprojlim \hat{M}/I^k\hat{M} \to \hat{M}$, and the third map pulls back along $A \to \hat{A}$.

Since the image of $A \to \hat{A}$ is dense in $\hat{A}$, and $\hat{A}$ is Hausdorff, it follows by Proposition 9 that the third map in the composition above that pulls back along $A \to \hat{A}$ is injective. Thus the inclusion that pulls back along $A \to \hat{A}$ is has a left inverse, and hence we have equality

$$D^{\leq n}_{\hat{A}/B}(\hat{A}, \hat{A}) = D^{\leq n}_{A/B}(A, \hat{A})$$

□

5 Hasse Derivatives

See Section 1.5 for multi-indexing notation.

5.1 Almost-quasi-regular sequences

We will define the notion of almost-quasi-regular sequences, which is where the general theory of Hasse derivatives begins.

Definition 12 (Almost-quasi-regular)**.** *Let R be a ring and I a proper ideal. A sequence of elements $f_1, \cdots, f_n \in I$ is called an almost-quasi-regular sequence of I if the R/I-algebra map*

$$(R/I)[X_1, \cdots, X_n] \to \bigoplus_{k \geq 0} I^k/I^{k+1}$$
$$X_i \mapsto f_i$$

sending the variable X_i to $f_i \in I/I^2$ in degree one is an isomorphism. In this case we say that I is an almost-quasi-regular ideal.

Lemma 5. *Let R be a ring and $I \subset R$ an almost-quasi-regular ideal. If $g_1, \cdots, g_n \in I$ map to an R/I-basis of I/I^2, then $g_1, \cdots, g_n \in I$ is an almost-quasi-regular sequence of I.*

Proof. Let $f_1, \cdots, f_n \in I$ be an almost-quasi-regular sequence of I, where we know the length of the sequence is n because the rank of free modules is well-defined. Let $\{c_{ij}\}_{ij}$ be the change of basis matrix of I/I^2 as a free R/I-module from $f_1, \cdots, f_n$ to $g_1, \cdots, g_n$, so that $g_i = \sum_{j=1}^n c_{ij} f_j$ in I/I^2. Because $\{c_{ij}\}_{ij} \in \mathrm{GL}_n(R/I)$, it follows that the map

$$\phi : (R/I)[X_1, \cdots, X_n] \to (R/I)[X_1, \cdots, X_n]$$
$$X_i \mapsto \sum_{j=1}^n c_{ij} X_j$$

is an R/I-algebra isomorphism. The composition of R/I-algebra isomorphisms

$$\bigoplus_{k \geq 0} I^k/I^{k+1} \xrightarrow{f_i \mapsto X_i} (R/I)[X_1, \cdots, X_n] \xrightarrow{\phi^{-1}} (R/I)[X_1, \cdots, X_n]$$

sends $g_i \in I/I^2$ in degree one to X_i. □

Let $B \to A$ be a ring map, and let I_Δ be the kernel of the multiplication map $A \otimes_B A \to A$.

Lemma 6. *Suppose that $y_1, \cdots, y_n \in I_\Delta$ is an almost-quasi-regular sequence of I_Δ . Then the module of principal parts $P^m_{A/B}$ is free as a left A-module with basis the monomials in the y_i's with weight up to m, that is, we have the internal direct sum*

$$P^m_{A/B} = \bigoplus_{|s|\le m} A \cdot y^{(s)}$$

where the direct sum is over multi-indices $s \in \mathbb{N}^n$ with weight $|s|$ up to m and $y^{(s)} := y_1^{s_1} \cdots y_n^{s_n}$.

Proof. Split the A-module exact sequence

$$0 \to \frac{I_\Delta}{I_\Delta^{m+1}} \to \frac{A \otimes_B A}{I_\Delta^{m+1}} \to A \to 0$$

with the section $A \to (A \otimes_B A)/I_\Delta^{n+1}$ sending $1 \mapsto 1 \otimes 1 = y^{(0)}$ (weight zero monomial to itself) to obtain the direct sum decomposition

$$\frac{A \otimes_B A}{I_\Delta^{m+1}} = A \oplus \frac{I_\Delta}{I_\Delta^{m+1}}$$

Next, split the A-module exact sequence

$$0 \to \frac{I_\Delta^2}{I_\Delta^{m+1}} \to \frac{I_\Delta}{I_\Delta^{m+1}} \to \frac{I_\Delta}{I_\Delta^2} \to 0$$

with the section $I_\Delta/I_\Delta^2 \to I_\Delta/I_\Delta^{m+1}$ sending the weight one monomial $y_i \to y_i$ to obtain

$$\frac{I_\Delta}{I_\Delta^{m+1}} = \frac{I_\Delta}{I_\Delta^2} \oplus \frac{I_\Delta^2}{I_\Delta^{m+1}}$$

Inductively, split the A-module exact sequence

$$0 \to \frac{I_\Delta^{k+1}}{I_\Delta^{m+1}} \to \frac{I_\Delta^k}{I_\Delta^{m+1}} \to \frac{I_\Delta^k}{I_\Delta^{k+1}} \to 0$$

with the section $I_\Delta^k/I_\Delta^{k+1} \to I_\Delta^k/I_\Delta^{m+1}$ sending the weight k monomials in the y_i's to themselves to obtain

$$\frac{I_\Delta^k}{I_\Delta^{m+1}} = \frac{I_\Delta^k}{I_\Delta^{k+1}} \oplus \frac{I_\Delta^{k+1}}{I_\Delta^{m+1}}$$

Thus we get the isomorphism (note this is not canonical and depends on the choice of almost-quasi-regular sequence)

$$P^m_{A/B} = A \oplus \frac{I_\Delta}{I_\Delta^2} \oplus \frac{I_\Delta^2}{I_\Delta^3} \oplus \cdots \oplus \frac{I_\Delta^m}{I_\Delta^{m+1}}$$

For $0 \le k \le m$, the factor $I_\Delta^k/I_\Delta^{k+1}$ in the above isomorphism maps to $P^m_{A/B}$ by sending the basis monomials of weight k in the y_i's to themselves in $P^m_{A/B}$. $\square$

5.2 Hasse derivatives

Let $B \to A$ be a ring map, and let I_Δ be the kernel of the multiplication map $A \otimes_B A \to A$.

Definition 13 (Hasse derivatives)**.** *Let $x_1, \dots, x_n \in A$ be such that $dx_1, \cdots, dx_n \in I_\Delta$ form an almost-quasi-regular sequence of I_Δ, where $dx_i := 1 \otimes x_i - x_i \otimes 1$. By Lemma 6, $P^m_{A/B}$ is a free A-module with basis the monomials in the dx_i's with weight up to m. For multi-index $s \in \mathbb{N}^n$, introduce the notation*

$$x^{(s)} := x_1^{s_1} \cdots x_n^{s_n} \in A \qquad dx^{(s)} := dx_1^{s_1} \cdots dx_n^{s_n} \in A \otimes_B A$$

so that $dx^{(s)}$ is a monomial with weight $|s|$. Define the collection of Hasse derivatives on the B-algebra A with respect to the (ordered) sequence $x_1, \dots, x_n$ to be the collection of differential operators $\{D^{(s)}\}_{s \in \mathbb{N}^n} \subset D_{A/B}$ (see Definition 8) such that for each $s \in \mathbb{N}^n$, the differential operator $D^{(s)} \in D^{\le |s|}_{A/B}$ corresponds to the coordinate projection on the weight $|s|$ monomial $dx^{(s)} \in P^{|s|}_{A/B}$.

Examples 1. *The Hasse derivatives on the polynomial ring $k[x_1, \dots, x_n]$ over k with respect to the sequence of variables $x_1, \dots, x_n$ are the standard Hasse derivatives, also called Hasse-Dieudonne derivatives, Hasse-Schmidt derivatives, or Hasse-Schmidt derivations (see [22]). In Section 9.3 we will see an example of Hasse derivatives on the germ of a smooth variety at a point with separably generated residue field.*

For the rest of this section, suppose that $x_1, \dots, x_n \in A$ are such that $dx_1, \cdots, dx_n \in I_\Delta$ form an almost-quasi-regular sequence of I_Δ. Let $\{D^{(s)}\}_{s \in \mathbb{N}^n} \subset D_{A/B}$ be the collection of Hasse derivatives on the B-algebra A with respect to the sequence $x_1, \dots, x_n \in A$.

By definition of the Hasse derivatives, for $f \in A$, we have the "Taylor expansion" up to order m:

$$d^m(f) = 1 \otimes f = \sum_{|s| \le m} D^{(s)}(f) dx^{(s)} \in P^m_{A/B}$$

The next proposition follows immediately from definitions.

Proposition 11. *$D^{\le m}_{A/B}$ is a free A-module with basis the Hasse derivatives $D^{(s)}$ with $|s| \le m$. The ring of differential operators $D_{A/B}$ is a free A-module with basis all the Hasse derivatives.*

Proof. By Lemma 6 and the definition of Hasse derivatives, we have

$$\begin{aligned} D^{\le m}_{A/B} &= \mathrm{Hom}_A(P^m_{A/B}, A) \\ &= \mathrm{Hom}_A\left(\bigoplus_{|s| \le m} A \cdot dx^{(s)}, A \right) \\ &= \bigoplus_{|s| \le m} \mathrm{Hom}_A(A \cdot dx^{(s)}, A) \\ &= \bigoplus_{|s| \le m} A \cdot D^{(s)} \end{aligned}$$

□

Before we prove various properties of the Hasse Derivative, we will need the following two lemmas.

Lemma 7. *Let $s, t \in \mathbb{N}^n$. Then*

$$D^{(t)}(x^{(s)}) = \binom{s}{t} x^{(s-t)}$$

Proof. Observe that

$$d^m(x^{(s)}) = \prod_{i=1}^n d^m(x_i^{s_i}) = \prod_{i=1}^n d^m(x_i)^{s_i} = \prod_{i=1}^n (x_i + dx_i)^{s_i}$$

where $x_i + dx_i$ is really $x_i dx^{(0)} + dx_i$. Expanding this product and extracting the coefficient of the $dx^{(t)}$ term, we obtain the result. $\square$

Lemma 8. *The set $\{d^m(x^{(s)})\}_{|s|\leq m} \subset P^m_{A/B}$ generates $P^m_{A/B}$ as a left A-module. Thus if two differential operators in $D^{\leq m}_{A/B}$ agree on $\{x^{(s)}\}_{|s|\leq m} \subset A$, then they must be equal.*

Proof. It suffices to show that the left A-submodule of $P^m_{A/B}$ generated by $\{d^m(x^{(s)})\}_{|s|\leq m}$ contains the basis monomials $dx^{(s)}$ for $|s| \leq m$. To do so, we proceed by induction on s. The base case is $d^m(x^{(0)}) = d^m(1) = 1 \otimes 1 = dx^{(0)}$. Now suppose $|s| > 0$ and that all monomials in the dx_i's with weight less than $|s|$ are in the left A-submodule generated by $\{d^m(x^{(s)})\}_{|s|\leq m}$. Note that

$$d^m(x^{(s)}) = \prod_{i=1}^n (x_i + dx_i)^{s_i}$$

We see that the only monomial that appears in $d^m(x^{(s)})$ with weight at least $|s|$ is $dx^{(s)}$, and its leading coefficient is one. By the inductive hypothesis, it follows that $dx^{(s)}$ can be written as an A-linear combination of the $\{d^m(x^{(s)})\}_{|s|\leq m}$, hence we are done. $\square$

Proposition 12 (See [5])**.**

(i) (Leibniz Rule) For $f_1, \ldots, f_m \in A$, we have that

$$D^{(s)}\left(\prod_{i=1}^m f_i\right) = \sum_{s^1+\cdots+s^m=s} \prod_{i=1}^m D^{(s^i)}(f_i)$$

In particular, for $f, g \in A$, we have

$$D^{(s)}(fg) = \sum_{s'+s''=s} D^{(s')}(f) D^{(s'')}(g)$$

(ii) (Hasse composition) Let $s, t \in \mathbb{N}^n$. Then

$$D^{(s)} \circ D^{(t)} = \binom{s+t}{s} D^{(s+t)}$$

In particular, $D^{(s)} \circ D^{(t)} = D^{(t)} \circ D^{(s)}$.

Proof.

(i) Take the coefficient of the $dx^{(s)}$ term of the following

$$d^{|s|}\left(\prod_{i=1}^m f_i\right) = \prod_{i=1}^m d^{|s|}(f_i) = \prod_{i=1}^m \sum_{|s^i|\leq|s|} D^{(s^i)}(f_i) dx^{(s^i)}$$

to obtain the result. $\square$

(ii) Note that both hand sides are differential operators of order $|s+t|$. Thus it suffices to check that they agree on all monomials $x^{(r)}$. Observe that

$$D^{(s)} \circ D^{(t)}(x^{(r)}) = D^{(s)}(\binom{r}{t} x^{(r-t)}) = \binom{r}{t}\binom{r-t}{s} x^{(r-t-s)}$$

and

$$\binom{s+t}{s} D^{(s+t)}(x^{(r)}) = \binom{s+t}{s}\binom{r}{s+t} x^{(r-t-s)}$$

It is easy to verify that

$$\binom{r}{t}\binom{r-t}{s} = \binom{s+t}{s}\binom{r}{s+t}$$

Thus we have the equality of differential operators. □

Let $e_i = (\ldots, 0, 1, 0, \ldots) \in \mathbb{N}^n$ be the ith coordinate vector. Let $me_i = (\ldots, 0, m, 0, \ldots) \in \mathbb{N}^n$ be the ith coordinate vector multiplied by $m \in \mathbb{N}$.

Definition 14 (Partial derivatives)**.** *For each $i = 1, \ldots, n$, let $\frac{\partial}{\partial x_i} := D^{(e_i)}$, and for $s_1, \ldots, s_n \in \mathbb{N}$, let*

$$\frac{\partial^{s_1+\cdots+s_n}}{\partial x_1^{s_1} \cdots \partial x_n^{s_n}} := \prod_{i=1}^{n} \left(\frac{\partial}{\partial x_i}\right)^{s_i}$$

where the product and power are taken using the associative algebra structure on the ring of differential operators $D_{A/B}$, that is, they are just composition.

Note that $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ form a basis for the A-module of B-derivations $\mathrm{Der}_B(A, A)$, where $\frac{\partial}{\partial x_i}$ corresponds to the coordinate projection on $dx_i \in \Omega_{A/B} = I_\Delta/I_\Delta^2$.

These partial derivatives are nothing but the familiar partial derivatives from calculus. The following corollary shows that the Hasse derivatives in characteristic zero are also familiar to us from calculus.

Corollary 4. *The ring of differential operators $D_{A/B}$ is generated as a left A-module (see Definition 8) by all possible products (that is, composition) of differential operators in the collection $\{D^{(me_i)}\}_{m\in\mathbb{N}, 1\le i\le n}$. If A has characteristic zero, then*

$$D^{(s)} = \frac{1}{s_1! \cdots s_n!} \frac{\partial^{s_1+\cdots+s_n}}{\partial x_1^{s_1} \cdots \partial x_n^{s_n}}$$

so that $D_{A/B}$ is generated as a left A-module by all possible products of the partial derivatives $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$, hence $D_{A/B}^{\le 1} D_{A/B}^{\le m} I = D_{A/B}^{\le m+1} I$ for all ideals $I \subset A$.

Proof. Corollary of Proposition 12 (ii). □

5.3 Hasse derivatives and adic completions

In this section we will see that Hasse derivatives behave well when extended to adic completions.

Just as in the previous section, let A be a B-algebra, I_Δ the kernel of the multiplication map $A \otimes_B A \to A$, and $x_1, \ldots, x_n \in A$ such that the sequence $dx_1, \ldots, dx_n \in I_\Delta$ is an almost-quasi-regular sequence of I_Δ. Let $\{D^{(s)}\}_{s\in\mathbb{N}^n}$ be the collection of Hasse derivatives on the B-algebra A associated to $x_1, \ldots, x_n \in A$.

Let $I \subset A$ be an ideal, and let $\hat{A}$ be the I-adic completion of A. Note that if $I = A$, then the I-adic topology on A is trivial, hence by [18, Proposition 1.3.2], $\hat{A} = 0$. Conversely, if $\hat{A} = 0$, then because the composition $A \to \hat{A} \to A/I$ of the completion map is the quotient map $A \to A/I$, we have $I = A$. Thus we will require $I \subset A$ to be proper since we want to rule out $\hat{A} = 0$.

Definition 15. *Define $\{\hat{D}^{(s)}\}_{s\in\mathbb{N}^n} \subset D_{\hat{A}/B}$ to be the collection of Hasse derivatives on the I-adic completion of the B-algebra A with respect to the (ordered) sequence $x_1, \ldots, x_n \in A$, where $\hat{D}^{(s)} \in D_{\hat{A}/B}$ is the unique extension of the Hasse derivative $D^{(s)} \in D_{A/B}$ (see Proposition 10).*

Proposition 13. *$D^{\leq m}_{\hat{A}/B}$ is a free $\hat{A}$-module with basis the Hasse derivatives $\hat{D}^{(s)}$ for $|s| \leq m$. Thus $\hat{D}^{(s)}$ is a differential operator of order $|s|$ and the ring of differential operators $D_{\hat{A}/B}$ is a free $\hat{A}$-module with basis all the Hasse derivatives.*

Proof. By Lemma 4, we have $\hat{A}$-module isomorphisms

$$D^{\leq m}_{\hat{A}/B}(\hat{A}, \hat{A}) = D^{\leq m}_{A/B}(A, \hat{A}) = \mathrm{Hom}_A(P^m_{A/B}, \hat{A})$$

where we give $D^{\leq m}_{\hat{A}/B}(\hat{A}, \hat{A})$, $D^{\leq m}_{A/B}(A, \hat{A})$, and $\mathrm{Hom}_A(P^m_{A/B}, \hat{A})$ all the structure of an $\hat{A}$-module by post-multiplication. Note that $\mathrm{Hom}_A(P^m_{A/B}, \hat{A})$ is a free $\hat{A}$-module with basis the coordinate projections $P^m_{A/B} \to \hat{A}$ that factor through the coordinate projections $P^m_{A/B} \to A$ associated to the Hasse derivatives, where we use that $1 \neq 0$ in $\hat{A}$, since otherwise all the coordinate projections to $\hat{A}$ would be the same. Observe that if $P^m_{A/B} \to A$ is associated to $D^{(s)}$, then $P^m_{A/B} \to A \to \hat{A}$ induces $\hat{D}^{(s)}$. Thus we are done. □

Example 1. *The previous proposition characterizes the ring of differential operators on formal power series $k[[x_1, \ldots, x_n]]$ over k in terms of the Hasse derivatives on the $(x_1, \ldots, x_n)$-adic completion of the polynomial algebra $k[x_1, \ldots, x_n]$ over k. Note that as a corollary, we have that the standard partial derivatives $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ form a $k[x_1, \ldots, x_n]$-basis for $\mathrm{Der}_k(k[[x_1, \ldots, x_n]], k[[x_1, \ldots, x_n]])$ ([20, Theorem 1.5.2]).*

Corollary 5. *If two differential operators in $D^{\leq m}_{\hat{A}/B}$ agree on $\{x^{(s)}\}_{|s|\leq m} \subset \hat{A}$, then they must be equal.*

Proof. Express the differential operators as $\hat{A}$-linear combinations of Hasse derivatives, then equate coefficients by evaluating at the monomials $\{x^{(s)}\}_{|s|\leq m}$. □

Proposition 14 (See [5])**.**

(i) (Leibniz Rule) Let $f_1, \cdots, f_m \in \hat{A}$. Then

$$\hat{D}^{(s)}\left(\prod_{j=1}^{m} f_j\right) = \sum_{s^1+\cdots+s^m=s} \prod_{j=1}^{m} \hat{D}^{(s^j)}(f_j)$$

(ii) (Hasse composition) Let $s, t \in \mathbb{N}^n$. Then

$$\hat{D}^{(s)} \circ \hat{D}^{(t)} = \binom{s+t}{s} \hat{D}^{(s+t)}$$

In particular, $\hat{D}^{(s)} \circ \hat{D}^{(t)} = \hat{D}^{(t)} \circ \hat{D}^{(s)}$.

Proof. (i) can easily be seen by the proof of extending differential operators to the completion. (ii) involve equating two differential operators, so one just verifies that the equality is true on the dense image of A in $\hat{A}$, and then use that $\hat{A}$ is Hausdorff and differential operators are continuous. $\square$

5.4 Algebraic independence of parameters

Let A be a B-algebra, I_Δ the kernel of the multiplication map $A \otimes_B A \to A$, and $x_1, \ldots, x_n \in A$ such that $dx_1, \ldots, dx_n \in I_\Delta$ is an almost-quasi-regular sequence of I_Δ. Let $I \subset A$ be a proper ideal and $\hat{A}$ the I-adic completion of A.

Proposition 15. *The elements $x_1, \cdots, x_n \in A$ are algebraically independent over B, that is, the subring $B[x_1, \cdots, x_n] \subset A$ is isomorphic as a B-algebra to the polynomial ring $B[X_1, \cdots, X_n]$, where $X_i \mapsto x_i$. Similarly, the image of $B[x_1, \cdots, x_n] \subset A \to \hat{A}$ in $\hat{A}$ is isomorphic as a B-algebra to the polynomial ring $B[X_1, \cdots, X_n]$, where $X_i \mapsto x_i$.*

Proof. Suppose $f = \sum b_s x^{(s)} = 0$, where $b_s \in B$. Let $t \in \mathbb{N}^n$ be such that $b_s = 0$ for all $|s| > |t|$. Then $D^{(s)}(f) = b_t$ and $\hat{D}^{(s)}(f) = b_t$, hence $b_t = 0$, and $f = 0$. $\square$

Thus we have the following diagrams of inclusions

$$\begin{array}{ccc} B[X] & \xrightarrow{d^m} & P^m_{B[X]/B} \\ \big\downarrow & & \big\downarrow \\ A & \xrightarrow{d^m} & P^m_{A/B} \end{array} \qquad \begin{array}{ccc} B[X] & \xrightarrow{d^m} & P^m_{B[X]/B} \\ \big\downarrow & & \big\downarrow \\ \hat{A} & \xrightarrow{d^m} & P^m_{A/B} \end{array}$$

where $B[X] = B[X_1, \ldots, X_n]$.

6 Sheaf of Principal Parts and Differential Operators

6.1 Conormal invariants of locally closed embeddings

Definition 16 (See [12, Definition 16.1.2])**.** *Let $f : Z \hookrightarrow X$ be a locally closed embedding of schemes. Let $\mathcal{I}_f$ be the kernel of the surjection $f^{-1}\mathcal{O}_X \twoheadrightarrow \mathcal{O}_Z$. Define the nth conormal invariant of f to be*

$$\mathcal{C}_n(f) := \mathcal{I}_f^n / \mathcal{I}_f^{n+1}$$

which has the structure of an $f^{-1}\mathcal{O}_X/\mathcal{I}_f = \mathcal{O}_Z$-module.

Observe that

$$\mathcal{C}_n(f) = \mathcal{I}_f^n / \mathcal{I}_f^{n+1} = \mathcal{I}_f \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{O}_X/\mathcal{I}_f = \mathcal{I}_f \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_Z$$

is an isomorphism of $\mathcal{O}_Z$-modules.

Lemma 9 (The conormal invariants are quasicoherent, see [12, Proposition 16.1.5(i)])**.** *The nth conormal invariant $\mathcal{C}_n(f)$ of a locally closed embedding $f : Z \hookrightarrow X$ is a quasicoherent $\mathcal{O}_Z$-module.*

Proof. Factor the locally closed embedding f as

$$Z \overset{j}{\hookrightarrow} U \overset{i}{\hookrightarrow} X$$

where j is a closed embedding and i an open embedding. Let $\mathcal{I}_U$ be the kernel of the surjection $\mathcal{O}_U \twoheadrightarrow j_*\mathcal{O}_Z$. Since inverse image is an exact functor, $\mathcal{I}_j = j^{-1}\mathcal{I}_U$ is the kernel of the surjection $j^{-1}\mathcal{O}_U \twoheadrightarrow \mathcal{O}_Z$. Observe that

$$f^{-1}\mathcal{O}_X = (i \circ j)^{-1}\mathcal{O}_X = j^{-1}i^{-1}\mathcal{O}_X = j^{-1}\mathcal{O}_U$$

Under this identification, we have $j^{-1}\mathcal{I}_U = \mathcal{I}_f$. Thus it suffices to show that the $\mathcal{C}_n(j)$ of the closed embedding $j : Z \hookrightarrow U$ is a quasicoherent $\mathcal{O}_Z$-module. Observe that

$$\begin{aligned}
\mathcal{C}_n(j) &= \mathcal{I}_j^n \otimes_{j^{-1}\mathcal{O}_X} \mathcal{O}_Z \\
&= (j^{-1}\mathcal{I}_U)^n \otimes_{j^{-1}\mathcal{O}_X} \mathcal{O}_Z \\
&= j^{-1}(\mathcal{I}_U^n) \otimes_{j^{-1}\mathcal{O}_X} \mathcal{O}_Z \\
&= j^*(\mathcal{I}_U^n)
\end{aligned}$$

Because $\mathcal{I}_U^n$ is a quasicoherent $\mathcal{O}_U$-module and pullback preserves quasicoherence, we have that $\mathcal{C}_n(f) = \mathcal{C}_n(j) = j^*(\mathcal{I}_U^n)$ is a quasicoherent $\mathcal{O}_Z$-module. □

Proposition 16 (Computing the conormal invariants affine-locally)**.** *Let $I \subset A$ be an ideal. Then the nth conormal invariant of the closed embedding $\operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A$ is the sheaf associated to the A/I-module I^n/I^{n+1}. Let $f : Z \hookrightarrow X$ be a locally closed embedding. Suppose we have the following pullback diagram:*

$$\begin{tikzcd}
\operatorname{Spec} A/I \arrow[r, hook, "c.e."] \arrow[d, hook, "o.e."] & \operatorname{Spec} A \arrow[d, hook, "o.e."] \\
Z \arrow[r, hook, "c.e."] & X
\end{tikzcd}$$

where c.e. and o.e. stand for closed and open embedding, respectively. Then $C_n(f)|_{\operatorname{Spec} A/I}$ is the sheaf associated to the A/I-module I^n/I^{n+1}.

Proof. Let $j : \operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A$ denote the closed embedding. The kernel of $\mathcal{O}_{\operatorname{Spec} A} \twoheadrightarrow j_*\mathcal{O}_{\operatorname{Spec} A/I}$ is the sheaf $\widetilde{I}$ associated to the A-module I. By the proof of Lemma 9, we then have

$$C_n(j)(\operatorname{Spec} A/I) = j^*(\widetilde{I}^n)(\operatorname{Spec} A/I) = I^n \otimes_A A/I = I^n/I^{n+1}$$

Let $i : \operatorname{Spec} A/I \hookrightarrow Z$ be the open embedding. Because inverse image is an exact functor, we have that $\mathcal{I}_{f\circ i} = i^{-1}\mathcal{I}_f$ is the kernel of $(f \circ i)^{-1}\mathcal{O}_X \to \mathcal{O}_{\operatorname{Spec} A/I}$. Thus

Then

$$\begin{aligned}
C_n(f)|_{\operatorname{Spec} A/I} &= (\mathcal{I}_f \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_Z)|_{\operatorname{Spec} A/I} \\
&= i^{-1}\mathcal{I}_f \otimes_{i^{-1}f^{-1}\mathcal{O}_X} i^{-1}\mathcal{O}_Z \\
&= \mathcal{I}_{f\circ i} \otimes_{(f\circ i)^{-1}\mathcal{O}_X} \mathcal{O}_{\operatorname{Spec} A/I} \\
&= C_n(f \circ i) \\
&= C_n(j) \quad \text{by proof of Lemma 9}
\end{aligned}$$

Thus $C_n(f)|_{\operatorname{Spec} A/I}$ is the sheaf associated to the A/I-module I^n/I^{n+1}. □

Remark 2. *Let $X \to Y$ be a morphism of schemes. Let $\Delta : X \hookrightarrow X \times_Y X$ be the diagonal embedding. Then the $\mathcal{O}_X$-module of Kähler differentials $\Omega_{X/Y}$ is the first conormal invariant of the diagonal $C_1(\Delta)$.*

6.2 Sheaf of Principal Parts

Let $X \xrightarrow{f} S$ be a scheme over S. Denoting the diagonal embedding $X \overset{\Delta}{\hookrightarrow} X \times_S X$, let $\mathcal{I}_\Delta$ be the kernel of the surjection $\mathcal{O}_{X \times_S X} \twoheadrightarrow \Delta_* \mathcal{O}_X$.

Definition 17 (Sheaf of Principal Parts, see [12, Definitions 16.1.2, 16.3.1, 16.3.6])**.** *Define the sheaf $\mathcal{P}^n_{X/S}$ of principal parts of order n to be*

$$\mathcal{P}^n_{X/S} := \frac{\Delta^{-1}\mathcal{O}_{X \times_S X}}{\Delta^{-1}\mathcal{I}_\Delta^{n+1}}$$

Let π_1 and π_2 be the first and second projection morphisms $X \times_S X \to X$, respectively. Then let ι and d^n be the morphisms $\mathcal{O}_X \to \mathcal{P}^n_{X/S}$ induced by π_1 and π_2 (apply Δ^{-1} to $\pi_i^{-1}\mathcal{O}_X \to \mathcal{O}_{X \times_S X}$), respectively. Observe that the following commutes

$$\begin{tikzcd}
f^{-1}\mathcal{O}_S \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[d, "d^n"] \\
\mathcal{O}_X \arrow[r, "\iota"] & \mathcal{P}^n_{X/S}
\end{tikzcd}$$

which endows $\mathcal{P}^n_{X/S}$ the structure of an $\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_S} \mathcal{O}_X$-algebra. We give $\mathcal{P}^n_{X/S}$ the structure of a left $\mathcal{O}_X$-module by the morphism ι induced by the first projection.

We give another description of the sheaf of principal parts.

Proposition 17. *Suppose that $f : X \to S$ is separated. Then we have the following natural isomorphism of $\mathcal{O}_X$-modules*

$$\mathcal{P}^n_{X/S} = (\pi_1)_* \left(\frac{\mathcal{O}_{X \times_S X}}{\mathcal{I}_\Delta^{n+1}} \right)$$

Proof. Consider the natural morphism from the adjunction pair (Δ^{-1}, Δ_*)

$$\eta : \mathcal{O}_{X \times_S X}/\mathcal{I}_\Delta^{n+1} \to \Delta_*\Delta^{-1}(\mathcal{O}_{X \times_S X}/\mathcal{I}_\Delta^{n+1})$$

We claim that this is an isomorphism of sheaves of rings. First observe that both the source and the target of η are supported on $V(\mathcal{I}_\Delta^{n+1}) = V(\mathcal{I}_\Delta) = \Delta(X)$. This is clear for the source. For the target, note that $\Delta_*\Delta^{-1}\mathcal{O}_{X \times_S X}/\mathcal{I}_\Delta^{n+1}$ is the extension by zero of $\Delta^{-1}\mathcal{O}_{X \times_S X}/\mathcal{I}_\Delta^{n+1}$ (see [15, Exercise 1.19]), hence the target is supported on $\Delta(X)$ as well. So we only have to check the isomorphism on $\Delta(X)$. The induced stalk maps of η on points of $\Delta(X)$ are isomorphisms because Δ is a closed embedding.

Now pushforward the isomorphism η by π_1

$$(\pi_1)_* \left(\frac{\mathcal{O}_{X \times_S X}}{\mathcal{I}_\Delta^{n+1}} \right) = (\pi_1)_*\Delta_*\Delta^{-1}\mathcal{O}_{X \times_S X}/\mathcal{I}_\Delta^{n+1} = \mathcal{P}^n_{X/S}$$

where the second equality follows because the diagonal Δ is a section of the projection π_1. $\square$

We observe that $\mathcal{O}_X$-module structure on the nth conormal invariant $\mathcal{C}_n(\Delta)$ of the diagonal embedding can also be obtained in an alternative way. A priori, before quotienting out by annihilators, $\mathcal{C}_n(\Delta)$ is an $\Delta^{-1}\mathcal{O}_{X \otimes_S X}$-module. We can restrict scalars to the left factor by the morphism $\iota : \mathcal{O}_X \to \Delta^{-1}\mathcal{O}_{X \otimes_S X}$ induced by the first projection. This coincides with the original $\mathcal{O}_X$-module structure on $\mathcal{C}_n(\Delta)$ because ι is a section of the surjection $\Delta^{-1}\mathcal{O}_{X \otimes_S X} \to \mathcal{O}_X$.

Proposition 18 (See [12, 16.3.2])**.** *$\mathcal{P}^n_{X/S}$ is a quasicoherent $\mathcal{O}_X$-module.*

Proof. We proceed by induction on n. The base case $n = 0$ is immediate because $\mathcal{P}^0_{X/S} = \mathcal{O}_X$ as an $\mathcal{O}_X$-module. So suppose $n > 0$ and the proposition true for $n-1$.

Consider the natural short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{C}_n(\Delta) \to \mathcal{P}^n_{X/S} \to \mathcal{P}^{n-1}_{X/S} \to 0$$

Since $\mathcal{C}_n(\Delta)$ is quasicoherent by 9 and $\mathcal{P}^{n-1}_{X/S}$ is quasicoherent by hypothesis, it follows that $\mathcal{P}^n_{X/S}$ is quasicoherent as well. $\square$

Now that we know the sheaf of principal parts is quasicoherent, we are in position to compute it affine-locally.

Proposition 19. *Let A be an B-algebra. Then $\mathcal{P}^n_{\operatorname{Spec} A/\operatorname{Spec} B}$ as a left $\mathcal{O}_{\operatorname{Spec} A}$-module is the sheaf associated to the left A-module $P^n_{A/B}$.*

Proof. Let $\Delta : \operatorname{Spec} A \hookrightarrow \operatorname{Spec} A \otimes_B A$ be the diagonal embedding and I_Δ be the kernel of the multiplication map $A \otimes_B A \to A$. Then the sheaf $\widetilde{I}_\Delta$ associated to the $A \otimes_B A$-module I_Δ is the kernel of $\mathcal{O}_{\operatorname{Spec} A \otimes_B A} \to \Delta_* \mathcal{O}_{\operatorname{Spec} A}$. Let $\pi_1 : \operatorname{Spec} A \otimes_B A \to \operatorname{Spec} A$ be the projection onto the first factor. Note that the projection π_1 is surjective. Since morphisms of affine schemes are separated, we have by Proposition 17 that

$$\begin{aligned}
\mathcal{P}^n_{\operatorname{Spec} A/\operatorname{Spec} B}(\operatorname{Spec} A) &= (\pi_1)_* \left(\frac{\mathcal{O}_{\operatorname{Spec} A \otimes_B A}}{\widetilde{I}^{n+1}_\Delta} \right) (\operatorname{Spec} A) \\
&= \left(\frac{\mathcal{O}_{\operatorname{Spec} A \otimes_B A}}{\widetilde{I}^{n+1}_\Delta} \right) (\operatorname{Spec} A \otimes_B A) \\
&= \frac{A \otimes_B A}{I^{n+1}_\Delta} \\
&= P^n_{A/B}
\end{aligned}$$

Since $\mathcal{P}^n_{\operatorname{Spec} A/\operatorname{Spec} B}$ is quasicoherent by Proposition 18, we are done. $\square$

Proposition 20 (Computing $\mathcal{P}^n_{X/S}$ affine-locally)**.** *Let* $\operatorname{Spec} A \hookrightarrow X$ *be an open affine mapping into the open affine* $\operatorname{Spec} B \hookrightarrow S$*. Then* $\mathcal{P}^n_{X/S}|_{\operatorname{Spec} A} = \mathcal{P}^n_{\operatorname{Spec} A/\operatorname{Spec} B}$

Proof. Consider the pullback diagram

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[r, hook, "\Delta_A"] \arrow[d, hook, "i"] & \operatorname{Spec} A \otimes_B A \arrow[d, hook, "j"] \\
X \arrow[r, hook, "\Delta_X"] & X \times_S X
\end{tikzcd}$$

Let I_Δ be the kernel of the multiplication map $A \otimes_B A \to A$ and $\widetilde{I}_\Delta$ the sheaf associated to the $(A \otimes_B A)$-module I_Δ. Then we have

$$\begin{aligned}
\mathcal{P}^n_{X/S}|_{\operatorname{Spec} A} &= i^{-1} \Delta_X^{-1} (\mathcal{O}_{X \times_S X} / \mathcal{I}^{n+1}_\Delta) \\
&= \Delta_A^{-1} j^{-1} (\mathcal{O}_{X \times_S X} / \mathcal{I}^{n+1}_\Delta) \\
&= \Delta_A^{-1} (\mathcal{O}_{\operatorname{Spec} A \otimes_B A} / \tilde{I}^{n+1}_\Delta) \\
&= \mathcal{P}^n_{\operatorname{Spec} A/\operatorname{Spec} B}
\end{aligned}$$

$\square$

Corollary 6 (See [12, Proposition 16.3.8]). *The image of the right $\mathcal{O}_X$-module homomorphism $d^n : \mathcal{O}_X \to \mathcal{P}^n_{X/S}$ generates $\mathcal{P}^n_{X/S}$ as a left $\mathcal{O}_X$-module.*

Proof. This is true over every affine of X mapping to an affine of S, from which the result follows. $\square$

6.3 Sheaf of Differential Operators

Let $X \xrightarrow{f} S$ be a scheme over S. Because the image of $d^\infty : \mathcal{O}_X \to \mathcal{P}^n_{X/S}$ generates $\mathcal{P}^n_{X/S}$ as a left $\mathcal{O}_X$-module, we have the inclusion

$$\mathscr{H}om_{\mathcal{O}_X}(\mathcal{P}^n_{X/S}, \mathcal{O}_X) \xhookrightarrow{\circ d^n} \mathscr{H}om_{f^{-1}\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_X)$$
$$\phi \mapsto \phi \circ d^\infty$$

where we give $\mathscr{H}om_{f^{-1}\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_X)$ its left $\mathcal{O}_X$-module structure by post-multiplication, i.e. post-composing with morphisms in $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) = \mathcal{O}_X$.

Definition 18 (Sheaf of Differential Operators, see [12, Definition 16.8.1]). *Define the $\mathcal{O}_X$-module $\mathcal{D}^{\leq n}_{X/S}$ of differential operators of order up to n to be the image under the inclusion of $\mathcal{O}_X$-modules*

$$\mathscr{H}om_{\mathcal{O}_X}(\mathcal{P}^n_{X/S}, \mathcal{O}_X) \hookrightarrow \mathscr{H}om_{f^{-1}\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_X)$$

Observe we have natural inclusions $\mathcal{D}^{\leq 0}_{A/B} \subset \mathcal{D}^{\leq 1}_{A/B} \subset \mathcal{D}^{\leq 2}_{A/B} \subset \cdots$ dual to the natural surjections $\mathcal{P}^0_{A/B} \twoheadleftarrow \mathcal{P}^1_{A/B} \twoheadleftarrow \mathcal{P}^2_{A/B} \twoheadleftarrow \cdots$.

Definition 19 (Derivatives of an ideal, see [14, Definition 3.73]). *Let $\mathcal{I} \subset \mathcal{O}_X$ be an ideal. Define $\mathcal{D}^{\leq n}_{X/S}\mathcal{I}$ to be the sheaf of ideals associated to the presheaf ideal $\mathcal{D}^{\leq n}_{pre}\mathcal{I}$ given by*

$$(\mathcal{D}^{\leq n}_{pre}\mathcal{I})(U) := \textit{ideal generated by } \operatorname{im}\left(\mathcal{D}^{\leq n}_{X/S}(U) \times \mathcal{I}(U) \xrightarrow{eval} \mathcal{O}_X(U)\right)$$

In plain english, $\mathcal{D}^{\leq n}_{X/S}\mathcal{I}$ is the ideal of $\mathcal{O}_X$ generated by the images of local sections under differential operators of order up to n.

Proposition 21 (See [12, Proposition 16.8.6]). *Let X/S be locally of finite presentation. Then $\mathcal{P}^n_{X/S}$ is an $\mathcal{O}_X$-module of finite presentation, implying that $\mathcal{D}^{\leq n}_{X/S}$ is quasicoherent. If we take $\mathcal{I} \subset \mathcal{O}_X$ to be quasicoherent, then $\mathcal{D}^{\leq n}_{X/S}\mathcal{I}$ is quasicoherent as well. If $X = \operatorname{Spec} A$ and $S = \operatorname{Spec} S$, then $\mathcal{D}^{\leq n}_{X/S}(\operatorname{Spec} A) = D^{\leq n}_{A/B}$ and $(\mathcal{D}^{\leq n}_{X/S}\mathcal{I})(\operatorname{Spec} A) = D^{\leq n}_{A/B}I$, where $I = \mathcal{I}(\operatorname{Spec} A)$.*

Proof. Follows from Proposition 5 and Theorem 10. $\square$

Remark 3. *Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Then by the adjunction (f^{-1}, f_*) and taking global sections, we have that*

$$\begin{aligned}\mathscr{H}om_{f^{-1}\mathcal{O}_{\operatorname{Spec} B}}(\mathcal{O}_{\operatorname{Spec} A}, \mathcal{O}_{\operatorname{Spec} A})(\operatorname{Spec} A) &= \operatorname{Hom}_{f^{-1}\mathcal{O}_{\operatorname{Spec} B}}(\mathcal{O}_{\operatorname{Spec} A}, \mathcal{O}_{\operatorname{Spec} A}) \\ &= \operatorname{Hom}_{\mathcal{O}_{\operatorname{Spec} B}}(f_*\mathcal{O}_{\operatorname{Spec} A}, f_*\mathcal{O}_{\operatorname{Spec} A}) \\ &= \operatorname{Hom}_B(A, A)\end{aligned}$$

The only thing to think about is given a B-linear map $A \to A$, constructing an $\mathcal{O}_{\operatorname{Spec} B}$-linear morphism $f_*\mathcal{O}_{\operatorname{Spec} A} \to f_*\mathcal{O}_{\operatorname{Spec} A}$, but this is easy because $f_*\mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} B[g^{-1}]) = \operatorname{Spec} A[g^{-1}]$ for $g \in B$.

This observation ties up the local and global theory of differential operators, since the global sections of the inclusion $\mathcal{D}^{\leq n}_{X/S} \subset \mathscr{H}om_{f^{-1}\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ is the inclusion $D^{\leq n}_{A/B} \subset \operatorname{Hom}_B(A, A)$ when X/S is locally of finite presentation.

7 Differential Operators on Smooth Varieties

7.1 General theory for smooth morphisms

Lemma 10. *Let $f : X \to S$ be a smooth morphism of schemes and $p \in X$. Then the kernel of the multiplication map $\mathcal{O}_{X,p} \otimes_{\mathcal{O}_{S,f(p)}} \mathcal{O}_{X,p} \to \mathcal{O}_{X,p}$ is almost-quasi-regular (see definition 12).*

Proof. Working affine-locally, we may replace X by $\operatorname{Spec} A$ and Y by $\operatorname{Spec} B$, and let $\phi : B \to A$ be the ring map associated to $f : X \to S$. Let $p = \mathfrak{p} \in \operatorname{Spec} A$. Because

$$\mathcal{O}_{X,p} \otimes_{\mathcal{O}_{S,f(p)}} \mathcal{O}_{X,p} = A_\mathfrak{p} \otimes_{B_{\phi^{-1}(\mathfrak{p})}} A_\mathfrak{p} = A_\mathfrak{p} \otimes_B A_\mathfrak{p}$$

it will suffice to show that the kernel of the multiplication map $A_\mathfrak{p} \otimes_B A_\mathfrak{p} \to A_\mathfrak{p}$ is almost-quasi-regular.

Let I be the kernel of the multiplication map $m : A \otimes_B A \to A$. Because the diagonal map $\Delta : \operatorname{Spec} A \to A \otimes_B A$ is a section of a smooth morphism, in particular the coordinate projections $\operatorname{Spec} A \times_{\operatorname{Spec} B} \operatorname{Spec} A \to \operatorname{Spec} A$, by [26, Tag 067R] it follows that Δ is a regular closed embedding. Thus the kernel $I_{m^{-1}(\mathfrak{p})}$ of the localized multiplication map $(A \otimes_B A)_{m^{-1}(\mathfrak{p})} \to A_\mathfrak{p}$ is generated by a regular sequence.

Let $S = \{s \otimes t \in A \otimes_B A | s, t \in A - \mathfrak{p}\}$, which is a multiplicative set in $A \otimes_B A$. It is easy to verify that

$$\begin{gathered} S^{-1}(A \otimes_B A) \to A_\mathfrak{p} \otimes_B A_\mathfrak{p} \\ \frac{f \otimes g}{s \otimes t} \mapsto \frac{f}{s} \otimes \frac{g}{t} \end{gathered}$$

is an isomorphism of rings. Since the image of S in A under the multiplication map is $A - \mathfrak{p}$, it follows that $S^{-1}A = A_\mathfrak{p}$. So $S^{-1}I$ is the kernel of the multiplication map $A_\mathfrak{p} \otimes_B A_\mathfrak{p} \to A_\mathfrak{p}$. We have a map of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & S^{-1}I \arrow[r] \arrow[d] & A_\mathfrak{p} \otimes_B A_\mathfrak{p} \arrow[r] \arrow[d, "\iota \otimes d^\infty"] & A_\mathfrak{p} \arrow[r] \arrow[d, "id"] & 0 \\ 0 \arrow[r] & I_{m^{-1}(\mathfrak{p})} \arrow[r] & (A \otimes_B A)_{m^{-1}(\mathfrak{p})} \arrow[r] & A_\mathfrak{p} \arrow[r] & 0 \end{tikzcd}$$

where $\iota, d^\infty : A_\mathfrak{p} \to (A \otimes_B A)_{m^{-1}(\mathfrak{p})}$ are given by $\iota(a/s) = (a \otimes 1)/(s \otimes 1)$ and $d^\infty(a/s) = (1 \otimes a)/(1 \otimes s)$.

Because $S \subset (A \otimes_B A) - m^{-1}(\mathfrak{p})$, the induced map

$$S^{-1}(I^k/I^{k+1}) \to (I^k/I^{k+1})_{m^{-1}(\mathfrak{p})}$$

is further localization for every k. But $I \subset A \otimes_B A$ annihilates I^k/I^{k+1}, and the images of S and $(A \otimes_B A) - m^{-1}(\mathfrak{p})$ in $(A \otimes_B A)/I = A$ are both equal to $A - \mathfrak{p}$ and in particular coincide, which means that further localization does not do anything. Thus

$$S^{-1}(I^k/I^{k+1}) \to (I^k/I^{k+1})_{m^{-1}(\mathfrak{p})}$$

is an isomorphism of $A_\mathfrak{p}$-modules. This means that the induced map of graded $A_\mathfrak{p}$-algebras

$$\bigoplus_{k\geq 0} S^{-1}I^k/S^{-1}I^{k+1} \to \bigoplus_{k\geq 0} I^k_{m^{-1}(\mathfrak{p})}/I^{k+1}_{m^{-1}(\mathfrak{p})}$$

is an isomorphism. Since $I_{m^{-1}(\mathfrak{p})}$ is generated by a regular sequence, it is almost-quasi-regular, hence by the above isomorphism we see that $S^{-1}I$, the kernel of the multiplication map $A_\mathfrak{p} \otimes_B A_\mathfrak{p} \to A_\mathfrak{p}$, is almost-quasi-regular. □

Corollary 7. *Let $X \to S$ be a smooth morphism. Then $\mathcal{P}^n_{X/S}$ is a locally finite free $\mathcal{O}_X$-module, hence $\mathcal{D}^{\leq n}_{X/S}$ is as well.*

Proof. $(\mathcal{P}^n_{X/S})_p = P^n_{\mathcal{O}_{X,p}/\mathcal{O}_{S,f(p)}}$ by Proposition 8 and Proposition 20. By Lemma 10 and Lemma 6, we obtain the result. □

7.2 Analysis of germs of smooth varieties with separable residue field

For this section, let X be a smooth variety over k of pure dimension n and $p \in X$ a point with residue field $\kappa(p)$ separably generated over k. Let $\mathfrak{m}_p$ be the maximal ideal of $\mathcal{O}_{X,p}$. Also, let $x_1, \ldots, x_r \in \mathcal{O}_{X,p}$ be a regular system of parameters at p and let the images of $x_{r+1}, \ldots, x_n \in \mathcal{O}_{X,p}$ in $\kappa(p)$ form a separating transcendence basis over k. See [15, Exercise 8.1]

Proposition 22. *Let $d : \mathcal{O}_{X,p} \to \Omega_{\mathcal{O}_{X,p}/k}$ be the universal derivation. Then $dx_1, \ldots, dx_n$ form a basis for $\Omega_{\mathcal{O}_{X,p}/k}$.*

Proof. The second fundamental exact sequence associated to $k \to \mathcal{O}_{X,p} \to \kappa(p)$.

$$0 \to \mathfrak{m}_p/\mathfrak{m}_p^2 \to \Omega_{\mathcal{O}_{X,p}/k} \otimes_{\mathcal{O}_{X,p}} \kappa(p) \to \Omega_{\kappa(p)/k} \to 0$$

is short exact by [26, Tag 00TU] because the residue field $\kappa(p)$ at p is separably generated over k. The image of $\mathfrak{m}_p/\mathfrak{m}_p^2$ in $\Omega_{\mathcal{O}_{X,p}/k}$ is generated by $dx_1, \ldots, dx_r$, and the images of $dx_{r+1}, \ldots, dx_n$ in $\Omega_{\kappa(p)/k}$ form a basis (see [19, Theorem 59]). Thus we see that $dx_1, \ldots, dx_n$ form a basis of $\Omega_{\mathcal{O}_{X,p}/k} \otimes_{\mathcal{O}_{X,p}} \kappa(p)$. So by Nakayama's lemma $dx_1, \ldots, dx_n$ generate $\Omega_{\mathcal{O}_{X,p}/k}$ hence form a basis because $\Omega_{\mathcal{O}_{X,p}/k}$ is a free $\mathcal{O}_{X,p}$-module of rank n. □

Corollary 8. *Let I_Δ be the kernel of the multiplication map $\mathcal{O}_{X,p} \otimes_k \mathcal{O}_{X,p} \to \mathcal{O}_{X,p}$. Then the sequence $dx_1, \ldots, dx_n \in I_\Delta$ is an almost-quasi-regular sequence of I_Δ, where $dx_i := 1 \otimes x_i - x_i \otimes 1$.*

Proof. By Proposition 22, the elements $dx_1, \ldots, dx_n$ form a basis for $\Omega_{\mathcal{O}_{X,p}/k} = I_\Delta/I_\Delta^2$. By Lemma 10 and Lemma 5, we have that $dx_1, \ldots, dx_n \in I_\Delta$ is an almost-quasi-regular sequence of I_Δ. □

Proposition 23 (Field of representatives. See [21, Exercise 3.9] and [14, page 158])**.** *There exists an element $a \in \hat{\mathcal{O}}_{X,p}$ such that $a \mod \hat{\mathfrak{m}}_p$ is the simple root of some polynomial $F(X)$ with coefficients in the field $k(x_{r+1}, \ldots, x_n)$ and such that the k-algebra inclusion $k(x_{r+1}, \ldots, x_n, a) \hookrightarrow \hat{\mathcal{O}}_{X,p}$ maps isomorphically onto the residue field $\kappa(p) = \hat{\mathcal{O}}_{X,p}/\hat{\mathfrak{m}}_p$. Thus $\hat{\mathcal{O}}_{X,p}$ has the field of representatives $k(x_{r+1}, \ldots, x_n, a)$ that is isomorphic to $\kappa(p)$ as a k-algebra. By the Cohen structure theorem, it follows that the continuous $k(x_{r+1}, \ldots, x_n, a)$-algebra map*

$$\begin{aligned} k(x_{r+1}, \ldots, x_n, a)[[X_1, \ldots, X_r]] &\to \hat{\mathcal{O}}_{X,p} \\ X_i &\mapsto x_i \end{aligned}$$

is an isomorphism, which identifies $\hat{\mathcal{O}}_{X,p}$ as a formal power series in the variables $x_1, \ldots, x_r$ over $\kappa(p) = k(x_{r+1}, \ldots, x_n, a)$.

Proof. Note that $x_{r+1}, \ldots, x_n$ are algebraically independent over k in $\mathcal{O}_{X,p}$. The elements $x_{r+1}, \ldots, x_n$ are also invertible in $\mathcal{O}_{X,p}$, since they are nonzero in the residue field. Thus we have the inclusion $k(x_{r+1}, \ldots, x_n) \hookrightarrow \mathcal{O}_{X,p}$ of a purely transcendental field extension in $n-r$ variables into $\mathcal{O}_{X,p}$.

Since $k(x_{r+1}, \ldots, x_n) \hookrightarrow \kappa(p)$ is a finite separable extension, by the primitive element theorem, there exists $\alpha \in \kappa(p)$ such that $\kappa(p)$ is obtained by adjoining α to $k(x_{r+1}, \ldots, x_n)$. Since α is separable over $k(x_{r+1}, \ldots, x_n)$, it is a simple root of some polynomial $F(X)$ over $k(x_{r+1}, \ldots, x_n)$. Consider $F(X)$ as a polynomial with coefficients in $\hat{\mathcal{O}}_{X,p}$ by the inclusions $k(x_{r+1}, \ldots x_n) \hookrightarrow \mathcal{O}_{X,p} \hookrightarrow \hat{\mathcal{O}}_{X,p}$. Because the reduction $\overline{F}(X) \in (\hat{\mathcal{O}}_{X,p}/\hat{\mathfrak{m}}_p)[X] = \kappa(p)[X]$ of F mod $\hat{\mathfrak{m}}_p$ has a simple root $\alpha \in \kappa(p)$, Hensel's lemma tells us that $F(X)$ has a root $a \in \hat{\mathcal{O}}_{X,p}$ such that $a \equiv \alpha \mod \hat{\mathfrak{m}}_p$. Thus

$$k(x_{r+1}, \ldots, x_n, a) \hookrightarrow \hat{\mathcal{O}}_{X,p}$$

is a k-algebra map mapping isomorphically onto the residue field $\kappa(p)$ of $\hat{\mathcal{O}}_{X,p}$. □

7.3 Hasse derivatives on germs of smooth varieties with separable residue field

Just as in Section 7.2, let X be a smooth variety over k of pure dimension n and $p \in X$ a point with residue field $\kappa(p)$ separably generated over k, where $x_1, \ldots, x_r \in \mathcal{O}_{X,p}$ is a regular system of parameters at p and the images of $x_{r+1}, \ldots, x_n \in \mathcal{O}_{X,p}$ in $\kappa(p)$ form a separating transcendence basis over k. Let $\mathfrak{m}_p$ be the maximal ideal of $\mathcal{O}_{X,p}$

Let the element $a \in \hat{\mathcal{O}}_{X,p}$ and the polynomial $F(X) \in k(x_{r+1}, \ldots, x_n)[X]$ be as in Proposition 23. So (1) $a \mod \hat{\mathfrak{m}}_p$ is a simple root of F, that is, $F'(a) \notin \hat{\mathfrak{m}}_p$, (2) the field $k(x_{r+1}, \ldots, x_n, a)$ maps isomorphically onto the residue field $\kappa(p)$ of $\hat{\mathcal{O}}_{X,p}$, and (3) $\hat{\mathcal{O}}_{X,p} = k(x_{r+1}, \ldots, x_n, a)[[x_1, ..., x_r]]$.

By Corollary 8, we can introduce the theory of Hasse derivatives (see Section 5) on $\mathcal{O}_{X,p}$. Let $\{D^{(s)}\}_{s \in \mathbb{N}^n} \subset D_{\mathcal{O}_{X,p}/k}$ be the collection of Hasse derivatives on the k-algebra $\mathcal{O}_{X,p}$ with respect to the sequence $x_1, \ldots, x_n$ (see Definition 13). Let $\{\hat{D}^{(s)}\}_{s \in \mathbb{N}^n} \subset D_{\hat{\mathcal{O}}_{X,p}/k}$ be the associated collection of Hasse derivatives on the $\mathfrak{m}_p$-adic completion of $\mathcal{O}_{X,p}$ with respect to the sequence $x_1, \ldots, x_n$ (see Definition 15).

We will analyze the behavior of the Hasse derivatives of the form $\hat{D}^{(s_1, \ldots, s_r, \vec{0})} \in D_{\hat{\mathcal{O}}_{X,p}/k}$.

Lemma 11. *Let $s \in \mathbb{N}^r$ and $0 \in \mathbb{N}^{n-r}$. Then $D^{(s,0)}$ is $k(x_{r+1}, \ldots, x_n)$-linear. By Proposition 10, $\hat{D}^{(s,0)}$ is $k(x_{r+1}, \ldots, x_n)$-linear as well.*

Proof. Suppose $f \in k[x_{r+1}, \ldots, x_n]$ and $g \in \mathcal{O}_{X,p}$. Then

$$D^{(s,0)}(fg) = \sum_{s'+s''=s} D^{(s',0)}(f) D^{(s'',0)}(g) = D^{(0,0)}(f) D^{(s,0)}(g) = f D^{(s,0)}(g)$$

because $D^{(s',0)}(f) = 0$ for all $|s'| > 0$. Now suppose $h \in k[x_{r+1}, \ldots, x_n]$ nonzero. Then by $k[x_{r+1}, \ldots, x_n]$-linearity, we obtain $k(x_{r+1}, \ldots, x_n)$-linearity:

$$D^{(s,0)}(\tfrac{f}{h} g) = \tfrac{h}{h} D^{(s,0)}(\tfrac{f}{h} g) = \tfrac{f}{h} D^{(s,0)}(\tfrac{h}{h} g) = \tfrac{f}{h} D^{(s,0)}(g)$$

□

Lemma 12. *Let $s \in \mathbb{N}^r$ be nonzero and $0 \in \mathbb{N}^{n-r}$. Then $\hat{D}^{(s,0)}(a) = 0$.*

Proof. We induct on $|s|$. If $|s| = 1$, then $\hat{D}^{(s,0)}$ is a $k(x_{r+1}, \dots, x_n)$-linear derivation, hence

$$\hat{D}^{(s,0)}(F(a)) = F'(a)\hat{D}^{(s,0)}(a)$$

so that $F'(a)\hat{D}^{(s,0)}(a) = 0$. Since $F'(a)$ is invertible in $\mathcal{O}_{X,p}$, we have $\hat{D}^{(s,0)}(a) = 0$.

Now suppose $|s| > 1$, and the lemma true when the order of the differential operator is less than $|s|$. By the Leibniz rule (Proposition 14 (i)), we have that

$$\hat{D}^{(s,0)}(a^m) = \sum_{s^1+\cdots+s^m=s} \prod_{i=1}^{m} \hat{D}^{(s^i,0)}(a)$$

By the inductive hypothesis, if some s^i in the sum $s^1 + \cdots + s^m = s$ is not s, then $\hat{D}^{(s^i,0)}(a) = 0$ for that i. Thus

$$\begin{aligned}\hat{D}^{(s,0)}(a^m) &= \sum_{i=1}^{m} \left\{ \prod_{j=1}^{m} \hat{D}^{(s^j,0)}(a) \middle| s_i = s, s_j = 0 \text{ for } j \neq i \right\} \\ &= ma^{m-1}\hat{D}^{(s,0)}(a)\end{aligned}$$

So by $k(x_{r+1}, \dots, x_n)$-linearity (Lemma 11), we have

$$\hat{D}^{(s,0)}(F(a)) = F'(a)\hat{D}^{(s,0)}(a)$$

As before, we arrive at $\hat{D}^{(s,0)}(a) = 0$. $\square$

Lemma 13. *For $s \in \mathbb{N}^r$ and $0 \in \mathbb{N}^{n-r}$, the Hasse derivative $\hat{D}^{(s,0)} \in D_{\hat{\mathcal{O}}_{X,p}/k}$ is $k(x_{r+1}, \cdots, x_n, a)$-linear, so that $\hat{D}^{(s,0)} \in D_{\hat{\mathcal{O}}_{X,p}/k(x_{r+1},\cdots,x_n,a)}$.*

Proof. If $s = 0 \in \mathbb{N}^r$, then $\hat{D}^{(s,0)}$ is the identity. So suppose $s \in \mathbb{N}^r$ is nonzero. By Lemma 12 and the Leibniz rule, we have that $\hat{D}^{(s,0)}$ is $k(x_{r+1}, \cdots, x_n)[a]$-linear. Then it follows that $\hat{D}^{(s,0)}$ is $k(x_{r+1}, \cdots, x_n, a)$-linear (see the end of the proof of Lemma 11). $\square$

We now present the culmination of all our previous analysis. The following theorem relates Hasse derivatives on the germ of a smooth variety at a point with separable residue field with Hasse derivatives on a ring of formal power series, which we know very well.

Theorem 11. *Identify $\kappa(p) = k(x_{r+1}, \dots, x_n, a)$ so that $\kappa(p) = k(x_{r+1}, \dots, x_n, a)$ maps identically onto the residue field $\kappa(p)$ of $\hat{\mathcal{O}}_{X,p}$. So we have the identification $\hat{\mathcal{O}}_{X,p} = \kappa(p)[[x_1, \dots, x_r]]$. Let $\{\hat{D}^{(s)}\}_{s\in\mathbb{N}^r} \subset D_{\kappa(p)[[x_1,\dots,x_r]]/\kappa(p)}$ be the Hasse derivatives on the completion of the $\kappa(p)$-polynomial algebra $\kappa(p)[x_1, \dots, x_r]$ with respect to the sequence $x_1, \dots, x_r$. Then $\hat{D}^{(s,0)} = \hat{D}^{(s)}$ for each $s \in \mathbb{N}^r$ and $0 \in \mathbb{N}^{n-r}$.*

Proof. By 13

$$\hat{D}^{(s,0)} \in D_{\kappa(p)[[x_1,\dots,x_r]]/\kappa(p)}$$

Note that for $t \in \mathbb{N}^r$, we have

$$\begin{aligned}\hat{D}^{(s,0)}(x^{(t)}) &= \hat{D}^{(s,0)}(x_1^{t_1} \cdots x_r^{t_r} \cdot x_{r+1}^0 \cdots x_n^0) \\ &= \hat{D}^{(s,0)}(x^{(t,0)}) \\ &= \binom{(t,0)}{(s,0)} x^{(t-s,0)} \\ &= \binom{t}{s} x^{(t-s)}\end{aligned}$$

By Lemma 5, we arrive at $\hat{D}^{(s,0)} = \hat{D}^{(s)}$. $\square$

Proposition 24. *Let $\mathcal{I} \subset \mathcal{O}_X$ a coherent ideal sheaf. If k is a field of characteristic zero, then*

$$\mathcal{D}^{\leq 1}_{X/k}\mathcal{D}^{\leq n}_{X/k}\mathcal{I} = \mathcal{D}^{\leq n+1}_{X/k}\mathcal{I}$$

Proof. See Corollary 4. □

8 Order of Vanishing on Smooth Varieties

Let k be a field.

8.1 Order of vanishing

Definition 20 (Order)**.** *Let $(A, \mathfrak{m})$ be a local ring. For $I \subset A$ be an ideal, define the order* $\operatorname{ord}_A I$ *of vanishing of the ideal I to be*

$$\operatorname{ord}_A I = \max\{r \geq 0 \mid I \subset \mathfrak{m}^r\}$$

If $f \in A$, define $\operatorname{ord}_A f$ *to be the order of the principal ideal $(f) \subset A$.*

Lemma 14 (Ideals vanish to higher order on subschemes)**.** *Let $(A, \mathfrak{m})$ be a local ring and $J \subset A$ an ideal contained in $\mathfrak{m}$, so A/J is also a local ring. Then if $I \subset A$ is an ideal, then*

$$\operatorname{ord}_A I \leq \operatorname{ord}_{A/J} I$$

Proof. If $I \subset \mathfrak{m}^k$, then $I \subset \mathfrak{m}^k \subset (\mathfrak{m} + J)^k \subset \mathfrak{m}^k + J$. Thus

$$\frac{I + J}{J} \subset \frac{\mathfrak{m}^k + J}{J} = \left(\frac{\mathfrak{m} + J}{J}\right)^k = \left(\frac{\mathfrak{m}}{J}\right)^k$$

Because $\mathfrak{m}/J$ is the maximal ideal of A/J and $(I + J)/J$ is the extension of the ideal I in A/J, we obtain the result that $\operatorname{ord}_A I \leq \operatorname{ord}_{A/J} I$. □

Lemma 15 (See [16, Proposition 8.8])**.** *Let $(A, \mathfrak{m})$ be a noetherian local ring and $(\hat{A}, \hat{\mathfrak{m}})$ the $\mathfrak{m}$-adic completion of A. Then for an ideal $I \subset A$, we have*

$$\operatorname{ord}_A I = \operatorname{ord}_{\hat{A}} \hat{I}$$

where $\hat{I}$ is the $\mathfrak{m}$-adic completion of I.

Proof. If $I \subset \mathfrak{m}^r$, then $\hat{I} \subset \hat{\mathfrak{m}}^r$ is immediate.

Before we show the converse, we make an observation. Let $\phi : A \to \hat{A}$ be the $\mathfrak{m}$-adic completion map, $J \subset A$ an ideal, and $\hat{J}$ the $\mathfrak{m}$-adic completion of J. Then

$$\phi^{-1}(\hat{J}) = \bigcap_{k=1}^{\infty}(\mathfrak{m}^k + J) = J$$

where the first equality follows because the map $A/J \to \hat{A}/\hat{J}$ induced by ϕ is the $\mathfrak{m}$-adic completion of A/J ([2, Proposition 10.12]), and the second equality follows because the $(\mathfrak{m}/J)$-adic topology on A/J is Hausdorff ([2, 10.19]).

Now suppose $\hat{I} \subset \hat{\mathfrak{m}}^r$. Thus we have

$$I \subset \phi^{-1}(\hat{I}) \subset \phi^{-1}(\hat{\mathfrak{m}}^r) = \phi^{-1}(\widehat{\mathfrak{m}^r}) = \mathfrak{m}^r$$

□

Definition 21 (Order of vanishing)**.** *Let X be a regular k-scheme and $\mathcal{I} \subset \mathcal{O}_X$ a coherent ideal sheaf. For $p \in X$, define the order of vanishing* $\operatorname{ord}_p \mathcal{I}$ *of the ideal $\mathcal{I}$ at p to be*

$$\operatorname{ord}_p \mathcal{I} := \max\{r \geq 0 \mid \mathcal{I}_p \subset \mathfrak{m}_p^r\}$$

where $\operatorname{ord}_p \mathcal{I} = \infty$ *if $\mathcal{I}_p$ is contained in all powers of $\mathfrak{m}_p^r$. If $f \in \mathcal{O}_{X,p}$, let*

$$\operatorname{ord}_p f := \max\{r \geq 0 \mid f \in \mathfrak{m}_p^r\}$$

The order of vanishing of an ideal $\mathcal{I}$ is a function on points of X taking values in $\mathbb{N} \cup \{\infty\}$. Note that $p \in V(\mathcal{I})$ if and only if $\operatorname{ord}_p \mathcal{I} \geq 1$. Also, $\operatorname{ord}_p \mathcal{I} = \infty$ if and only if $\mathcal{I}_p = 0$ if and only if $\mathcal{I}$ vanishes on the irreducible component of X containing p by Krull's intersection theorem [2, Corollary 10.18].

8.2 Upper semicontinuity of order of vanishing

Theorem 12. *Let X be a regular k-scheme and $\mathcal{I} \subset \mathcal{O}_X$ a coherent sheaf of ideals. Then for every nonnegative integer r, there exists an open subset U_r of X such that $x \in X$ belongs to U_r if and only if* $\operatorname{ord}_p \mathcal{I} \leq r$*. That is, the order of vanishing function $p \mapsto \operatorname{ord}_p \mathcal{I}$ on X is upper-semicontinuous.*

Proof. [17, Corollary III.3.1, page 220]. □

Definition 22. *Let X be a noetherian regular k-scheme and $\mathcal{I} \subset \mathcal{O}_X$ a coherent sheaf of ideals. Define the maximal order* $\operatorname{max-ord} \mathcal{I}$ *of vanishing of $\mathcal{I}$ on X to be*

$$\operatorname{max-ord} \mathcal{I} := \max\{\operatorname{ord}_p \mathcal{I} \mid p \in X\}$$

Note that $\operatorname{max-ord} \mathcal{I} = \infty$ iff $\mathcal{I}$ vanishes on some irreducible component of X. If $\mathcal{I}$ does not vanish on some irreducible component of X, then because order of vanishing is upper semicontinuous and X is noetherian, we have that $\operatorname{max-ord} \mathcal{I} < \infty$.

For an integer r, what scheme structure should we endow on the locus of points where $\mathcal{I}$ vanishes to order at least r? For arbitrary fields k, the answer is not clear, but when the field k is perfect, the answer is provided by the following theorem, in which case we obtain a separate proof from Hironaka's Theorem 12 that the order of vanishing is upper semicontinuous.

Theorem 13 (See [21, Theorem A.19])**.** *Let X be a smooth variety over k, and $\mathcal{I} \subset \mathcal{O}_X$ a coherent ideal sheaf. If $p \in X$ is a point with residue field $\kappa(p)$ separably generated over k, then $p \in V(\mathcal{D}^{\leq r-1}_{X/k}\mathcal{I})$ if and only if* $\operatorname{ord}_p \mathcal{I} \geq r$*. So if the field k is perfect, the locus where $\mathcal{I}$ vanishes to order at least r is cut out by $\mathcal{D}^{\leq r-1}_{X/k}\mathcal{I}$.*

Proof. Let $p \in X$. Note that $\operatorname{ord}_p \mathcal{I} \geq r$ iff $\mathcal{I}_p \subset \mathfrak{m}_p^r$ iff $f \in \mathfrak{m}_p^r$ for every $f \in \mathcal{I}_p$. Also note that $p \in V(\mathcal{D}^{r-1}_{X/k}\mathcal{I})$ iff $D^{\leq r-1}_{\mathcal{O}_{X,p}/k}\mathcal{I}_p \subset \mathfrak{m}_p$ iff $D^{\leq r-1}_{\mathcal{O}_{X,p}/k}(f) \subset \mathfrak{m}_p$ for every $f \in \mathcal{I}_p$. Thus it suffices to prove that $f \in \mathfrak{m}_p^r$ iff $D^{\leq r-1}_{\mathcal{O}_{X,p}/k}(f) \subset \mathfrak{m}_p$ for $f \in \mathcal{O}_{X,p}$.

Let $f \in \mathcal{O}_{X,p}$. If $f \in \mathfrak{m}_p^r$ and $D \in D^{\leq r-1}_{\mathcal{O}_{X,p}/k}$, then by Proposition 9, we have

$$D^{\leq r-1}_{\mathcal{O}_{X,p}/k}(f) \subset D^{\leq r-1}_{\mathcal{O}_{X,p}/k}(\mathfrak{m}_p^r) \subset \mathfrak{m}_p^{r-(r-1)} = \mathfrak{m}_p$$

Conversely, suppose that $D^{\leq r-1}_{\mathcal{O}_{X,p}/k}(f) \subset \mathfrak{m}_p$. For the sake of contradiction, assume that $f \notin \mathfrak{m}_p^r$. By Lemma 15, we have $\operatorname{ord}_{\hat{\mathcal{O}}_{X,p}} f = \operatorname{ord}_p f < r$. Now we use that $\kappa(p)$ is separably generated over k.

Download and install Section 7.3 into this proof as follows. As in Section 7.3, let $x_1, \ldots, x_r \in \mathcal{O}_{X,p}$ be a regular system of parameters, the images of $x_{r+1}, \ldots, x_n \in \mathcal{O}_{X,p}$ in the residue field separably generate $\kappa(p)$ over k, where n is the dimension of the irreducible component of X containing p, and a field of representatives $\kappa(p) \subset \hat{\mathcal{O}}_{X,p}$ such that $\kappa(p)$ is a finite separable extension of $k(x_{r+1}, \ldots, x_n) \subset \hat{\mathcal{O}}_{X,p}$. Identify $\hat{\mathcal{O}}_{X,p} = \kappa(p)[[x_1, \ldots, x_r]]$. Let $\{D^{(s)}\}_{s\in\mathbb{N}^n}$ be the Hasse derivatives on the k-algebra $\mathcal{O}_{X,p}$ with respect to $x_1, \ldots, x_n$, and let $\{\hat{D}^{(s)}\}_{s\in\mathbb{N}^n}$ be the associated Hasse derivatives on the $\mathfrak{m}_p$-adic completion of the k-algebra $\mathcal{O}_{X,p}$ with respect to $x_1, \ldots, x_n$. Also, let $\{\hat{D}^{(s)}\}_{s\in\mathbb{N}^r}$ be the Hasse derivatives on the $(x_1, \ldots, x_r)$-adic completion of the $\kappa(p)$-algebra $\kappa(p)[x_1, \ldots, x_r]$ with respect to $x_1, \ldots, x_r$. By Theorem 11, $\hat{D}^{(s,0)} = \hat{D}^{(s)}$ for every $s \in \mathbb{N}^r$ and $0 \in \mathbb{N}^{n-r}$.

Let $b = \operatorname{ord}_{\hat{\mathcal{O}}_{X,p}} f$. Since b is the largest number such that $f \in \hat{\mathfrak{m}}_p^b = (x_1, \ldots, x_r)^b$, there is some $t \in \mathbb{N}^r$ with weight $|t| = b$ such that the monomial term $x^{(t)}$ has nonzero coefficient in f expressed as a power series in $\kappa(p)[[x_1, \ldots, x_r]]$. Since this coefficient is equal to the constant term in the power series expression of $\hat{D}^{(t)}(f)$, we see that $\hat{D}^{(t)}(f)$ is invertible in $\kappa(p)[[x_1, \ldots, x_r]]$. Thus $\hat{D}^{(t,0)}(f) = \hat{D}^{(t)}(f)$ is invertible in $\hat{\mathcal{O}}_{X,p}$, that is, $\hat{D}^{(t,0)}(f) \notin \hat{\mathfrak{m}}_p$. Since $D^{(t,0)}(f) = \hat{D}^{(t,0)}(f)$ and $\mathcal{O}_{X,p} \hookrightarrow \hat{\mathcal{O}}_{X,p}$ is a local ring homomorphism, we have that $D^{(t,0)}(f) \notin \mathfrak{m}_p$. But $D^{(t,0)} \in D^{\leq r-1}_{\mathcal{O}_{X,p}/k}$, hence we have a contradiction. □

Remark 4. *The previous theorem above may not be true when the ground field k is not perfect. For example, let k be a field with characteristic p that is not perfect. So there is an element $a \in k$ having no pth roots. Let $X = \mathbb{A}^1_k$. Since $x^p - a$ is an irreducible polynomial over k, let $q = [(x^p - a)] \in \mathbb{A}^1_k$ and $I = (x^p - a) \subset k[x]$. The order of vanishing of I is an indicator function on X at the closed point q, where $\operatorname{ord}_q I = 1$. However,*

$$D^{\leq 0}_{k[x]/k}(I) = D^{\leq 1}_{k[x]/k}(I) = \cdots = D^{\leq p-1}_{k[x]/k}(I) = I \qquad D^{\leq p}_{k[x]/k}(I) = (1)$$

Definition 23 (b-singular locus)**.** *Let k be a perfect field and X a smooth variety over k. Let b be a nonnegative integer. Define the b-singular locus $V(\mathcal{I}, b)$ to be the closed subscheme of X cut out by the ideal $\mathcal{D}^{\leq b-1}_{X/k}\mathcal{I}$. $V(\mathcal{I}, b)$ is the scheme-theoretic locus of points where $\mathcal{I}$ vanishes to order at least b. If $b = \text{max-ord}\,\mathcal{I}$, we call $V(\mathcal{I}, b)$ the top locus. $V(\mathcal{I}, b)$ also goes by Sing$(\mathcal{I}, b)$ (see [24, Definition 3.2]), Sing$_b V(\mathcal{I})$ (see [7]), Sing$_b\mathcal{I}$ (see [6]), or cosupp$(\mathcal{I}, b)$ (see [14, Definition 3.59]), where the scheme structure is usually left out.*

9 Explicit Differentiation of Ideals on Smooth Affine Varieties

We will show how to explicitly compute derivatives of ideals on smooth varieties.

9.1 Explicit affine-local basis of first order differential operators

Lemma 16 (See [3, Lemma 3.18]). *Suppose that $A = k[x_1, \dots, x_N]/(f_1, \dots, f_r)$ is such that* $\operatorname{Spec} A$ *is a smooth variety over k of pure dimension n. Let*

$$J = [df_1 \ \cdots \ df_r]^T = \left(\frac{\partial f_i}{\partial x_j}\right)_{ij} \in k[x_1, \dots, x_N]^{r \times N}$$

be the Jacobian. Let $ROW \subset \{1, \dots, r\}$ and $COL \subset \{1, \dots, N\}$ be subsets of size $N-n$, and M be the $(N-n) \times (N-n)$ submatrix of the Jacobian J involving the rows indexed by ROW and the columns indexed by COL.

$$M = \left(\frac{\partial f_i}{\partial x_j}\right)_{i \in ROW, j \in COL}$$

and let $h = \det M \in k[x_1, \dots, x_N]$. Let

$$C = \{C_{ij}\}_{i \in ROW, j \in COL} = \{(-1)^{i+j} M_{ij}\}_{i \in ROW, j \in COL}$$

be the matrix of cofactors of M, where M_{ij} is the determinant of the matrix obtained by removing the row $i \in ROW$ and the column $j \in COL$ of M. For each $j' \notin COL$, define the derivation $D_{j'} \in \operatorname{Der}_k(k[x_1, \dots, x_N], k[x_1, \dots, x_N])$ given by

$$D_{j'} = h \frac{\partial}{\partial x_{j'}} - \sum_{\substack{i \in ROW \\ j \in COL}} \frac{\partial f_i}{\partial x_{j'}} C_{ij} \frac{\partial}{\partial x_j}$$

Then the $D_{j'}$'s extend to a basis for $\operatorname{Der}_k(A[h^{-1}], A[h^{-1}])$. Thus the extension of the ideal

$$(f) + \left(\ \{ \ D_{j'}(f) \ \}_{j' \notin COL} \ \right) \subset k[x_1, \dots, x_N]$$

in $A[h^{-1}]$ coincides with the ideal $D^{\leq 1}_{A[h^{-1}]/k}(f)$.

(Note: for this lemma, i will always indicate an index in ROW, j an index in COL, and an apostrophe on i or j will always indicate an index in the complement, that is, $i' \notin$ ROW and $j' \notin$ COL. Indexing in this way should slightly mitigate the heavy use of indices)

(Sanity check: $N - n \geq 0$. To see why, consider the embedding $\operatorname{Spec} A \hookrightarrow \mathbb{A}^N_k$. The number r of defining equations is at least the codimension $N-n$ in affine space: just take a closed point $p \in \operatorname{Spec} A$ and observe that $\mathcal{O}_{\operatorname{Spec} A, p} = \mathcal{O}_{\mathbb{A}^N_k, p}/(f_1, \dots, f_r)$ is a regular local ring of dimension n)

Proof. For every $\mathfrak{p} \in \operatorname{Spec} A$, the Jacobian J as a $\kappa(\mathfrak{p})$-valued matrix has rank $N-n$ because $\Omega_{A/k}$ is locally free of rank n.

Because $C^T M = hI$, we have for every $j \in$ COL that

$$\sum_{i \in \text{ROW}} C_{ij} df_i = h dx_j + \sum_{j' \notin \text{COL}} \left(\sum_{i \in \text{ROW}} C_{ij} \frac{\partial f_i}{\partial x_{j'}} \right) dx_{j'}$$

Thus we have the inclusion of submodules of $\Omega_{k[x_1,\dots,x_N]/k}$

$$\left\{hdx_j + \sum_{j'\notin \mathrm{COL}} \sum_{i\in \mathrm{ROW}} C_{ij}\frac{\partial f_i}{\partial x_{j'}} dx_{j'}\right\}_{j\in\mathrm{COL}} \subset \{df_i\}_{i\in \mathrm{ROW}} \subset (df_1,\dots,df_r)$$

In $\Omega_{A[h]/k}$, these inclusions become an equality. To see why, let $\widetilde{df_i}$ be the image of df_i under the following map

$$\begin{aligned}\bigoplus_{k=1}^{N} A[h^{-1}]dx_k &\to \bigoplus_{j'\notin\mathrm{COL}} A[h^{-1}]dx_{j'}\\ dx_k &\mapsto \begin{cases} dx_k & \text{if } k\notin \mathrm{COL}\\ -h^{-1}\sum\limits_{j'\notin\mathrm{COL}}\sum\limits_{i\in\mathrm{ROW}} C_{ik}\frac{\partial f_i}{\partial x_{j'}}dx_{j'} & \text{if } k\in\mathrm{COL}\end{cases}\end{aligned}$$

This induces an isomorphism

$$\Omega_{A[h^{-1}]/k} = \frac{\bigoplus_{k=1}^{N} A[h^{-1}]dx_k}{(df_1,\dots,df_r)} = \frac{\bigoplus_{j'\notin\mathrm{COL}} A[h^{-1}]dx_{j'}}{(\widetilde{df_1},\dots,\widetilde{df_r})}$$

For every $\mathfrak{p}\in \operatorname{Spec} A[h^{-1}]$, the vector space $\Omega_{A[h]/k}\otimes_{A[h^{-1}]}\kappa(\mathfrak{p})$ is free of rank n with basis $dx_1,\dots,dx_n$. Thus

$$(\widetilde{df_1},\dots,\widetilde{df_r}) \subset \bigcap_{\mathfrak{p}\in\operatorname{Spec}A[h^{-1}]}\bigoplus_{j'\notin\mathrm{COL}} \mathfrak{p}dx_{j'} = \bigoplus_{j'\notin\mathrm{COL}} \mathrm{Nil}(A[h^{-1}])dx_{j'}$$

where $\mathrm{Nil}(A[h^{-1}])$ is the nilradical of $A[h^{-1}]$. But $\operatorname{Spec} A$ is smooth, hence in particular reduced, so that $\mathrm{Nil}(A[h^{-1}])=0$. Thus $(\widetilde{df_1},\dots,\widetilde{df_r})=0$, which implies the equality

$$\left(\left\{hdx_j + \sum_{j'\notin\mathrm{COL}}\left(\sum_{i\in\mathrm{ROW}} C_{ij}\frac{\partial f_i}{\partial x_{j'}}\right)dx_{j'}\right\}_{j\in\mathrm{COL}}\right) = (df_1,\dots,df_r)$$

in $\Omega_{A[h^{-1}]/k}$. This means that we have the isomorphism

$$\Omega_{A[h^{-1}]/k} = \bigoplus_{j'\notin\mathrm{COL}} A[h^{-1}]dx_{j'}$$

Let $f\in k[x_1,\dots,x_N]$. Then

$$\begin{aligned} hdf &= h\sum_{j'\notin\mathrm{COL}}\frac{\partial f}{\partial x_{j'}}dx_{j'} + h\sum_{j\in\mathrm{COL}}\frac{\partial f}{\partial x_j}dx_j\\ &= h\sum_{j'\notin\mathrm{COL}}\frac{\partial f}{\partial x_{j'}}dx_{j'} - \sum_{j\in\mathrm{COL}}\sum_{j'\notin\mathrm{COL}}\frac{\partial f}{\partial x_j}\left(\sum_{i\in\mathrm{ROW}} C_{ij}\frac{\partial f_i}{\partial x_{j'}}\right)dx_{j'}\\ &= \sum_{j'\notin\mathrm{COL}}\left(h\frac{\partial f}{\partial x_{j'}} - \sum_{\substack{i\in\mathrm{ROW}\\ j\in\mathrm{COL}}}\frac{\partial f_i}{\partial x_{j'}}C_{ij}\frac{\partial f}{\partial x_j}\right)dx_{j'}\\ &= \sum_{j'\notin\mathrm{COL}} D_{j'}(f)dx_{j'}\end{aligned}$$

Thus the derivation in $\mathrm{Der}_k(A[h^{-1}],A[h^{-1}])$ corresponding to the coordinate projection on $dx_{j'}$ in $\Omega_{A[h^{-1}]/k} = \bigoplus_{j'\notin\mathrm{COL}} A[h^{-1}]dx_{j'}$ is induced by $h^{-1}D_{j'}$. $\square$

9.2 Algorithm for taking derivatives of an ideal

By the previous Lemma 16 and Lemma A.19, we obtain the following algorithm.

Algorithm 1: Taking derivatives on a smooth affine variety of pure dimension

Input: $f_1, \dots, f_r \in k[x_1, \dots, x_N]$
$g_1, \dots, g_m \in k[x_1, \dots, x_N]$
such that $\operatorname{Spec} A$ is a smooth variety over k of pure dimension n
where $A = k[x_1, \dots, x_N]/(f_1, \dots, f_r)$

Output: An ideal of $k[x_1, \dots, x_N]$ whose extension in A is equal to $D^{\leq 1}_{A/k}(g_1, \dots, g_m)$

if $f_1 = \dots = f_r = 0$ **then**
  return $(g_1, \dots, g_m, \frac{\partial g_1}{\partial x_1}, \dots, \frac{\partial g_i}{\partial x_j}, \dots, \frac{\partial g_m}{\partial x_N})$
else
  initialization
  - $J = [df_1 \cdots df_r]^T = (\frac{\partial f_i}{\partial x_j})_{ij}$
  - $L = \{$ $N-n$ by $N-n$ square submatrices of the Jacobian matrix J whose determinants are non-zero in A $\}$
  - $I = (1) \subset k[x_1, \dots, x_N]$

  Because $\operatorname{Spec} A$ is smooth over k of pure dimension n, we have

$$\operatorname{Spec} A = \bigcup_{M \in L} \operatorname{Spec} A[\det M^{-1}]$$

  for *each* $M \in L$ **do**
  - $h = \det M$
  - let $\mathrm{ROW} \subset \{1, \dots, r\}$ be the row indices and $\mathrm{COL} \subset \{1, \dots, N\}$ the column indices of the Jacobian J that M involve
  - let C be the matrix of cofactors of M (see Lemma 16)
  - set

$$I_M = (g_1, \dots, g_m) + \left(\left\{ h \frac{\partial f}{\partial x_j} - \sum_{\substack{i \in \mathrm{ROW} \\ j \in \mathrm{COL}}} \frac{\partial g_s}{\partial x_j} C_{ij} \frac{\partial f_i}{\partial x_{j'}} \middle| s = 1, \dots, m,\ j' \notin \mathrm{COL} \right\} \right)$$

  which is an ideal of $k[x_1, \dots, x_N]$
  - $I = I \cap (I_M : h^\infty)$ (see [4, A.3.1])

  return I

In the literature on resolution of singularities, the ideal $\mathcal{D}^{\leq 1}_{X/k}\mathcal{I}$ also goes by the name $\Delta(\mathcal{I})$. I learned about the algorithm above from [7, Page 312], in which the algorithm goes by Algorithm Delta. There is an implementation of Delta in SINGULAR [8] that applies the Delta operator Δ to ideals of smooth varieties.

9.3 Algorithm for maximal order of vanishing

In characteristic zero, this algorithm allows us to compute the maximal order of an ideal by repeatedly applying $D^{\leq 1}$ until the unit ideal is obtained.

Algorithm 2: Maximal order of vanishing of an ideal on a smooth variety

---

Let k be a field of characteristic zero.

Input: X a smooth variety over k covered by affine opens $U_1, \dots, U_N$ and $\mathcal{I} \subset \mathcal{O}_X$ a coherent sheaf of ideals

Output: the maximal order of vanishing $\text{max-ord}\,\mathcal{I}$ (see Definition 22)

if $N=1$ **then**

  So $X = U_1 = \operatorname{Spec} A$

  Let $I = \Gamma(X, \mathcal{I}) \subset A$

  Let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be the minimal primes of A, which can be computed using primary decomposition (see [13])

  if $I \subset \mathfrak{p}_i$ *for some* i **then**

    return $\text{max-ord}\,\mathcal{I} = \infty$

  else

    In this case, $\mathcal{I}$ does not vanish on any irreducible component of X, so $\text{max-ord}\,\mathcal{I} < \infty$

    Thus there exists a smallest integer $a \in \mathbb{N}$ such that $\mathcal{D}^{\leq a}_{X/k}\mathcal{I} = \mathcal{O}_X$ (see Algorithm 1 and Proposition 24)

    return $\text{max-ord}\,\mathcal{I} = a$

else

  We can compute $\text{max-ord}\,\mathcal{I}|_{U_i}$ for each i since U_i is affine

  return $\max\{\text{max-ord}\,\mathcal{I}|_{U_1}, \dots, \text{max-ord}\,\mathcal{I}|_{U_N}\}$

---

10 Maximal Contact and their Explicit Computation

10.1 General theory

Let X be a smooth variety over a field k.

Definition 24. *Let $p \in X$. We say that $f \in \mathcal{O}_{X,p}$ is a regular parameter if it is part of some regular system of parameters at p. Equivalently,* $\operatorname{ord}_p f = 1$.

Lemma 17. *Let $p \in X$ and $f \in \mathcal{O}_{X,p}$ be a regular parameter. Then f lifts to an open neighborhood $U \subset X$ of p such that f is a regular parameter at every point of $V(f) \subset U$, so that $V(f)$ is a regular subvariety of X. If k is perfect, then $V(f)$ is a smooth subvariety of U.*

Proof. We may assume that f lifts to a function on X, and let $\mathcal{I} = (f) \subset \mathcal{O}_X$ be the ideal sheaf defined by f. By upper semicontinuity of order of vanishing, the locus of points with order less than two is open. Since $\operatorname{ord}_p \mathcal{I} = 1 < 2$, there exists an open neighborhood U of p such that $\operatorname{ord}_q \mathcal{I} < 2$ for every $q \in U$. But this means that for every point $q \in V(f) \cap U$, the function $f \in \mathcal{O}_{X,q}$ must be a regular parameter. □

Definition 25 (Pure Codimension and Hypersurface)**.** *Let X be a topological space and Z a closed subset of X. We say that Z has pure codimension c in X if the codimension of each of Z's irreducible components in X is equal to c. A hypersurface on X is a closed subset $H \hookrightarrow X$ of pure codimension one.*

For empty set lovers: the empty set $\varnothing \hookrightarrow X$ is a hypersurface because each irreducible component of $\varnothing$ has codimension one.

Definition 26 (Maximal contact element and hypersurface)**.** *Let k be a perfect field. Suppose $\mathcal{I} \subset \mathcal{O}_X$ is a coherent sheaf of ideals not vanishing on any irreducible component of X. Let $p \in V(\mathcal{I}, a)$, where $a = \text{max-ord}\,\mathcal{I}$. We say that a regular parameter $f \in \mathcal{O}_{X,p}$ is a maximal contact element of $\mathcal{I}$ at p if $f \in D^{\leq a-1}_{\mathcal{O}_{X,p}/k}\mathcal{I}$. We say that a smooth hypersurface $H \hookrightarrow X$ is a maximal contact hypersurface of $\mathcal{I}$ if H scheme-theoretically contains $V(\mathcal{I}, a)$, that is, $\mathcal{D}^{\leq a-1}_{X/k}\mathcal{I}$ contains the ideal cutting H out of X. If $U \subset X$ is an open subscheme, we say that a smooth hypersurface $H \hookrightarrow U$ is a local maximal contact hypersurface of $\mathcal{I}$ on U if H scheme-theoretically contains $V(\mathcal{I}, a)|_U$.*

Proposition 25 (Maximal contact elements exist in characteristic zero)**.** *Let k be a field of characteristic zero and $\mathcal{I} \subset \mathcal{O}_X$ a coherent sheaf of ideals not vanishing on any irreducible component of X. Let $p \in V(\mathcal{I}, a)$, where $a = \text{max-ord}\,\mathcal{I}$. Then there exists a maximal contact element of $\mathcal{I}$ at p.*

Proof. Let $\mathfrak{m}_p$ be the maximal ideal of $\mathcal{O}_{X,p}$. Because $p \in V(\mathcal{I}, a)$, we have $D^{\leq a-1}_{\mathcal{O}_{X,p}/k}\mathcal{I}_p \subset \mathfrak{m}_p$, and because $a = \text{max-ord}\,\mathcal{I}$, we have $D^{\leq a}_{\mathcal{O}_{X,p}/k}\mathcal{I}_p = \mathcal{O}_{X,p}$. Since k is a field of characteristic zero, we have that $D^{\leq a}_{\mathcal{O}_{X,p}/k}\mathcal{I}_p = D^{\leq 1}_{\mathcal{O}_{X,p}/k}D^{\leq a-1}_{\mathcal{O}_{X,p}/k}\mathcal{I}_p$. Thus there exists an element $f \in D^{\leq a-1}_{\mathcal{O}_{X,p}/k}\mathcal{I}_p$ and a differential operator $D \in D^{\leq 1}_{\mathcal{O}_{X,p}/k}$ such that $D(f) \in \mathcal{O}_{X,p}$ is invertible. Note that f cannot be in $\mathfrak{m}_p^2$, since otherwise $D(f) \subset \mathfrak{m}_p$. Thus f is a maximal contact element of $\mathcal{I}$ at p. □

Corollary 9 (Local maximal contact hypersurfaces exist in characteristic zero)**.** *Let k be a field of characteristic zero and $\mathcal{I} \subset \mathcal{O}_X$ a coherent sheaf of ideals not vanishing on any irreducible component of X. Then there exists an open cover $X = \bigcup_{i=1}^r U_i$ by finitely many open subsets such that there is a local maximal contact hypersurface $H_i \hookrightarrow U_i$ on those U_i that meet the top locus $V(\mathcal{I}, a)$. If U_i does not meet the top locus $V(\mathcal{I}, a)$, then the empty hypersurface on U_i contains $V(\mathcal{I}, a)|_{U_i} = \varnothing$.*

Proof. Let $a = \text{max-ord}\,\mathcal{I}$. Let $p \in V(\mathcal{I}, a)$. By Proposition 25, there is a maximal contact element $f \in \mathcal{O}_{X,p}$. Since $f \in D^{\leq a-1}_{\mathcal{O}_{X,p}/k}\mathcal{I}_p$, we may lift f to $\widetilde{f} \in \mathcal{O}_X(U_p)$ on a sufficiently small affine neighborhood U of p such that $\widetilde{f} \in (\mathcal{D}^{\leq a-1}_{X/k}\mathcal{I})|_U$. By upper semicontinuity of order of vanishing, the set of points on U_p where $\widetilde{f}$ vanishes to order less than two is an open set. Since $\widetilde{f}$ vanishes to order one at p, we may assume that $\widetilde{f}$ vanishes to order less than two at every point on U_p by shrinking U_p, still keeping $p \in U_p$. But this means that $\widetilde{f}$ vanishes to order one at every point on $V(\widetilde{f})$, that is, $V(\widetilde{f})$ is a regular subvariety of U_p. Since k is perfect, $V(\widetilde{f})$ is a smooth hypersurface of U_p, and since $\widetilde{f} \in (\mathcal{D}^{\leq a-1}_{X/k}\mathcal{I})|_{U_p}$, we have $V(\widetilde{f})$ contains the top locus restricted to U. Thus $V(\widetilde{f})$ is a local maximal contact hypersurface on U_p.

By noetherianity of X, finitely many of the U_p's cover $V(\mathcal{I}, a)$. Add the open set $X \setminus V(\mathcal{I}, a)$ to this finite set of U_p's to obtain the desired open cover of X. □

10.2 Algorithm(s) for maximal contact hypersurfaces

We now demonstrate how to explicitly obtain such an open covering with maximal contact hypersurfaces.

Remark 5 (Smooth Hypersurface Computational Check)**.** *Let* $\operatorname{Spec} A$ *be a smooth variety over* k*. Let* $f \in A$*. Let* $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ *be the minimal primes of* A*, which can be computed in* Singular *using primary decomposition [13]. Because* $\operatorname{Spec} A$ *is smooth,* $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ *are precisely the prime ideals belonging to* 0*, so by Krull's Principal Ideal theorem,* $\operatorname{Spec} A/f$ *is a hypersurface of* $\operatorname{Spec} A$ *if and only if* $f \notin \bigcup_{i=1}^m \mathfrak{p}_i$ *(see [2, Proposition 4.7]). If the ground field* k *is perfect, the singular locus of any variety over* k *is explicitly computable using [9, Algorithm 5.7.8]. Thus when* k *is perfect, it can be verified computationally whether* $\operatorname{Spec} A/f$ *is a smooth hypersurface.*

Algorithm 3: Maximal Contact

Let k be a field of characteristic zero.

Input: $X = \operatorname{Spec} A$ a smooth variety over k of pure dimension n, where $A = k[x_1, \dots, x_N]/(f_1, \dots, f_r)$, and $\mathcal{I} \subset \mathcal{O}_X$ an ideal such that $a = \text{max-ord}\,\mathcal{I} < \infty$ (this can be checked with 2)

Output: Two finite lists $MC = \{(\operatorname{Spec} A[G_i^{-1}], F_i)\}_i$ and $E = \{\operatorname{Spec} A[g_j^{-1}]\}_j$, where $G_i, F_i, g_j \in k[x_1, \dots, x_N]$ such that

- $$\operatorname{Spec} A = \bigcup_i \operatorname{Spec} A[G_i^{-1}] \cup \bigcup_j \operatorname{Spec} A[g_j^{-1}]$$
- $\operatorname{Spec} A[G_i^{-1}]$ meets the top locus $V(\mathcal{I}, a)$
- $\operatorname{Spec} A[g_j^{-1}]$ does not meet the top locus $V(\mathcal{I}, a)$
- F_i is a local maximal contact hypersurface of $\mathcal{I}$ on $\operatorname{Spec} A[G_i^{-1}]$

Let $F_1, \dots, F_m \in k[x_1, \dots, x_N]$ generate the ideal $\mathcal{D}_{X/k}^{\leq a-1}\mathcal{I}$ (see Algorithm 1)

Using Remark 5, check whether any F_s is a smooth hypersurface on $\operatorname{Spec} A$, in which case we have a global maximal contact hypersurface

if *there exists* s *such that* $V(F_s) := \operatorname{Spec} A/F_s$ *is a smooth hypersurface of* $\operatorname{Spec} A$ **then**

  return $MC = \{(\operatorname{Spec} A, F_s)\},\ E = \varnothing$

Initialization

- $\mathrm{MC} = \varnothing$
- $\mathrm{E} = \varnothing$
- $J = [df_1 \cdots df_r]^T = (\frac{\partial f_i}{\partial x_j})_{ij}$
- $L = \{N - n$ by $N - n$ submatrices of J with determinant not in $(f_1, \dots, f_r)\}$

 L consists of $N - n$ by $N - n$ submatrices of the Jacobian J with nonzero determinant in A. Since $\operatorname{Spec} A$ is smooth and has pure dimension n, we have

$$\operatorname{Spec} A = \bigcup_{M \in L} \operatorname{Spec} A[\det M^{-1}]$$

For *each* $M \in L$

  As in Lemma 16, let

  - $h = \det M$
  - $\mathrm{ROW} \subset \{1, \dots, r\}$ and $\mathrm{COL} \subset \{1, \dots, N\}$ be the row and column indices that M involve as a submatrix of the Jacobian J
  - for each $j' \notin \mathrm{COL}$, let $D_{j'} \in \operatorname{Der}_k(k[x_1, \dots, x_N], k[x_1, \dots, x_N])$ be given by

$$D_{j'} = h\frac{\partial}{\partial x_{j'}} - \sum_{\substack{i \in \mathrm{ROW} \\ j \in \mathrm{COL}}} \frac{\partial f_i}{\partial x_{j'}} C_{ij} \frac{\partial}{\partial x_j}$$

Because $a = \text{max-ord}\,\mathcal{I}$, we have that

$$D^{\leq 1}_{A[h^{-1}]/k}(F_1, \ldots, F_m) = D^{\leq 1}_{A[h^{-1}]/k} D^{\leq a-1}_{A[h^{-1}]/k} I = D^{\leq a}_{A[h^{-1}]/k} I = (1)$$

By Lemma 16,

$$(F_1, \ldots, F_m) + \big(\{D_{j'}(F_1), \ldots, D_{j'}(F_m)\}_{j' \notin \text{COL}}\big) = A[h^{-1}]$$

Introducing a new variable y, we express $A[h^{-1}]$ as follows:

$$A[h^{-1}] = \frac{k[x_1, \ldots, x_N, y]}{(f_1, \ldots, f_r, 1 - yh)}$$

Using [4, 5.1.78 lift] in Singular, we can then compute polynomials $a_s, b_{sj'} \in k[x_1, \ldots, x_N, y]$ such that

$$\sum_{s=1}^{m} a_s F_s + \sum_{j' \notin \text{COL}} \sum_{s=1}^{m} b_{sj'} D_{j'}(F_s) \equiv 1 \mod (f_1, \ldots, f_r, 1 - yh)$$

Thus we have

$$\operatorname{Spec} A[h^{-1}] = \bigcup_{a_s \neq 0 \text{ in } A[h^{-1}]} \operatorname{Spec} A[h^{-1}, F_s^{-1}] \;\cup \bigcup_{b_{sj'} \neq 0 \text{ in } A[h^{-1}]} \operatorname{Spec} A[h^{-1}, D_{j'}(F_s)^{-1}]$$

• Since $F_s \in \mathcal{D}^{\leq a-1}_{X/k}\mathcal{I}$, the top locus $V(\mathcal{I}, a) = V(\mathcal{D}^{\leq a-1}_{X/k}\mathcal{I})$ does not meet $\operatorname{Spec} A[h^{-1}, F_s^{-1}] = \operatorname{Spec} A[(hF_s)^{-1}]$.

for *each s such that $a_s \neq 0$ in $A[h^{-1}]$* **do**

$$\text{E} = \text{E}, \big\{ \operatorname{Spec} A[(hF_s)^{-1}]\big\}$$

• By Algorithm A.10, let $e_1, \ldots, e_d \in k[x_1, \ldots, x_N]$ be orthogonal idempotents of A. So $\operatorname{Spec} A$ has d irreducible components, and its tth component can be obtained by localizing at e_t.

for *each $j' \notin COL$* **do**

for *each s such that $b_{sj'} \neq 0$ in $A[h^{-1}]$* **do**

• F_s is smooth on $\operatorname{Spec} A[h^{-1}, D_{j'}(F_s)^{-1}]$ by [26, Tag 07PF].

for *each t such that F_s is nonzero on* $\operatorname{Spec} A[h^{-1}, D_{j'}(F_s)^{-1}, e_t^{-1}]$ **do**

• By Krull's Principal Ideal theorem, F_s is a hypersurface on $\operatorname{Spec} A[h^{-1}, D_{j'}(F_s)^{-1}, e_t^{-1}] = \operatorname{Spec} A\big[(hD_{j'}(F_s)e_t)^{-1}\big]$.

• Since $F_s \in \mathcal{D}^{\leq a-1}_{X/k}\mathcal{I}$, we have that F_s is a local maximal contact hypersurface of $\mathcal{I}$ on $\operatorname{Spec} A\big[(hD_{j'}(F_s)e_t)^{-1}\big]$

$$\text{MC} = \text{MC}, \{(\operatorname{Spec} A\big[(hD_{j'}(F_s)e_t)^{-1}\big], F_s)\}$$

return MC, E

The following algorithm will be needed in Section 11.2 for Algorithm 5.

Algorithm 4: Maximal Contact on Subvariety

Let k be a field of characteristic zero.

Input: • $A = k[x_1, \ldots, x_N]/(f_1, \ldots, f_r)$ such that $X = \operatorname{Spec} A$ is a smooth variety over k of pure dimension,

• polynomials $h_1, \ldots, h_m \in k[x_1, \ldots, x_N]$ such that $Z := \operatorname{Spec} A/(h_1, \ldots, h_m)$ is also a smooth variety over k of pure dimension

• $\mathcal{I} \subset \mathcal{O}_Z$ a coherent ideal sheaf on Z such that $a = \operatorname{max-ord} \mathcal{I} < \infty$

Output: Two finite lists $\mathrm{MC}_{\mathrm{lift}} = \{(\operatorname{Spec} A[G_i^{-1}], F_i)\}_i$ and $\mathrm{E}_{\mathrm{lift}} = \{\operatorname{Spec} A[g_j^{-1}]\}_j$, where $G_i, F_i, g_j \in k[x_1, \ldots, x_N]$ such that

•
$$\operatorname{Spec} A = \bigcup_i \operatorname{Spec} A[G_i^{-1}] \cup \bigcup_j \operatorname{Spec} A[g_j^{-1}]$$

• $(\operatorname{Spec} A[G_i^{-1}])\big|_Z$ meets the top locus $V(\mathcal{I}, a)$

• $(\operatorname{Spec} A[g_j^{-1}])\big|_Z$ does not meet the top locus $V(\mathcal{I}, a)$

• F_i is a local maximal contact hypersurface of $\mathcal{I}$ on $(\operatorname{Spec} A[G_i^{-1}])\big|_Z$

Let $B = k[x_1, \ldots, x_N]/(f_1, \ldots, f_r, h_1, \ldots, h_m)$.

initialization

• $\mathrm{MC}_{\mathrm{lift}} = \varnothing$

• $\mathrm{E}_{\mathrm{lift}} = \varnothing$

• Obtain MC and E by applying Algorithm 3 on $\operatorname{Spec} k[x_1, \ldots, x_N]/(f_1, \ldots, f_r, h_1, \ldots, h_m)$.

for *each* $(\operatorname{Spec} B[G^{-1}], F) \in MC$ **do**

$$\mathrm{MC}_{\mathrm{lift}} = \mathrm{MC}_{\mathrm{lift}}, \{(\operatorname{Spec} A[G^{-1}], F)\}$$

for *each* $\operatorname{Spec} B[g^{-1}] \in E$ **do**

$$\mathrm{E}_{\mathrm{lift}} = \mathrm{E}_{\mathrm{lift}}, \{\operatorname{Spec} A[g^{-1}]\}$$

Note that the complement of Z in X is covered by $\operatorname{Spec} A[h_1^{-1}], \ldots, \operatorname{Spec} A[h_m^{-1}]$. Thus the restrictions of these distinguished opens to Z are empty and hence do not meet the top locus $V(\mathcal{I}, a)$.

for *each* $i = 1, \ldots, m$ **do**

$$\mathrm{E}_{\mathrm{lift}} = \mathrm{E}_{\mathrm{lift}}, \{\operatorname{Spec} A[h_i^{-1}]\}$$

return MC_{lift}, E_{lift}

---

11 The Lexicographic Order Invariant, the Associated Parameters, and their Explicit Computation

11.1 General theory

Let k be a field of characteristic zero. Let X be a smooth variety over k of pure dimension n.

Definition 27 (Coefficient Ideal, [7, Definition 57])**.** *Let A be a k-algebra. Let $I \subset A$ be an ideal* *and $b > 0$ an integer. Define the coefficient ideal*

$$C_A(I,b) := \sum_{i=0}^{b-1} \left(D^{\leq i}_{A/k} I\right)^{b!/(b-i)} \subset A$$

If X is a variety over k and $\mathcal{I} \subset X$ an coherent sheaf of ideals, then define the coefficient ideal

$$\mathcal{C}_X(\mathcal{I},b) := \sum_{i=0}^{b-1} \left(\mathcal{D}^{\leq i}_{X/k} I\right)^{b!/(b-i)} \subset \mathcal{O}_X$$

Remark 6. *In [1, Definition 4.1.1], they use the following larger coefficient ideal*

$$\mathcal{C}(\mathcal{I},b) := \sum_{\sum_{i=1}^{b-1}(b-i)s_i \geq b!} I^{s_0} \cdot (D^{\leq 1}_{X/k}\mathcal{I})^{s_1} \cdots (D^{\leq b-1}_{X/k}\mathcal{I})^{s_{b-1}}$$

where the sum runs over all monomials n the ideals $\mathcal{I}, \dots, \mathcal{D}^{\leq b-1}_{X/k}\mathcal{I}$ of weighted degree

$$\sum_{i=1}^{b-1}(b-i)s_i \geq b!$$

Observe that $\mathcal{C}_X(I,b) \subset \mathcal{C}(I,b)$. There is no problem using this smaller coefficient ideal. The larger ideal, also called the Włodarczyk ideal or $W(\mathcal{I})$ (see [14, 3.54.1] and [25]), is useful for theoretical proofs (see [1]). However, It is more efficient to utilize the smaller coefficient ideal $\mathcal{C}_X(\mathcal{I},b)$ for computational resolution of singularities.

For the following definition, we use the convention that the positive natural numbers are $\mathbb{N}_{>0} = \{1,2,\dots\}$ and the nonnegative rational numbers are $\mathbb{Q}_{\geq 0}$.

Definition 28 (See [1, Section 5])**.** *Let*

$$\mathbb{Q}^{\leq n}_{\geq 0} := \bigsqcup_{0 \leq k \leq n} \mathbb{Q}^k_{\geq 0}$$

be the disjoint union of the products $\mathbb{Q}^k$ for $k = 1, \dots, n$. Order elements of $\mathbb{Q}^{\leq n}_{\geq 0}$ lexicographically, with truncated sequences considered larger. For example,

$$(1,1,1) < (1,1,2) < (1,2,1) < (1,2) < (2,2,1) < ()$$

where $() \in \mathbb{Q}^0_{\geq 0}$ is the empty sequence. In this way, $\mathbb{Q}^{\leq n}_{\geq 0}$ becomes totally ordered. Also, define

$$\mathbb{N}^{\leq n}_{>0} := \bigsqcup_{k=0}^{n} \mathbb{N}^k_{>0}$$

ordered lexicographically, where truncated sequences are considered larger, just as with $\mathbb{Q}^{\leq n}_{\geq 0}$. In this way, $\mathbb{N}^{\leq n}_{>0}$ becomes a well-ordered set.

Let $\mathcal{I} \subset \mathcal{O}_X$ be an ideal sheaf. We will define a function called the b-invariant with respect to $\mathcal{I}$ taking values in a well-ordered set

$$\begin{aligned} b\text{-inv}(\mathcal{I}) : X &\to \mathbb{N}^{\leq n}_{>0} \\ p &\mapsto b\text{-inv}_p(\mathcal{I}) \end{aligned}$$

From this we will define the lexicographic order invariant

$$\begin{aligned} \mathrm{inv}(\mathcal{I}) : X &\to \mathbb{Q}_{\geq 0}^{\leq n} \\ p &\mapsto \mathrm{inv}_p(\mathcal{I}) \end{aligned}$$

We will need to consider the following function

Definition 29 (b-to-a function)**.** *The function b-to-a* $: \mathbb{N}_{>0}^{\leq n} \to \mathbb{Q}_{\geq 0}^{\leq n}$ *sends the empty sequence* $()$ *to itself and* $(b_1, ..., b_k) \in \mathbb{N}_{>0}^{\leq n}$ *to* $(a_1, \ldots, a_k) \in \mathbb{Q}_{\geq 0}^{\leq n}$*, where* $a_1 = b_1$ *and* $a_i = b_i/(b_{i-1} - 1)!$ *for* $i > 1$*.*

We will also associate to every point in X a sequence of regular parameters.

Definition 30 (Lexicographic order invariant, b-invariant, and associated parameters, see [1])**.** *Let* $p \in X$*. Let* $\mathcal{I}[1] := \mathcal{I}$*. Set*

$$b_1 := \mathrm{ord}_{\mathcal{O}_{X,p}}(\mathcal{I}[1])$$

If $b_1 = \infty$*, set b-inv*$_p(\mathcal{I}) = ()$ *to be the empty sequence. Otherwise, let* $x_1 \in \mathcal{O}_{X,p}$ *be a maximal contact element of* $\mathcal{I}[1]$ *at* p*. Inductively, write*

$$\mathcal{I}[i+1] := C_{\mathcal{O}_{X,p}}(\mathcal{I}[i], b_i) \subset \mathcal{O}_{X,p}$$

Set

$$b_{i+1} = \mathrm{ord}_{\mathcal{O}_{X,p}/(x_1,\ldots,x_i)}(\mathcal{I}[i+1]|_{V(x_1,\ldots,x_i)})$$

If $b_{i+1} = \infty$*, set b-inv*$_p(\mathcal{I}) = (b_1, \ldots, b_i)$ *and associate the sequence of parameters* $(x_1, \ldots, x_i)$ *to the point* p*. Otherwise, let* $x_{i+1} \in \mathcal{O}_{X,p}$ *be the lift of a maximal contact element of* $\mathcal{I}[i+1]|_{V(x_1,\ldots,x_i)}$ *as an ideal of the local ring* $\mathcal{O}_{X,p}/(x_1, \ldots, x_{i+1})$*. Define the lexicographic order invariant by*

$$\mathit{inv}_p(\mathcal{I}) := (\mathit{b\text{-}to\text{-}a})(\mathit{b\text{-}inv}_p(\mathcal{I}))$$

Note that the associated sequence of parameters $(x_1, \ldots, x_i)$ form linearly independent vectors in the Zariski cotangent space $\mathfrak{m}_p/\mathfrak{m}_p^2$ at p. Also, note that $\mathrm{inv}_p \mathcal{I} = ()$ if and only if $\mathcal{I}$ vanishes on the irreducible component of X containing p.

Definition 31 (Maximal lexicographic order invariant, see [1])**.** *Let* $\mathcal{I} \subset \mathcal{O}_X$ *be a coherent ideal sheaf on* X*. Then define the maximal lexicographic order invariant by*

$$\mathrm{maxinv}\, \mathcal{I} = \max_{p \in X} \mathrm{inv}_p \mathcal{I}$$

Proposition 26 (See [1, Section 1.1 and Theorem 5.1.1])**.** *The lexicographic order invariant of a coherent ideal sheaf* $\mathcal{I} \subset \mathcal{O}_X$ *is independent of the choices and is upper-semicontinuous. Suppose that* $V(\mathcal{I})$ *is pure codimension* c *in* X *(see Definition 25). Then inv*$_p(\mathcal{I}) = (1, \ldots, 1)$ *is the constant sequence of ones with length* c *iff* $V(\mathcal{I})$ *is regular at* p*, and otherwise the invariant is bigger.*

Proof. See [1, Section 1.1 and Theorem 5.1.1]. $\square$

11.2 Algorithm for maximal lexicographic order invariant

We now show how to explicitly compute the maximal lexicographic invariant and the associated parameters on some open cover. The algorithm proceeds as a competition among open sets covering the smooth variety. Every descent in dimension winnows the contenders into losers and the remaining contenders, until eventually terminating once the maximal lexicographic invariant is found, in which case the surviving contenders are now winners.

Algorithm 5: Maximal lexicographic order invariant and associated parameters

---

Let k be a field of characteristic zero.

Input: X a smooth variety over k of pure dimension covered by open affines $\operatorname{Spec} A_1, \ldots, \operatorname{Spec} A_N$, and $\mathcal{I} \subset \mathcal{O}_X$ a coherent ideal sheaf

Output: $(a_1, \ldots, a_r) \in \mathbb{Q}_{\geq 0}^{\leq n}$ and finite lists WINNERS $= \{(U_i, [f_{i1}, ..., f_{ir}])\}_i$ and LOSERS $= \{W_j\}_j$, such that

- U_i and W_j are affine opens that cover X
- $\operatorname{maxinv} \mathcal{I} = (a_1, \ldots, a_r)$
- the maximal lexicographic order invariant is attained on each U_i but not on any W_j
- $f_{i1}, ..., f_{ir} \in \mathcal{O}(U_i)$
- if $p \in U_i$ is a point at which the maximal lexicographic order is attained, i.e. $\operatorname{inv}_p \mathcal{I} = (a_1, \ldots, a_r)$, then the image of the ordered sequence $[f_{i1}, \ldots, f_{ir}]$ in $\mathcal{O}_{X,p}$ is an associated sequence of parameters (see Definition 30)

initialization

- maxbinv = ()
- maxinvfound = FALSE
- CONTENDERS $= \{(\operatorname{Spec} A_1, \mathcal{I}|_{\operatorname{Spec} A_1}, [\ ]\ ), \ldots, (\operatorname{Spec} A_N, \mathcal{I}|_{\operatorname{Spec} A_N}, [\ ]\ )\}$
- WINNERS $= \varnothing$
- LOSERS $= \varnothing$

(continue to next page for the rest of the algorithm)

---

while *NOT maxinvfound* **do**

bmax $= -\infty$

for $(U, I, [h_1, \ldots, h_m]) \in \mathit{CONTENDERS}$ **do**

- $h_1, \ldots, h_m \in \mathcal{O}(U)$ are smooth hypersurfaces on the affine open $U \subset X$
- $Z := \operatorname{Spec} \mathcal{O}(U)/(h_1, \ldots, h_m)$ is a smooth pure codimension m subvariety
- $I \subset \mathcal{O}_Z$ is an ideal

bcurr = maxinv I, the maximal lexicographic order invariant of I on Z

bmax $= \max\{\text{bmax}, \text{bcurr}\}$

SURVIVORS $= \varnothing$

for $(U, I, [h_1, \ldots, h_m]) \in \mathit{CONTENDERS}$ **do**

if maxinv $I = \mathit{bmax}$ **then**

SURVIVORS = SURVIVORS, $\{(U, [h_1, \ldots, h_m])\}$

else

LOSERS = LOSERS, $\{U\}$

if $\mathit{bmax} = \infty$ **then**

maxinvfound = TRUE

for $(U, I, [h_1, \ldots, h_m]) \in \mathit{SURVIVORS}$ **do**

WINNERS = WINNERS, $\{(U, [h_1, \ldots, h_m])\}$

else

maxbinv = maxbinv, bmax

NEWCONTENDERS $= \varnothing$

for $(U, I, [h_1, \ldots, h_m]) \in \mathit{SURVIVORS}$ **do**

- $Z := \operatorname{Spec} \mathcal{O}(U)/(h_1, \ldots, h_m)$
- $I \subset \mathcal{O}_Z$
- Apply Algorithm 4 on the closed embedding $Z \hookrightarrow U$, obtaining $\{(U_i, f_i)\}_i$, $\{W_j\}_j$ such that U is covered by the U_i's and W_j's, and f_i is a local maximal contact hypersurface of I on $U_i\big|_Z$
- For each i, let $V(f_i) := \operatorname{Spec} \mathcal{O}(U)/(h_1, \ldots, h_m, f_i)$

NEWCONTENDERS = NEWCONTENDERS, $\{(U_i, C_{V(f_i)}(I, \text{bmax}), [h_1, \ldots, h_m, f_i])\}_i$

LOSERS = LOSERS, $\{W_j\}_j$

CONTENDERS = NEWCONTENDERS

return *WINNERS, LOSERS,* $(b\text{-}to\text{-}a)(\mathit{maxbinv})$*; see Definition 29*

12 Algorithmic Weighted Resolution of Singularities

12.1 Weighted blowing up algorithm

We now describe the algorithm that partially computes the stack-theoretic weighted blowing up of a variety along a reduced center (see [1]).

Algorithm 6: Weighted blowup along reduced center

Input: $\operatorname{Spec} A$ a smooth irreducible variety over k
$f_1, \ldots, f_r \in k[x_1, \ldots, x_N]$
positive integers $w_1, \ldots, w_r$
positive integer c, called the control
$I \subset A$ an ideal
such that $(f_1^{1/w_1}, \ldots, f_r^{1/w_r})$ is a reduced center on $\operatorname{Spec} A$ (see [1])

Output: a list $CHARTS = \{(U_i, I_i)\}_{i=1}^r$, where $U_i = \operatorname{Spec} A_i$ and $I_i \subset A_i$ is an ideal such that U_i cover the weighted blowup of $\operatorname{Spec} A$ along the reduced center $(f_1^{1/w_1}, \ldots, f_r^{1/w_r})$ and I_i is the strict transform of the closed subscheme $V(I) = \operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A$ restricted to U_i.

Remark: it remains to compute the data of the group action on the charts U_i, in which case we will have completely described the stack-structure of the weighted blowing up of $\operatorname{Spec} A$ along the reduced center

Initialization
- CHARTS $= \varnothing$

Define the A-algebra map $\phi : A[y_1, \ldots, y_r] \to A[T]$ give by $y_i \mapsto f_i T^{w_i}$
Let $\widetilde{A}$ be the integral closure of $A[y_1, \ldots, y_r]/\ker\phi$ (see [10] and [4, 5.1.65 kernel])
for *each* $i = 1, \ldots, r$ **do**
- Let $A_i = \widetilde{A}/(y_i - 1)$
- Then $U_i = \operatorname{Spec} A_i$ is the ith chart of the weighted blowup of $\operatorname{Spec} A$ along the reduced center $(f_1^{1/w_1}, \ldots, f_r^{1/w_r})$
- Let E_i be the extension of the ideal $(f_1, \ldots, f_r) \subset A$ in A_i
- Let Tot_i be the extension of the ideal $I \subset A$ in A_i
- E_i is the restriction of the exceptional divisors to the chart U_i and Tot_i is the restriction of the total transform of $V(I) \hookrightarrow \operatorname{Spec} A$ to the chart U_i
- Define the ideal $I_i = (\mathrm{Tot}_i : E_i^\infty) \subset A_i$, where we remove the exceptional divisors up to all multiplicities
- CHARTS = CHARTS, $\{(U_i, I_i)\}$

return *CHARTS*

Elimination methods provided by the library [10] in Singular can reduce the number of variables in A_i. Taking the integral closure $\widetilde{A}$ may adjoin many variables to A, but usually almost all of them can be eliminated in the quotient $A_i = \widetilde{A}/(y_i - 1)$.

12.2 Weighted resolution algorithm

Let X be a smooth variety of pure dimension n over a field k of characteristic zero and $Z \hookrightarrow X$ a pure codimension c subvariety. The goal is to resolve singularities of Z embedded in X. See [1, Theorem 1.1.1].

We now present an explicitly computable stack-theoretic resolution of singularities!

Algorithm 7: Weighted Resolution

Let k be a field of characteristic zero.

Input: X a smooth irreducible variety over k,
$Z \hookrightarrow X$ a pure codimension c reduced subvariety

Output: A list EVOLUTION of size $n+1$, such that the ith entry EVOLUTION[i] (using zero indexing) for $i = 0, \ldots, n$ is a collection of charts of the pair $(Z_i \subset X_i)$ in

$$(Z_n \subset X_n) \to (Z_{n-1} \subset X_{n-1}) \to \cdots \to (Z_0 \subset X_0) = (Z \subset X)$$

where
• $(Z_i \subset X_i) = F_{er}(Z_{i-1} \subset X_{i-1})$. The functor F_{er} is described in [1, Theorem 1.1.1].
• $(Z_i \subset X_i) \to (Z_{i-1} \subset X_{i-1})$ is the weighted blowup of X_i along the reduced center associated to Z_{i-1} and Z_i is the proper transform of Z_{i-1} (see [1, Section 5]) and n the smallest integer for which Z_n is smooth

Initialization
• YEAR = 0
• singularities_resolved = FALSE
• EVOLUTION[YEAR] = $(Z \subset X)$

if *Z is smooth* **then**
  return *EVOLUTION*

• $(\widetilde{Z} \subset \widetilde{X}) = (Z \subset X)$

while *NOT singularities_resolved* **do**
  • YEAR = YEAR+1
  • $(\widetilde{Z} \subset \widetilde{X})$ is the collection of charts that cover the embedded pair of stacks $\widetilde{Z} \subset \widetilde{X}$
  • Let $\mathcal{I} \subset \mathcal{O}_{\widetilde{X}}$ be the ideal of $\widetilde{Z} \hookrightarrow \widetilde{X}$
  • Use Algorithm 5 to obtain the maximal lexicographical order invariant and associated parameters on $\widetilde{X}$ associated to $\mathcal{I}$. Let $\operatorname{maxinv} \mathcal{I} = (a_1, \ldots, a_r)$ be the maximal lexicographical order invariant. In the notation of Algorithm 5, if $U \subset \widetilde{X}$ is a chart with associated parameters $[f_1, \ldots, f_r]$, then the reduced center associated to $\mathcal{I}$ on U is

$$(f_{i1}^{1/w_1}, \ldots, f_{ir}^{1/w_r})$$

  where $(1/w_1, \ldots, 1/w_r) = (a_1, \ldots, a_r)/\operatorname{lcm}(a_1, \ldots, a_r)$ (see [1]).
  • Use Algorithm 8 to compute the charts of the weighted blowup along the determined reduced center of $\widetilde{X}$ as well as the proper transform of $\widetilde{Z}$, and let $(\widetilde{Z}_{\text{new}} \subset \widetilde{X}_{\text{new}})$ represent this data of charts and proper transform.
  • EVOLUTION[YEAR] = $(\widetilde{Z}_{\text{new}} \subset \widetilde{X}_{\text{new}})$
  • Let $\mathcal{I}_{\text{new}}$ be the ideal of $\widetilde{Z}_{\text{new}} \hookrightarrow \widetilde{X}_{\text{new}}$.
  • Use Algorithm 5 to obtain $\operatorname{maxinv} \mathcal{I}_{\text{new}}$
  • The proper transform $\widetilde{Z}_{\text{new}}$ is smooth if and only $\operatorname{maxinv} \mathcal{I}_{\text{new}} = (1, \ldots, 1)$ is a constant sequence of ones with length the codimension c
  if *$\widetilde{Z}_{new}$ is smooth* **then**
    singularities_resolved = TRUE
  • $(\widetilde{Z} \subset \widetilde{X}) = (\widetilde{Z}_{\text{new}} \subset \widetilde{X}_{\text{new}})$

return *EVOLUTION*

A Appendix

A.1 Saturation and Zariski Closure

Definition 32 (Scheme-Theoretic Image and Closure)**.** *Let $f : X \to Y$ be a morphism of schemes. The scheme-theoretic image of f is the smallest closed embedding $Z \hookrightarrow Y$ through which f factors through. The scheme-theoretic image Z is cut out by the sum of all quasicoherent ideals contained in $\ker(\mathcal{O}_Y \to f_*\mathcal{O}_X)$. If $f : X \to Y$ is a locally closed embedding, then we call the scheme-theoretic image of f the scheme-theoretic closure or the Zariski closure, and we denote the Zariski closure of $X \hookrightarrow Y$ by $\overline{X}$.*

If $B \to A$ is a ring map, then it is not hard to show that the scheme-theoretic image of $\operatorname{Spec} A \to \operatorname{Spec} B$ is cut out of $\operatorname{Spec} B$ by by the kernel of $B \to A$.

Lemma 18. *Let $f : X \to Y$ be a morphism of ringed spaces and let $X = \bigcup_{i \in I} X_i$ be an open covering of X by open sets X_i indexed by $i \in I$. then*

$$\bigcap_{i \in I} \ker\left(\mathcal{O}_Y \to (f|_{X_i})_*\mathcal{O}_{X_i}\right) = \ker(\mathcal{O}_Y \to f_*\mathcal{O}_X)$$

Proof. This directly follows from the definition of sheaves. □

Corollary 10. *Let $f : X \to Y$ be a morphism of schemes and let $X = \bigcup_{i=1}^r X_i$ be an open covering of X by finitely many open subschemes X_i. If $\mathcal{I} \subset \mathcal{O}_Y$ cuts out the scheme-theoretic image of $f : X \to Y$ and $\mathcal{I}_i \subset \mathcal{O}_Y$ cuts out the scheme-theoretic image of $f|_{X_i} : X_i \hookrightarrow X \to Y$, then $\mathcal{I} = \bigcap_{i=1}^r \mathcal{I}_i$.*

Proof. Just note that the finite intersection of quasicoherent ideals is still quasicoherent. □

Definition 33 (Saturation)**.** *Let A be a ring and $I, J \subset A$ ideals. Define the saturation $(I : J^\infty)$ of I with respect to J to be the following ideal*

$$(I : J^\infty) := \bigcup_{n=1}^{\infty} (I : J^n)$$

If $h \in A$, then define $(I : h^\infty) := (I : (h)^\infty)$. Saturation can be computed in Singular *[4, A.3.1 Saturation].*

Note that $(I : h^\infty) = \ker(A \to (A/I)[h^{-1}])$.

For the following proposition, let *o.e.* and *c.e.* stand for open and closed embedding, respectively, and note that by [23] Exercise 8.1.M, morphisms that factor as an open embedding followed by a closed embedding is a locally closed embedding.

Proposition 27 (Geometric Interpretation of Saturation)**.** *Let A be a ring and $I, J \subset A$ ideals, where J is a finitely generated ideal. Then the Zariski closure of $V(I) \setminus V(J) \xhookrightarrow{o.e.} V(I) \xhookrightarrow{c.e.} \operatorname{Spec} A$ is cut out by $(I : J^\infty) \subset A$.*

Proof. First suppose that J is a principal ideal generated by $h \in A$. The open embedding $V(I) \setminus V(h) \hookrightarrow V(I)$ is precisely the inclusion of the distinguished affine open $\operatorname{Spec}(A/I)[h^{-1}] \hookrightarrow \operatorname{Spec} A/I$. Thus the Zariski closure of $V(I) \setminus V(h)$ in $\operatorname{Spec} A$ is cut out by $\ker(A \to (A/I)[h^{-1}]) = (I : h^\infty)$.

Now suppose that $J = (h_1, \ldots, h_r)$ is finitely generated and not necessarily principal. Then

$$V(I) \setminus V(J) = V(I) \setminus \bigcap_{i=1}^{r} V(h_r) = \bigcup_{i=1}^{r} V(I) \setminus V(h_r)$$

so that $V(I) \setminus V(J)$ is covered by finitely many open affines $V(I) \setminus V(h_r)$. Thus the Zariski closure of $V(I) \setminus V(J)$ in $\operatorname{Spec} A$ is cut out by $\bigcap_{i=1}^r (I : h_i^\infty)$. We claim that

$$\bigcap_{i=1}^{r} (I : h_i^\infty) = (I : (h_1, \dots, h_r)^\infty)$$

To see this, first let $f \in (I : (h_1, \dots, h_r)^\infty)$. Then $f(h_1, \dots, h_r)^n \subset I$ for some n, so that $f \in (I : h_i^n)$ for every i. Conversely, suppose $f \in \bigcap_{i=1}^r (I : h_i^\infty)$. Then for sufficiently large N, we have $fh_i^N \in I$ for each i, so that $f \in (I : (h_1, \dots, h_r)^{rN})$. Thus we are done. □

Lemma 19 (Gluing Ideals)**.** *Let A be a ring and $h_1, \dots, h_r \in A$ be elements generating the unit ideal, i.e. $(h_1, \dots, h_r) = A$. Let $I \subset A$ and $I_i \subset A$ for $i = 1, \dots, r$ be ideals such that I and I_i have the same extension in $A[h_i^{-1}]$ for each i. Then*

$$I = \bigcap_{i=1}^{r} (I_i : h_i^\infty)$$

Proof. First we give a geometric picture. Since $V(I)$ and $V(I_i)$ agree on $D(h_i)$, we have $V(I) \subset \cup V(I_i)$. There may be components of $V(I_i)$ that lie entirely in $V(h_i)$ that do not appear in $V(I)$. So we remove these extraneous components by taking the Zariski closure of $V(I_i) \setminus V(h_i)$. Thus $V(I) = \bigcup_{i=1}^r \overline{V(I_i) \setminus V(h_i)}$. Scheme-theoretically, this translates to precisely the algebraic statement.

Now the algebraic proof. If $f \in I$, then because I and I_i have the same extension in $A[h_i^{-1}]$, there is N_i such that $fh_i^{N_i} \in I_i$, hence $f \in (I_i : h_i^\infty)$.

Conversely, assume that $f \in \bigcap_{i=1}^r (I_i : h_i^\infty)$. Then for some sufficiently high N, we have $fh_i^N \in I_i$ for each i. Since I and I_i have the same extension in $A[h_i^{-1}]$, by taking sufficiently larger N, we have $fh_i^N \in I$ for each i. Because $(h_1, \dots, h_r) = A$, there exists $a_i \in A$ such that $\sum_{i=1}^r a_i h_i^N = 1$ ("partition of unity" trick). Thus

$$f = \sum_{i=1}^{r} a_i f h_i^N \in I$$

□

A.2 Orthogonal Idempotents

Let $\mathfrak{a}_1, \dots, \mathfrak{a}_n \subset A$ be ideals such that $\operatorname{Spec} A$ is the (scheme-theoretic) disjoint union of its closed subschemes $V(\mathfrak{a}_i) := \operatorname{Spec} A/\mathfrak{a}_i$

$$\operatorname{Spec} A = \bigsqcup_{i=1}^{n} V(\mathfrak{a}_i)$$

This means we are in the situation of [2, Proposition 1.10]. Thus for each $i = 1, \dots, n$, we have

$$\mathfrak{a}_i + \sum_{j \neq i} \mathfrak{a}_j = (1)$$

So there exists $e_i \in \sum_{j \neq i} \mathfrak{a}_j$ such that $e_i \equiv 1 \mod \mathfrak{a}_i$. Then $\phi(e_i)$ is equal to the standard ith coordinate vector $\phi(e_i) = (\dots, 0, 1 + \mathfrak{a}_i, 0, \dots)$. Thus $A[e_i^{-1}] = A/\mathfrak{a}_i$. The e_i's are called orthogonal idempotents (see [15, Exercise 2.19]). This leads us to the following algorithm that explicitly computes the orthogonal idempotents on a smooth affine variety.

Algorithm 8: Orthogonal idempotents on a Smooth affine variety

Input: $I \subset k[x_1, \ldots, x_N]$ an ideal such that $\operatorname{Spec} A$ a smooth variety over k, where $A = k[x_1, \ldots, x_N]/I$.

Output: Polynomials $e_1, \ldots, e_n \in k[x_1, \ldots, x_N]$ such that the irreducible components of $\operatorname{Spec} A$ are $\operatorname{Spec} A[e_1^{-1}], \ldots, \operatorname{Spec} A[e_n^{-1}]$

$$\operatorname{Spec} A = \bigsqcup_{i=1}^{n} \operatorname{Spec} A[e_i^{-1}]$$

Begin:

Using primary decomposition (see [13]), compute the minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \subset k[x_1, \ldots, x_N]$ of I. Because $\operatorname{Spec} A$ is a smooth variety, its irreducible components are disjoint. So scheme-theoretically we have

$$\operatorname{Spec} k[x_1, \ldots, x_N]/I = \bigsqcup_{i=1}^{n} \operatorname{Spec} \frac{k[x_1, \ldots, x_N]}{I + \mathfrak{p}_i}$$

Thus for each i, we have that

$$I + \mathfrak{p}_i + \sum_{j \neq i} \mathfrak{p}_j = (1)$$

Using [4, 5.1.75 lift] from SINGULAR, we can then compute a polynomial $e_i \in I + \sum_{j \neq i} \mathfrak{p}_j$ such that $e_i \equiv 1 \mod (\mathfrak{p}_i + I)$.

return $e_1, \ldots, e_n \in k[x_1, \ldots, x_N]$

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