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Honors Thesis

Quantifying the Inhomogeneous Distribution of Matter by Perturbing the Smooth Universe

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1 An Introduction to Cosmology

The study of the large-scale structure of the universe is one that has always fascinated cosmologists. Since the beginning of human existence, people have been staring to the heavens and contemplating the cosmos. Indeed, we have come far in our understanding of the universe. From Copernicus to Einstein, physicists have combined mathematical tools with physical insight to explain our experimental observations of the stars. These innovations in cosmology have culminated in the theory today known as the Standard Model.

The Standard Model is a powerful cosmological model that explains key properties of universe. It will provide the foundation for the theory that we will develop later in this paper. My goal in this section is to introduce the Standard Model and outline some of its important physical consequences. We will focus in particular on the smooth expansion of the universe predicted by the Standard Model.

1.1 The Smooth, Expanding Universe

The expansion of the universe is a well-documented fact. There is ample evidence to suggest the distance between two points in the universe grows with time. To quantify this effect, we employ a scale factor $a(t)$. The scale factor relates the coordinate distance to physical distance in the expansion of the universe. If we represent space in the universe as a coordinate grid, then the co-moving distance is the distance between two points on the grid. This distance remains constant. The physical distance, however, is the co-moving distance between two points multiplied by the scale factor. Thus, the scale factor quantifies the expansion of the universe. It expands in time, allowing us to convert coordinate distance into the physical distance of the universe. By convention, the value of the scale factor today is set to unity.

An important distance scale that relates to the scale factor is conformal time. Conformal time is defined as the co-moving distance that light could travel in a given time interval. By convention, the speed of light is set to 1. Thus, the physical distance light travels is $dx_1 = dt$ in a time interval dt. The co-moving distance light travels $dx_2 = adx_1$ where dx_2 is the infinitesimal co-moving distance. Thus, the co-moving distance light travels in an infinitesimal time interval dt is equal to $dx_2 = dt/a$. The total conformal time is given by

$$\eta = \int_0^t \frac{dt}{a(t)} \tag{1}$$

This is a very important distance because no information can travel faster than the speed of light. Conformal time creates cone of causality. Any two points separated by a co-moving distance greater than η cannot be causally connected. Points within the cone of causality can communicate, those outside of it cannot. For this reason, conformal time is understood as a co-moving horizon. We will often employ of conformal time as a substituted time parameter to simplify equations later in this paper.

The universe does not expand at a constant rate. The evolution of the scale factor is a complicated function of time. It is useful to introduce the Hubble rate $H(t)$, which quantifies how the scale factor evolves with time. The Hubble rate is defined by

$$H(t) = \frac{\dot{a}}{a} \tag{2}$$

The Hubble rate quantifies how the scale factor evolves in time. It has units of s^{-1}, and is by definition intimately connected to the expansion of the universe. The Hubble rate is a useful quantity in probing the age of the universe.

To understand the smooth, expanding universe, it is necessary to have a grasp of the underlying physical processes that govern its expansion. General relativity is just one of these essential processes in cosmology. It relates gravity, whose source is matter and energy, to the structure of space-time in the universe. With general relativity, we are able to describe the expansion of the universe by considering the matter and energy density of the universe.

General relativity has two important axioms. The first is that gravity is incorporated into the structure of space-time and can be described by a metric. The metric converts coordinate dependent quantities into invariants.

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{3}$$

Here, ds^2 represents the invariant interval or proper time, and dx^μ represents a 4-dimensional coordinate point. The first coordinate is time, and the last three are space coordinates. $g_{\mu\nu}$ is the metric. Again, it converts coordinate intervals that are frame dependent into invariant intervals. This is a necessity in an expanding universe that does not have globally consistent measurements of time and space. The metric given by the Standard Model is

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t) & 0 \\ 0 & 0 & 0 & a^2(t) \end{bmatrix} \tag{4}$$

This is known as the Friedman-Robertson-Walker (FRW) metric. This metric encodes the expansion of the universe. Using equation 2, it says the physical distance between two points at a time t is the coordinate distance times the scale factor. This is precisely the relationship we discussed when we introduced the scale factor.

The second axiom of general relativity relates the metric to the matter and energy density of the universe. These relations are given by Einstein's equations

$$G_{\mu\nu} = 8\pi \mathrm{G} T_{\mu\nu} \tag{5}$$

$G_{\mu\nu}$ is a complicated function of the metric known as Einstein's tensor. G is Newton's constant, and $T_{\mu\nu}$ is the energy-momentum tensor. It encodes the energy density and the pressure of the different matter and energy constituents of the universe. In the Standard Model, the energy-momentum tensor is modeled as a perfect isotropic fluid.

$$T_{\mu\nu} = \begin{bmatrix} -\rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix} \tag{6}$$

ρ is the energy density and P is the pressure of the fluid. This is a very simplified version of the energy momentum tensor. It is equivalent to assuming the universe is smooth and homogeneous. In later section, we will complicate this assumption and introduce perturbations to the energy-momentum tensor. Nevertheless, the Standard Model is a useful place to begin our discussion of the structure of the Universe. The tensor is diagonalized, which simplifies the mathematics while still providing valuable physical insight.

The relationship between energy density and the metric is incredibly powerful. It allows us to predict the evolution of the scale factor by considering the energy density of the universe. In fact, if we input the FRW metric and the energy-momentum tensor into Einstein's equation, we arrive at a two very significant cosmological equations

$$H^2(t) = \frac{8\pi \mathrm{G}}{3}\rho(t) \tag{7}$$

$$\frac{\ddot{a}}{a} + \frac{1}{2}H^2(t) = -4\pi G P \tag{8}$$

General relativity has enabled us to relate the Hubble rate with a functions of the energy density of the universe and the cosmic pressure. We have mathematically connected the dots between the expansion of the universe and the evolution of its energy density. This equation gives us a powerful base from which we can delve into the expansion and structure of the universe.

1.2 Cosmic Constituents

General relativity underscores the fundamental role the energy density plays in determining the cosmology of the universe. Indeed, the total energy density determines the curvature of the universe. Using general relativity, we find the universe has an energy density threshold known as the critical density. If the energy density is greater than the critical density, the universe is closed. Particles traveling along parallel paths will eventually converge. If the total energy density is less than the critical density, the universe is open. Particles travelling along parallel paths will eventual diverge, as if falling off a saddle. If the total energy density is equal to the critical density, then particles travelling along parallel paths will remain parallel. The critical density underscores the fundamental role that energy density plays in cosmology. In this section, we will outline the cosmic constituents of that make up the energy in the universe. They fall under three board categories: matter, radiation, and dark energy. Matter is composed of dark matter and baryonic matter. Radiation is composed of photons and neutrinos. Dark energy is a relative unknown, but seems to act as a cosmological constant. We will examine each of these forms of energy and explore how their energy density changes in a homogeneous universe.

Before we continue our discussion of cosmic constituents, we must state some conventions. We assume the energy densities of all the cosmic constituents sum to the critical density, equivalent to assuming the universe is flat. We will also employ the variable Ω_i. It is defined as the ratio of the energy density of the ith cosmic constituent to the critical density. Note the ratios of the different energy densities will vary with time. Ω_i is defined in terms of the present day value of the energy densities.

$$\Omega_i = \frac{\rho_i}{\rho_{cr}} \tag{9}$$

Although the ratios of the different energy densities will vary with time, they will always sum to the critical density (the critical density will actually also change with time, but for the universe to remain flat, the different energy densities must always sum to the critical density no matter the age of the universe). With the aid of these conventions, we now turn to the specifics of matter and radiation energy density in our universe.

1.2.1 Matter

Nonrelativistic matter is the cosmic constituent with which we are most familiar. As discovered by Einstein, the energy of a nonrelativistic particle is equal to its rest mass. So long as the particle remain intact, this energy is unchanged. The total energy in the form of matter is equal to the total number of massive particles times their rest mass energy. The energy density of matter is therefore equal to the number density of the particles multiplied by their rest mass.

$$\rho_m(t) = m_e \frac{n}{V} \tag{10}$$

The above equation shows the number density of the massive particles is inversely proportion to the volume they inhabit. As the universe expands, the number density of the massive particles decreases. The volume of the universe expands with the cube of the scale factor, so the number density goes as a^{-3}. The rest mass energy remains constant. Thus, the energy density of matter is inversely proportional to the cube of the scale factor.

$$\rho_m(t) \propto \frac{1}{a^3} \tag{11}$$

By relating the energy density of matter to the scale factor, we have just derived how the matter density evolves in the universe. This is an important result. To utilize the Friedman equation we must know how the energy density evolves in time.

Matter itself can be divided into two groups - baryonic matter and dark matter.

Baryonic matter is the matter that we interact with every day. It is made of atoms whose fundamental components are protons, neutrons, and electrons. An important property of baryons is that they interact with some of the other cosmic constituents. Not only are baryons connected to the other cosmic constituents through the metric, but electrons are connected to photons through a process known as Compton scattering. In this process, a photon and electron bounce off of each other exchanging momenta and energy. In the early universe, when it was very dense and hot, electrons and protons existed in equilibrium because of Compton scattering. Once the rate of this process dropped below the Hubble rate as the universe expanded, photons and electrons fell out of equilibrium. They still interact today, just not at the rate required to maintain a chemical equilibrium. Baryons are a complicated form of energy to study. As a result, the energy density of the baryons must be measured imperially. It has been found that baryons contribute roughly two percent of the critical density.

$$\Omega_b \simeq 0.02 \tag{12}$$

The other form of nonrelativistic matter in the universe is dark matter. Dark matter gets its name because, unlike baryonic matter, it does not interact with light. This makes dark matter somewhat tricky to study. Most cosmological experiments rely upon electromagnetic radiation to garner information about the universe, but these techniques do not work for dark matter. Instead, the presence of dark matter is probed using gravity. Discrepancies between the visible matter in the galaxies and the strength of their gravitational fields points towards dark matter. Leading models have estimated that dark matter is somewhere around thirty percent of the critical density, roughly five time the amount of baryonic matter

$$\Omega_{DM} \simeq 0.30 \tag{13}$$

Dark matter is distinct from baryonic matter and must have a different particulate nature. The leading candidate for a dark matter particle is known as the Weakly Interacting Massive Particle, or WIMP. These WIMPs are hypothesized to have been produced very shortly after the Big Bang They are relics of the early universe. When the universe was very hot, the dark matter particles existed in equilibrium with the cosmic plasma. However, very early on, when the temperature of the universe dropped below the mass scale of the WIMPs, the WIMPs froze out. This means they were no longer in equilibrium with the cosmic plasma.

We can use Boltzman statistics to probe the relic abundance of dark matter particles at the time of the freeze out. If we know the abundance of dark matter particles when the fall out of equilibrium with the cosmic plasma, it is straightforward to determine their abundance today. This knowledge combined with the empirical data that $\Omega_D M \simeq 0.02$ allows us to gain insight into the fundamental properties of the dark matter particle, like its cross section and mass.

We will not delve much into the formalism of the Boltzmann statistics. It is highly technical and tangentially related to the purpose of this paper. Instead, we will roughly sketch its logic. The Boltzmann equation relates the change in the number density of a particle over time to its creation and annihilation rates. The creation and annihilation rates are in turn related to the number density and equilibrium number density of the particle. The equilibrium number density is the density a particle would have if it remained in equilibrium with the cosmic plasma. It is defined as zero order number density with no chemical potential, equivalently

$$n^{(0)} \equiv g \int \frac{d^3p}{(2\pi)^3} e^{-E/T} \tag{14}$$

To greatly simply the rest of a complicated derivation, we end up with a differential equation relating the change in the number density of the particle to a function of its number density and equilibrium number density. For WIMPs, this equation becomes

$$a^{-3}\frac{d(n_X a^3)}{dt} = \langle \sigma v \rangle [({n_X}^{(0)})^2 - {n_X}^2] \tag{15}$$

Here, n_X is the number density of the WIMPs, ${n_X}^{(0)}$ is the equilibrium number density of the WIMPs. The term in the brackets is a thermally average cross section. It is necessary to determine the creation and annihilation rates of the dark matter particles.

To fully solve differential equation requires some rigorous mathematics that we will avoid. However, the intuition of the solution is not difficult to understand if we note the equilibrium number density is usually damped by an $e^{-\frac{m}{T}}$ term. Thus, the equilibrium term drops out of the equation at late times. This leaves us with a different equation only in n_X, which we can integrate from freeze out until late times. The result is

$$\frac{n_X(\infty)}{{T_\infty}^3} \simeq \frac{m_{DM}}{\lambda {T_f}^4} \tag{16}$$

$n_X(\infty)$ gives the number density of WIMPs at very late times. λ is a term related to the thermally averaged cross section of the WIMPs. This equation, when input with the appropriate values, gives a relatively good approximation of the abundance of the WIMPs today. When this relic abundance is related to the energy density of dark matter, we can derive a value for the cross section of the WIMP particles. The best estimate using this method is $10^{-39} cm^2$. This value is in reasonable agreement with the cross section of massive particles predicted in theoretical models.

A final important property of dark matter is that it plays a key role in the large scale structure of the universe. So far, we have assumed that the universe is homogeneously distributed. We will see in later section that inhomogeneties do exist on smaller scales. Dark matter is an important source of these inhomogeneties. Because dark matter particles froze out of equilibrium at such an early time, the attractive force of gravity clumped the particles together. These clumps are an important source of the inhomogeneous structure of the universe, which we will explore in more depth later in this paper.

1.2.2 Radiation

Relativistic radiation is a less familiar cosmic constituent than matter. It is composed of photons and neutrinos zooming around at high speeds in the empty vacuum that is space. These particles have no mass and travel as waves. Their energy is inversely proportional to their wavelength.

$$E_r = \frac{\hbar c}{\lambda} \tag{17}$$

The energy density of radiation is the number density of these particles times the energy of the particle.

$$\rho_r = n_r \frac{\hbar c}{\lambda} \tag{18}$$

Here, n_r is the number density of the radiation particles. We would like to know how the value scales as the universe expands. As with matter, the number density of the radiation particles must scale as a^{-3}. The number density is inversely proportional to the volume. As the universe expands, its volume goes as the cube of the scale factor. What differentiates the evolution of radiation from matter is the evolution of the energy of its particles as the universe expands. The rest mass energy of massive particles remain constant; this is not true of relativistic particles. The energy of relativistic particles is inversely related to the wavelength. As the universe expands, radiation undergoes the peculiar phenomenon that its wavelength grows proportionally to the expansion of the universe. On the co-moving grid, the wavelength of radiation is constant, but the physical distance of the wavelength increases as $a(t)$. This causes the energy of the radiation to decreases as a^{-1} as the universe expands. The energy density of radiation is inversely proportional to the fourth power of the scale factor.

$$\rho_r \propto \frac{1}{a^4} \tag{19}$$

As the universe expands, the energy density of radiation decreases faster than the energy density of matter by a factor of a^{-1}. Whereas matter contributes roughly 30 percent of the critical density today, radiation contributes something to the order of 0.01 percent of the critical density.

$$\Omega_r \simeq 8.47\times10^{-5} \tag{20}$$

Because the matter density is so much larger than the radiation density, we say today we are in a matter dominated era. This claim is complicated by the possible presence of dark energy, which we will discuss briefly. It is generally accepted, however, that we are living in a matter dominated era because the energy density of matter so greatly outweighs the energy density of radiation. This was not always the case. Indeed, the early universe was radiation dominated.

Moving forward in time, the energy density of radiation decreases faster than the energy density of matter. If we move backwards in time, the energy density of radiation increases faster than matter. Thus, if we move far enough back in time to the early universe, the energy density of radiation will greatly outweigh that of matter. We can infer that the cosmic plasma of the early universe, when the cosmic constituents were in equilibrium and the temperature was very hot, the energy density was dominated by radiation.

If the early universe was dominated by radiation energy, and today the universe is dominated by matter energy, then there must have been a time period sometime in between when the energy density were roughly equal. This period is known as the epoch of matter-radiation equality. It is useful to known the time the equality occurred because inhomogeneities in the universe grow at different rates before and after. Thus, the epoch of matter-radiation equality has consequences on large-scale structure of the universe.

We can calculate the value of the scale factor at the time of matter-radiation equality, denoted a_{eq}, by noting how the energy density of matter and radiation vary with the scale factor.

$$\rho_r = \Omega_r \rho_{cr} a^{-4} \tag{21}$$

$$\rho_m = \Omega_m \rho_{cr} a^{-3} \tag{22}$$

Setting these two equations equal and solving for the scale factor yields:

$$a_{eq} = \frac{\Omega_r}{\Omega_m} \simeq 2.8 \times 10^{-4} \tag{23}$$

Just like matter, radiation can be divided into two groups - photons and neutrinos. Photons are the particles of electromagnetic radiation, or the particles of light. Neutrinos are a theoretical Fermi-Dirac particle that have never been observed experimentally. They are, however, a commonly proposed particle in most theoretical models.

We will begin our discussion of radiation starting with photons. A powerful way of exploring the properties of photons is to model it as a cosmic gas. Unlike our discussion of matter, we are able to describe photons as a gas who distribution is defined by its temperature and chemical potential. We can employ the occupation function of statistical mechanics to find the energy density and pressure of photons.

The occupation function of a species describes the number of particles within of phase space region of position and momentum. The occupation function gives the distribution of particles along the position and momentum axes.

Photons are a Bose-Einstein particle, meaning they have integer spin. Their occupation function is therefore given by

$$f_{BE} = f_\gamma = \frac{1}{e^{(E-\mu)/T} - 1} \tag{24}$$

Here, E is the energy of the photon, μ is the chemical potential, and T is the temperature. We can simplify this occupation function further by noting the chemical potential of photon is zero. This is a consequence of the fact that the photon number was not conserved in the early universe. This assumption has also been verified empirically.

A final note on the occupation function is in order. Because the energy of a photon is equal to the magnitude of its momentum, the occupation function we just described only depends upon the magnitude of the momentum. This is a consequence of a homogeneous, smooth universe. When we examine the inhomogeneities of the universe, we will have to perturb the occupation function. This perturbation will depend on the magnitude and direction of momentum and position.

Once we know the occupation function of photons in the universe, it is straightforward to calculate both the energy density and pressure of the photons. To find the energy density, we must integrate the occupation function times the energy $E(p)$ of a photon over all possible momentum states. To find the pressure, we must integrate the occupation function times $p^2/E(p)$ over all possible momentum states. A factor of three must appear in the denominator to account for the 3 possible directions of pressure. Thus, the energy density and pressure are given by

$$\rho_\gamma = 2 \int \frac{d^3p}{(2\pi)^3} \frac{p}{e^{(p/T)} - 1} \tag{25}$$

$$P_\gamma = 2 \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3p\left[e^{(p/T)} - 1\right]} \tag{26}$$

The factor of two in the equations account for the degeneracy in the spin of a photon (two spin states). An immediate and obvious consequence of this formalism is that

$$P_\gamma = \frac{\rho_\gamma}{3} \tag{27}$$

The second and perhaps more important consequence of this statistical mechanics formulation is that the energy density of photons can be expressed as a function of temperature. The above integrals can be evaluated. The integral for the energy density yields

$$\rho_\gamma = \frac{\pi^2}{15} T^4 \tag{28}$$

We know the energy density of radiation scales as a^{-4}. Thus, we have just shown that the temperature is inversely proportional to the scale factor in a smooth, expanding universe.

Before we go about evaluating equation 28 by inputting the temperature of photons today, we must know something about the Cosmic Microwave Background (CMB). The CMB is the source of the photon energy density of the universe. It is comprised of photons that fell out of equilibrium with the cosmic plasma in the very early universe. This process is known as decoupling. The only other cosmic constituent that interacts with photons are electrons. Photons last scattered off of electrons when the scale factor was roughly 9×10^4 times smaller than it is today. Thus, the photon decoupling happened very early on in the life span of the universe. Since then, photons have been travelling freely through space in the CMB.

Studies of the CMB offer a powerful look into the early structure of the universe. Indeed, it was by studying the CMB that led physicist to hypothesize that the universe was smooth and homogeneous. They observed the CMB to be isotropic, suggesting that the universe had been that way dating back to the very earliest epochs. Only recently have studies begun to find small anisotropies in the CMB. We will touch upon this topic when we discuss perturbations to the smooth universe.

Physicists have measured the present day temperature of the CMB to be $T = 2.725 \pm 0.0002$K. Plugging this temperature into equation, we arrive at a value for the energy density of photos in the universe. Today, photons make up roughly 0.005 percent of the critical density.

$$\Omega_\gamma \simeq 5.04\times10^{-5} \tag{29}$$

Neutrinos are the second component of radiation in the universe. Neutrinos, like dark matter, are difficult to observe experimentally because they do not interact with electromagnetic forces, but it is still useful to work through its energy density and pressure as we did with photons. Just like photons, we can model neutrinos as a cosmic gas with temperature and zero chemical potential, and use statistical mechanics to evaluate the energy density and pressure.

The first step is to identify an occupation function for the neutrinos. Neutrinos are Fermi-Dirac particles with zero chemical potential, and as a result have a slightly different occupation function, given by

$$f_{FD} = f_\nu = \frac{1}{e^{E/T} + 1} \tag{30}$$

Now that we have the occupation function, we can apply formalism we developed with the photons to neutrinos. The equations for finding the energy density and pressure are the same, with the neutrino occupation function substituted photon one. These equations are given by

$$\rho_\nu = 6 \int \frac{d^3p}{(2\pi)^3} \frac{p}{e^{(p/T)} + 1} \tag{31}$$

$$P_\nu = 6 \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3p\left[e^{(p/T)} + 1\right]} \tag{32}$$

The equations are strikingly similar to those of the photons. Just like photons, neutrinos are massless and relativistic, and their energy is equal to the magnitude of their momentum. Also just like photons, the pressure exerted by the neutrinos is equal to a third of the energy density. We have just shown that for all forms of radiation $P = \rho/3$.

The differences between equation 31 and 25 arise in the prefactor and the occupation function. The difference in the occupation function occurs because neutrinos are Fermi-Dirac particles. The prefactor occurs because neutrinos have a spin degeneracy of two, and there are three generations of neutrinos. This means the overall degeneracy factor of the neutrinos is 6.

These equations are integrable. The result we find for the energy density of neutrinos is

$$\rho_\nu = \frac{7\pi^2}{40} T_\nu^4 \tag{33}$$

It is important to note that the temperature of the neutrinos is not the temperature of the CMB. Neutrinos fell out of equilibrium with the cosmic plasma before the photons. Within this time period, electron positron annihilation occurred. This annihilation released energy into the plasma and raised the temperature of the photons. The neutrinos missed this heating process. Furthermore, there is no empirical temperature upon which we can draw.

We are saved because we can relate the temperature of the neutrinos to the temperature of the photons through the entropy of the universe. The entropy scales as a^{-3} and is therefore proportional to the cube of the temperature. Before the positron electron annihilation, the entropy is given by

$$s(a_1) = \frac{43\pi^2}{90} T^3 \tag{34}$$

Here T is the temperature of the plasma before annihilation. The plasma contains both neutrinos and photons. Directly after annihilation, the entropy is given by

$$s(a_2) = \frac{2\pi^2}{45}[2{T_\gamma}^3 + 5.25{T_\nu}^3] \tag{35}$$

Setting these two equation equal to each other, we are able to solve for the temperature of the neutrinos in terms of the photon temperature.

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3} \tag{36}$$

Combining equations 33 and 36 with our knowledge of the present day photon temperature, we are able to find a value for the neutrino energy density. It comes about to be about 0.003 percent of the critical density.

$$\Omega_\gamma \simeq 3.43{\times}10^{-5} \tag{37}$$

Dark energy is the last form of energy in the universe. We do not know much about it, but there are two compelling reasons to believe in its existence. First, the energy densities of matter and radiation do not sum to the critical density. If the universe is flat, and there is good evidence to suggest that it is, then there must be a third form of energy that fills this gap. The second comes from plotting theoretical distance versus redshift curves. The redshift z of light quantifies how much its wavelength has increased in the time after it was emitted until it was observed due to the expansion of the universe. It is defined in the below equation

$$1 + z = \frac{\lambda_{emit}}{\lambda_{obs}} = \frac{1}{a} \tag{38}$$

As can be seen by the its relation to the scale factor, the redshift is also a measure of how old light radiation is. The higher the redshift, the earlier in the universe the light was emitted.

Using the redshift parameter, we can create distance versus redshift curves that depend upon the composition of energy in the universe. For different theoretical compositions energy, the universe will expand at different rates (from equation 7) leading to distinct distance versus redshift curves. Cosmologists have equation experimental data to test these theoretical curves. The curve that fits the experimental data best incorporates a dark energy component.

With these two compelling pieces of evidence, it is not hard to accept that dark energy exists. But that does not bring us any closer to understanding what it is. Indeed, there is not much scientific consensus on what dark energy is. A popular theory is that dark energy acts as a cosmological constant. As the universe expands, its energy density remains constant. Another theory is that dark energy can be described as a time dependent scalar field. Neither of these theories have much evidence to support them, however, so dark energy remains a mystery. The only concrete property of dark energy that we can point to is that must have negative pressure. This must be true if the universe is flat.

2 Perturbations to the Smooth Background

In the previous section, we have discussed the cosmology of a smooth expanding universe. We outlined a fairly simple metric that could encode both gravity and the expansion of the universe, the Freidman-Robertson-Walker metric. It depended only upon the scale factor, and even then only the spacial components depended upon it. We assumed a homogeneous distribution of matter and an isotropic distribution of the radiation in the universe.

This smooth model of the universe led to important cosmological insights. We developed notions of the scale factor, conformal time, and the Hubble rate. We derived the evolution of matter and radiation energy density in the universe. But while assuming the universe to be smooth is a useful starting point, it is an ultimately the flawed model. Matter has is not distributed evenly throughout our universe, and anisotropies in the Cosmic Microwave Background have been experimentally observed. The universe is very smooth, but there exist small perturbations to the general smoothness; very important physics emerges from a study of these perturbations.

In this section, we complicate our smooth model of the universe. We add perturbations to the metric to account for slight changes in gravity, and we add perturbations to our models of radiation and matter to account for slight irregularities in their distribution. Our ultimate goal will be to develop a theoretical prediction for the inhomogeneous distribution of matter in the universe. To achieve this will require a long process, and accordingly this section is broken into two parts. First, we derive the Boltzmann-Einstein equations, a set of six differential equations, that couples the evolution of the different perturbations (metric, matter, and radiation perturbations) to first order. Second, we use the theory of inflation to arrive at the initial conditions of the Boltzmann-Einstein equations.

2.1 The Boltzmann-Einstein Evolution Equations

Before we venture into the Boltzmann-Einstein formalism, we need to know what changes we are actually adding to our cosmological models. We will define first the perturbations to the metric, then to matter, and finally to radiation. Each will have two perturbation variables. With these definitions in place, we can begin relating their coupled evolution. It is important to know that these perturbations are all assumed to be small on cosmological scales, for the universe is generally smooth. When we work through the formalism, we will only keep terms that are first order in the perturbations.

The variables that are needed to perturb the FRW metric are the Newtonian potential (Ψ) and a spacial curvature (Φ) perturbations. These two variables both depend upon space and time. The new gravitational metric is given by

$$g_{\mu\nu} = \begin{bmatrix} -1-2\Psi & 0 & 0 & 0 \\ 0 & a^2(t)[1+2\Phi] & 0 & 0 \\ 0 & 0 & a^2(t)[1+2\Phi] & 0 \\ 0 & 0 & 0 & a^2(t)[1+2\Phi] \end{bmatrix} \tag{39}$$

In our convention, $\Psi < 0$ represents a region with above average gravitational potential, accordingly a region with greater than average mass (over-dense). $\Phi > 0$ represents a region with above average spacial curvature, also an over-dense region. In the limit in which our perturbations go to zero, this metric simplifies to the FRW metric that characterizes a smooth, expanding universe. In this absence of the scale factor, this metric describes a weak gravitational field. Ψ is the weak gravitational potential of some source mass, and Φ is the spacial curvature caused by the source mass. These limits corroborate our choice of perturbation variables.

A final note on our choice of perturbations to metric â the perturbations we have added to the FRW metric are scalar perturbations. Because the metric is still diagonalized, the perturbations qualify as a scalar perturbations. Vector and tensor perturbations to the metric are possible; they would affect the off-diagonal terms of the metric. Because of the decomposition theorem, however, scalar, vector, and tensor perturbations cannot interact with each other. Thus, we choose to focus on scalar perturbations because they have the greatest effect on the development of inhomogeneities and anisotropies in the universe.

Next come the perturbations to the matter distribution. Given an average density of matter in the universe n_m, we define the first perturbation variable $\delta(x)$ to be

$$\delta(\vec{x}) = \frac{n(\vec{x}) - \bar{n}_m}{\bar{n}_m} \tag{40}$$

Thus $\delta(x)$ characterizes the over- and under-densities of matter at a given position x in the universe.

To define the second perturbation variable to matter, we first need a notion of a matter distribution function f_m analogous to that of radiation. We can define it implicitly by

$$n_m = \int \frac{d^3p}{(2\pi)^3} f_m \tag{41}$$

This immediately leads to the definition of the second perturbation variable v

$$v^i = \frac{1}{n_m} \int \frac{d^3p}{(2\pi)^3} \frac{f_m \times p^i}{E} \tag{42}$$

This perturbation variable v is known as the matter velocity. In an inhomogeneous universe, over-dense and under-dense regions of energy will induce cohesive flows in the matter. We understand this cohesive flow of matter as the distribution's velocity, and it is the second variable we need to perturb our homogeneous model.

These perturbation variables, δ and v, are usually reserved to describe the over-densities and velocities of dark matter. We can define exactly analogous variables for baryons δ_b and v_b. We will ignore the contributions of the baryonic perturbations to the overall matter distribution. This is justified because baryons make up a very small portion of the total matter in the universe; most of it is dark matter. It will also help us to simplify the Boltzmann equations when we solve for the inhomogeneities in the distribution of matter.

The final perturbations are those to the radiation distribution. Isolating two distinct (but simple) perturbation variables for radiation is more difficult than in the previous two cases. We begin by defining a perturbation to the photon distribution function. Recall that photons are Bose-Einstein particles, so

$$f(x, p, \hat{p}, t) = \left[exp \left[\frac{p}{T\,(1 + \Theta(x, \hat{p}, t))} \right] - 1 \right]^{-1} \tag{43}$$

where T is the temperature of the CMB. Thus, the perturbation characterizes small changes to the photon distribution's temperature. These perturbations depend upon not only the age of the universe, but also the direction and position of the photon distribution. The perturbation does not depend upon the magnitude of the photon's momentum because the momentum remains almost unchanged after Compton scattering.

We will see that this perturbation to the distribution function quickly becomes messy, and it is useful to break down the perturbation Θ in terms of its poles. In fact, it will only be necessary for us to keep the monopole and dipole terms of the photon distribution. These two terms, which we will define exactly later, act as our perturbation variables for electromagnetic radiation.

We will justify dropping the higher moments more rigorously using Boltzmann statistics, but there is a simple physical intuition that undergirds the result. Before recombination, photons were tightly coupled to electron. This mixture of particles acted as a cosmic fluid. This simple cosmic fluid can be described completely with only the monopole and dipole moments of the photon perturbation. Compton scattering, which couples photons and electrons, suppressed the higher order moments. The higher moments of the photon perturbation do become important in describing the photon distribution after recombination, but recombination occurs well into the matter dominated era. Thus, the higher moments of the photon perturbation only start to matter once the radiation energy density has become negligible. The higher order moments of the photon perturbation will therefore not have an effect on the matter distribution in the universe.

We can define an exactly analogous perturbation to the distribution function of neutrinos. For this perturbation as well, we will assume the higher moments of the perturbation are negligible and focus on the monopole and dipole terms. These two terms are exactly analogous to the two photon perturbation terms. We will focus our discussion on the evolution of photon perturbations. We will see that the Boltzmann equations that emerge for photons will be the same for neutrinos. Indeed, we will eventually lump the photon and neutrino perturbations together as radiation perturbations.

2.1.1 Boltzmann Equations for Radiation

Now that we have a list of all the perturbation variables, we can begin to derive the Boltzmann equations that their evolution as the universe expands. This is a complicated task because of how all the different cosmic constituents interact. To systematically account for the interactions amongst different species, we must relate the evolution of the distribution function to the collision terms of each species. We start with the unintegrated Boltzmann equation

$$\frac{df}{dt} = C[f] \tag{44}$$

We will solve this equation for dark matter and photons. The evolution of neutrinos will follow from our discussion of photons. We begin with photons.

We first consider the left-hand side of equation, and rewrite the total time derivative as a sum of partial derivatives over time, position, and momentum. It becomes

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f}{\partial p}\frac{dp}{dt} \tag{45}$$

where we have ignored the term that involves the momentum unit vector. This term will necessarily be second order in the photon perturbation, and we are interested only in first order terms.

We can use general relativity to solve for $\frac{dx^i}{dt}$ and $\frac{dp}{dt}$ coefficients of the partial derivatives. The square of a photon's four momentum is zero. Examining the spacial components of the photon's four momentum immediately leads to

$$\frac{dx^i}{dt} = \frac{P^i}{P^0} = p^i \frac{1-\Phi}{ap(1-\Psi)} = \frac{\hat{p}}{a}(1+\Psi-\Phi) \tag{46}$$

If we remember our conventions, the above equation suggest that photons slow down upon entering an over-dense region. This agrees with our physical intuition. Nevertheless, this term multiplies $\frac{\partial f}{\partial x^i}$, which is first order, so we must drop the Φ and Ψ terms.

Evaluating the time component of the momentum geodesic equation and keeping only first order terms leads yields a result for dp/dt

$$\frac{dp}{dt}\frac{1}{p} = -H - \frac{\partial \Phi}{\partial t} - \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i} \tag{47}$$

This differential equation accounts for the change in the momentum of a photon in perturbed, expanding universe. The first term in the equation accounts for a photon's loss of energy (and momentum) in an expanding universe. The second term accounts for a photon's loss of energy as it passes through a deepening potential well, while the third term says that a photon traveling into a potential well gains energy through gravity. Again, these results match our intuition of how gravity effects light.

To fully expand the left-hand side of equation 45, we must deal with the perturbed Bose-Einstein distribution function of photons given in equation 24. We are interested in terms expanded to zero and first order in Θ. The appropriate expansion is given by

$$f \simeq f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta \tag{48}$$

where $f^{(0)}$ is the zero order BE distribution given in equation 24.

With equations 46, 47, and 48, we now have all the tools we need to expand the left-hand side of the unintegrated Boltzmann equation to zero and first order in the relevant perturbation variables. We are interested in comparing the expanded version of the unintegrated Boltzmann equation to derive the differential equations that guide our perturbation variables. Thus, the zero order terms will not be very useful. By definition, the zero order terms are unaffected by the perturbation variables. Furthermore, the zero order collision term of the Boltzmann equation vanishes. Compton scattering between photons and electrons are the source of the collision term. As we noted early, photons and electrons exist in equilibrium in the cosmic plasma during the radiation-dominated era. Because this process is in equilibrium at zero order, it will drive the corresponding collision term to zero. No useful information is gained about the evolution of our perturbation. To derive our guiding equation, we must investigate the first order terms of the Boltzmann equation.

Although the zero order collision term of the Boltzmann equation was zero, the first order term will not be. The collision term incorporates the effect Compton scattering will have upon the photon distribution. Compton scattering is the process by which an electron and a photon collide and exchange momentum:

$$e^-(q) + \gamma(p) \leftrightarrow e^-(q^\dagger) + \gamma(p^\dagger) \tag{49}$$

To turn Compton scattering into an expression for the change in the distribution function of a photon, we need a few ingredients from Boltzmann statistics. First of all, energy and momentum must be conserved in the scattering process. Second, the change in the distribution function is related to the production rate of photons ($\gamma(p)$) minus the loss rate of photons ($\gamma(p)$). From the schematic equation above, the production rate must be proportional to $f_e(q^\dagger)f_\gamma(p^\dagger)$ while the loss rate must be proportional to $f_e(q)f_\gamma(p)$. Third, the effect of the Compton scattering will be related to its quantum mechanical amplitude M, which is related to the cross section of the interaction. Finally, we must sum over all possible momentum states of each particle. Putting these ingredients together, we arrive at an equation for the Compton scattering collision term.

$$C[f(p)] = \frac{1}{p}\int \frac{d^3q d^3q^\dagger d^3p^\dagger}{8E_e(q)E_e(q^\dagger)E_\gamma(p^\dagger)}\frac{|M|^2}{(2\pi)^5} \times \delta^3[p+q-p^\dagger-q^\dagger]$$
$$\times\, \delta[E_\gamma(p)+E_e(q)-E_\gamma(p^\dagger)-E_e(q^\dagger)] \times [f_e(q^\dagger)f_\gamma(p^\dagger)-f_e(q)f_\gamma(p)] \quad (50)$$

The above formula incorporates all of the Boltzmann statistics we outlined above into an equation for the Compton scattering collision term. The delta-Dirac functions ensure conservation of energy and momentum. The $|M|^2$ term accounts for the quantum mechanical amplitude of the process. The last term accounts for the production and loss rates. We have ignored Bose enhancement and Pauli exclusion because those will only affect the distribution to second order. Equation 50 simplifies drastically when we plug in the appropriate distribution functions and expand to first order. It reduces to

$$C[f(p)] = -p\frac{\partial f^{(0)}}{\partial t} n_e\sigma_T[\Theta_0 - \Theta(\hat{p}) + \hat{p}\cdot v_e]\Theta_0(x,t) = \frac{1}{4\pi}\int d\Omega^\dagger \Theta(p^\dagger, x, t) \quad (51)$$

A few notes on the above equations. First, the $n_e\sigma_T$ term comes from the quantum mechanical amplitude; n_e is the free electron density and σ_T is the Compton cross section. Second, Θ_0 is defined as the monopole moment of the photon perturbation. This term, along with the dipole, will be crucial in simplifying the perturbation, as we noted earlier in this section.

We can now equate the first order parts of the left and right-hand side of the Boltzmann equation. Substituting in conformal time η as our time variable, the equation becomes

$$\dot{\Theta} + \hat{p}^i\frac{\partial \Theta}{\partial x^i} + \dot{\Phi} + \hat{p}^i\frac{\partial \Psi}{\partial x^i} = a \times n_e\sigma_T[\Theta_0 - \Theta + \hat{p}\cdot v_e] \quad (52)$$

This is the final result towards which our formalism regarding the photon perturbation has been building. We have a differential equation coupling the evolution of three of our perturbation variables. This is a linear differential equation. Furthermore, the x dependence in the different equation is contained within the perturbation variables themselves. If we transform into Fourier space, then the Fourier modes of the differential equation evolve independently of one another. The Fourier transform serves to decouple an infinite set of coupled differential equations. Transforming to

Fourier space, equation 52 becomes

$$\dot{\Theta} + \hat{p}^i \frac{\partial \Theta}{\partial x^i} + \dot{\Phi} + \hat{p}^i \frac{\partial \Psi}{\partial x^i} = a \times n_e \sigma_T [\Theta_0 - \Theta + \hat{p} \cdot v_e] \tag{53}$$

The perturbation variables are now functions of μ rather than x. μ is defined as

$$\mu \equiv \frac{k \cdot \hat{p}}{k} \tag{54}$$

We have just derived the Boltzmann equation coupling the evolution of the photon perturbation to the evolution of the gravitation and spacial curvature perturbations of the metric. We are, however, primary concerned about understanding the inhomogeneities in the distribution of matter, not radiation. We need understand the evolution of radiation only so much as it effects the evolution of the metric perturbations. Armed with this physical logic, we can make to assumptions that will drastically simplify equation 53.

First, only the monopole and dipole moments of Θ will affect the distribution of matter. Consider equation 53. When the electrons in the cosmic plasma lack a bulk velocity ($v_b = 0$), the perturbation is driven to equal its monopole moment. All other moments vanish. If the electrons do carry a bulk velocity, it is necessary to introduce the dipole moment of the photon perturbation. Still in this case, however, all moments higher than the dipole will vanish. Thus, the tight coupling of photons and electrons through Compton scattering erases all but the monopole and dipole moments of the photon perturbation. The high moments of the photon perturbation only become important after the photons and electrons decouple. As we noted earlier, this happens at recombination. Recombination occurs deep into the epoch of matter domination, so that higher moments of the radiation distribution have a negligible effect. Thus, we need only consider the monopole and dipole moments of Θ. They are defined as follows

$$\Theta_0 \equiv \int_{-1}^{1} \frac{d\mu}{2} \Theta(\mu) \tag{55}$$

$$\Theta_1 \equiv \int_{-1}^{1} \frac{d\mu}{2} \mu \Theta(\mu) \tag{56}$$

Second, we will drop the baryon source term ($n_e \sigma_T$). As we discussed early in this section, we are working under the approximation of small baryon density. The baryonic density is very small compared to the more abundant dark matter. In the radiation dominated era, the baryonic density is much smaller than the photon density. With some formalism, it is possible to show that the baryonic source term is proportional to ratio of the baryonic energy density to the photon energy density. This ratio is negligibly small for all times during which the photon perturbation has an effect on the matter distribution. Thus, we drop the baryonic source term on the right side of equation 53. Note, this simplification only holds specifically because we are studying the distribution of matter in the universe. Had we, for instance, been interested in the evolution of radiation, the baryonic source term would play an important role.

These two assumptions lead to two simple equations for the evolution of the monopole and dipole terms of the photon perturbation. Dropping the baryonic source term and isolating the appropriate pole, we arrive at

$$\dot{\Theta}_0 + k\Theta_1 = -\dot{\Phi} \tag{57}$$

$$\dot{\Theta}_1 - \frac{k}{3}\Theta_0 = -\frac{k}{3}\Psi \tag{58}$$

We have completed our derivation of the Boltzmann equations for Θ that concern the matter distribution. We have not touched upon the evolution of neutrinos, the other form of radiation. This is because, for our purposes, we will assume neutrinos obey the same Boltzmann equations as photons. Neutrinos have a distribution function very similar to photons, and neutrinos have no collision term. If we were to retrace the steps of our derivation for photons, we would find that neutrinos also obey equation 53, with the term on the right equal to zero. This is equivalent to setting the baryonic source term to zero. If we assume neutrinos have small higher moments, then the Boltzmann equations governing neutrinos will be precisely equations 57 and 58. In fact, for the rest of this paper we will combine the neutrino and photon perturbations, treating them as radiation perturbations. This is justifiable if photons and neutrinos have the same initial condition. Indeed, when we discuss inflation, the primordial mechanism that produced the initial conditions, we will assume it does not differentiate between photons and neutrinos. For the rest of this paper, we will replace Θ with Θ_r, our new symbol for radiation perturbation.

2.1.2 Boltzmann Equations for Matter

Now that we have solved the Boltzmann equations for radiation, it is time to turn our attention to matter. Again, we will focus our discussion solely on the dark matter perturbations . We follow the same general steps outlined for photons. We expand the left-hand side of the unintegrated Boltzmann equation in terms of partial derivatives. We use the equations of general relativity to solve for dx^i/dt, dE/dt, and $d\hat{p}^i/dt$. One difference from the photon derivation is the constraint equation. Photons are massless particles, and therefore the norm of its four momentum is zero. The norm of the four-momentum of dark matter particles is given by

$$g_{\mu\nu}P^\mu P^\nu = -m^2 \tag{59}$$

where m is the mass of a dark matter particle. This is the constraint equation for dark matter.

The general relativity formalism leads to an expanded form of the distribution function for dark matter.

$$\frac{\partial f_{dm}}{\partial t} + \frac{\partial f_{dm}}{\partial x^i}\frac{p^i}{aE} - \frac{\partial f_{dm}}{\partial E}\left[H\frac{p^2}{E} + \frac{p^2}{E}\frac{\partial \Phi}{\partial t} + \frac{p^i}{a}\frac{\partial \Psi}{\partial x^i}\right] = 0 \tag{60}$$

These are the terms that are zero in first order in the gravitational perturbations. Dark matter has no collision term, and as a result we set the time derivative of the distribution function equal to zero. The presence of the p^2/E factors differentiate the above equation from the time derivative of the photon distribution function. Indeed, in the limit of no mass, both equations are equal.

To solve this equation for the dark matter perturbations, we must use the fact dark matter is assumed to be non-relativistic. This means that p/E is small, and we can therefore ignore the thermal fluctuations of dark matter. The important consequence of these approximations is that we do not need a form for the dark matter distribution function. Instead, we may take moments of the unintegrated Boltzmann equation. The quantities that become important are δ and v, the perturbations we defined earlier in this section. δ quantifies the over and under-densities of the dark matter. v quantifies the velocity of the dark matter. Although we can ignore thermal fluctuations, the inhomogeneous matter distribution will induce coherent velocities.

We take the first moment by integrating equation 60 over momentum phase space units $d^3/(2\pi)^3$. The integral of the zero order terms leads to a cosmological continuity equation, leading to the conclusion that the zero order number density of dark matter scales a^{-3}. This is what we surmised in section I. The integral of the first order terms contains the information about the evolution of the perturbations. The first order part leads to

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\frac{\partial v^i}{\partial x^i} + 3\frac{\partial \Phi}{\partial t} = 0 \tag{61}$$

We take the second moment by integrating equation 60 over momentum phase space unit, tacking on a factor of p^j/E. This factor weights each term in the integral by its momentum. Ignoring the zero order terms and evaluating the integrals of the first order terms leads to

$$\frac{\partial v^i}{\partial t} + Hv^i + \frac{1}{a}\frac{\partial \Psi}{\partial x^i} = 0 \tag{62}$$

Equations 61 and 62 are the Boltzmann equations that guide the evolution of dark matter perturbations. It is important to note that only the first two moments of the dark matter distribution were needed. This is, again, because dark matter is non-relativistic. Higher order moments depend upon p/E to higher and higher powers. This factor is very small in non-relativistic particles, so we need only focus on the first two moments.

We would like to rewrite both equations in terms of conformal time and Faurier transform each into k-space. As was the case with radiation perturbations, this simplifies the time derivatives and decouples the differential equations. The final versions of the two Boltzmann equations for matter perturbations are

$$\dot{\delta} + ikv + 3\dot{\Phi} = 0 \tag{63}$$

$$\dot{v} + Hv + ik\Psi = 0 \tag{64}$$

where our perturbation variables are now functions of the mode k rather than x.

2.1.3 Einstein Equations for Gravity

Using the Boltzmann formalism, we have derived four differential equations that couple the evolution of six perturbation variables. We need to more differential equations to solve this system. To find this last two equations, we turn to Einstein's equation. Einstein's equation $G^{\mu}_{\nu} = 8\pi \mathrm{G} T^{\mu}_{\nu}$ relates the metric to the energy-momentum tensor. We explored the zero-order consequences of this equation in section I, and arrived at two fundamentally important results relating the scale factor to the energy and pressure of the universe. We will now incorporate our perturbations into Einstein's equations. We will follow these perturbations through first order to derive two new differential equations relating their evolution.

We first consider the time-time component of Einstein's equation. The time-time component of Einstein's tensor is given by

$$G^{0}{}_{0} = g^{00}[R_{00} - \frac{1}{2}g_{00}R] = (-1+2\Psi)R_{00} - \frac{R}{2} \tag{65}$$

where $R_{\mu\nu}$ is the Ricci tensor and R is the Ricci scalar. We are interested in the first order change to this tensor caused by the gravitation (Ψ) and spacial curvature (Φ) perturbations to the metric. This change is given by

$$\delta G^{0}{}_{0} = -6H\Phi_{,0} + 6\Psi H^2 - 2\frac{k^2\Phi}{a^2} \tag{66}$$

We must now equate this first order change to Einstein's tensor with the first order change to the energy momentum tensor. Referring to equation 6, the time-time component of the energy momentum tensor ($-T^{0}{}_{0}$) is the total energy density of all the cosmic constituents in the universe. We would like to find the first order change to this value caused by matter (δ) and radiation (Θ_r) perturbations.

The energy density of matter to first order in δ is trivial to find. By definition

$$T^{0}{}_{0} = \rho_{dm}[1+\delta] \tag{67}$$

To determine the energy density of radiation to first order in Θ_r, we must return to Boltzmann statistics. The energy density of radiation can be found by integrating the distribution function times the energy over momentum space. To first order, this is given by

$$T^0{}_0 = -2\int \frac{d^3}{(2\pi)^3} p \left[f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta_r \right] = -\rho_r[1 + 4\Theta_{r,0}] \tag{68}$$

The first term in the integral picks out the zero order energy density of the radiation. The second term requires an angular integral, which ends up picking out the monopole term of the perturbation with a factor of four.

Equating the first order changes in the energy density of the cosmic constituents with the first order change in Einstein's tensor, we have

$$k^2\Phi + 3H(\dot{\Phi} - H\Psi) = 4\pi G a^2[\rho_{dm}\delta + 4\rho_r\Theta_{r,0}] \tag{69}$$

where the time derivatives are with respect to conformal time η. We have found the evolution equation for Φ and Ψ using Einstein's equations. In the absence of expansion, this equation simplifies to a simple Poisson differential equation. The expansion terms play a key role, however, in determining the evolution of modes that are on the order of the Hubble radius H^{-1}.

To find our last equation guiding the evolution of our cosmic perturbations, we focus on the spacial components of Einstein's equation. The formalism pertaining to the spacial components of Einstein's equations is very similar to the time components, so I just quote the result. The second equation guiding the evolution of Φ and Ψ is

$$k^2(\Phi + \Psi) = -32\pi G a^2[\rho_\gamma\Theta_2] \tag{70}$$

Note that the sum of our gravitational perturbations depends upon the quadrupole moment of the radiation perturbation. Continuing with our assumption that all moments higher than the dipole moment our negligible, the right-hand side of equation 70 becomes zero. Our second Einstein equation then tells us that $\Psi = -\Phi$. This is a nice result and will make the calculations that follow in the paper much simpler.

We have two equations that determine how the gravitational perturbations to the metric evolve in a perturbed universe. We have found these two equations, however, without considering all of the components of the Einstein tensor. It turns out that considering the other components of the tensor will not give us any new information. Everything that we learn will be redundant. Nevertheless, the equations we find by considering the other components of the Einstein tensor will be in a different form. There is one form, known as the algebraic form of Einstein's evolution equations, that will be particularly later on in this paper. It is given by

$$k^2\Phi = 4\pi G a^2 \left[\rho_{dm}\delta + \rho_r\Theta_{r,0} + \frac{3aH}{k}\left(i\rho_{dm}v + 4\rho_r\Theta_{r,1}\right) \right] \tag{71}$$

We have now derived all six Einstein-Boltzmann equations that govern the evolution of the six perturbation variables. They are summarized in the list below.

$$\dot{\Theta}_{r,0} + k\Theta_{r,1} = -\dot{\Phi} \tag{72}$$

$$\dot{\Theta}_{r,1} - \frac{k}{3}\Theta_{r,0} = -\frac{k}{3}\Psi \tag{73}$$

$$\dot{\delta} + ikv + 3\dot{\Phi} = 0 \tag{74}$$

$$\dot{v} + Hv + ik\Psi = 0 \tag{75}$$

$$k^2\Phi + 3H(\dot{\Phi} - H\Psi) = 4\pi Ga^2[\rho_{dm}\delta + 4\rho_\gamma\Theta_{r,0}] \tag{76}$$

$$\Psi = -\Phi \tag{77}$$

The first two equations are radiation evolution equations, the second two are matter evolution equations, and the last two are gravitational evolution equations. Our next objective is to establish the initial conditions of these six differential equations.

2.2 Determining Initial Conditions With Inflation

We have derived the six Boltzmann-Einstein equations that govern the evolution of the cosmic perturbation. We now turn our attention briefly to the phenomenon of inflation. Inflation refers to the very rapid expansion of the universe at very early times, and can be used to explain the initial conditions of our perturbations. To fully derive the initial conditions set by inflation requires scalar field theory, a topic beyond the scope of this paper. Instead, we will focus upon how inflation solves the horizon problem, and then quote the numerical result of the initial conditions.

Before we delve into the theory of inflation, however, we will narrow our task. By considering our six evolution equations at early time, we will relate the initial conditions of all to perturbation variable to Φ. Thus, we only need determine the initial conditions of Φ to set the initial conditions for our evolution equations.

2.2.1 Relating the Initial Conditions

Inflation occurs at very early time scales, so our initial conditions arise when η is small. We will assume $k\eta \ll 1$ to be true at the times of interest. This inequality allows us to simplify our equations. Consider equations 72 and 74. In each, the first term is proportional to η^{-1} while the second is proportional to k. Thus, the first term is greater than the second by a factor of $(k\eta)^{-1}$, exactly the factor we assumed was large. We may drop the second term in each equation. This leaves

$$\dot{\Theta}_{r,0} = -\dot{\Phi} \tag{78}$$

$$\dot{\delta} = -\dot{\Phi} \tag{79}$$

The velocity of dark matter and the first moment of radiation are both smaller than the zero order moments by a factor of $k\eta$, and we set the initial conditions for these perturbations to be zero.

The last equation remaining is equation (we already reasoned how the second Einstein equation simplifies to $\Psi = -\Phi$, there is no more work to be done here). Immediately, we can drop the first term because it is proportional to k^2. We drop the matter terms on the right because the initial conditions are set in the radiation dominated era. The scale factor evolves proportionally with conformal time in this era. Thus, $H = 1/\eta$ for times concerned. Finally, $16\pi G\rho_{cr} = 6H^2$ by the Friedman equation. Note, that the critical density is equal to the radiation energy density in the radiation dominated era. These equations and simplifications lead to

$$\dot{\Phi}\eta - \Psi = 2\Theta_{r,0} \tag{80}$$

We can eliminate Ψ from this equation using equation 77, and we can eliminate $\Theta_{r,0}$ by differentiating and using equation 78. This leads to the following differential equation for Φ in the early universe

$$\ddot{\Phi}\eta + 4\dot{\Phi} = 0 \tag{81}$$

If we assume that the gravitational perturbation is a power of the conformal time, then we arrive at $\Phi = \eta^0, \eta^{-3}$. In the early universe, the gravitational potential has two modes. However, the second mode (η^{-3}) will decay as the universe ages, and will not affect the universe. We will therefore focus our attention on the first mode (η^0) as it may be the seed that blossoms into the perturbations we observe today. Plugging $\Phi = \eta^0$ into equation 81, we find

$$\Phi = 2\Theta_{r,0} \tag{82}$$

It is important to note that this equation only holds at some early time in the universe, after which the perturbations will evolve with the Boltzmann-Einstein equations. The equation only relates the initial conditions of our perturbations.

We relate the initial condition of the density perturbation to the initial condition of the radiation perturbation by assuming our perturbations are adiabatic. Adiabatic perturbations assume that the ratio of matter to radiation number density is a constant independent of space. This ratio is given by

$$\frac{n_{dm}}{n_r} = \frac{{n_{dm}}^{(0)}}{{n_r}^{(0)}} \left[\frac{1+\delta}{3\Theta_{r,0}}\right] \tag{83}$$

The right side of this equation is only independent of space if the second quotient equals one. This leads to

$$\delta = 3\Theta_{r,0} \tag{84}$$

We have now successfully related the initial conditions of our perturbations to the initial condition of the gravitational perturbation Φ or argued that the initial conditions of a perturbation is zero because it is negligible in the early universe. Equation 82 relates the initial conditions of $\Theta_{r,0}$ to Φ, equation 84 relates the initial conditions of δ to Φ, and equation 77 relates Ψ to Φ. We argued that the velocity and dipole moment of the radiation were negligibly small in the early universe.

2.2.2 Inflation

We still are not any closer at determining the mechanism, or for that matter the initial value, that causes Φ to be nonzero. For these answers, we must turn to inflation. Inflation, as we already detailed, is the theory that the universe rapidly expanded early on in its lifetime. Originally, inflation had no connection to the initial values of the cosmic perturbations. Instead, it was supposed to explain another quandary in cosmology - the uniform temperature of the Cosmic Microwave Background radiation.

The uniform temperature of the Cosmic Microwave Background temperature was a problem we glossed over in section I. Indeed, on its face it does not seem like much of a problem at all. But some digging into the physical processes that govern the CMB raise some serious issues. To understand this process, we need to develop the idea of the cosmic horizon.

As we explained in section I, conformal time η is the maximum co-moving distance that light can travel since the universe began. η sets the horizon. Points separated by a distance greater than η are not causally connected. The physics of one cannot effect the physics of another. When we made the approximation $k\eta \ll 1$, this means that a given mode of a perturbation is larger than the cosmic horizon (note, $\lambda = 2\pi/k$). $k\eta$ is roughly equal to the ratio of the co-moving horizon to the commoving wavelength. When this ratio is much less than one, this indicates the mode in question has a wavelength larger than the horizon. Causal physic cannot affect this mode.

How do modes that were once causally disconnected from the rest of the universe reconnect? Because the conformal time grows as the universe ages, whereas the co-moving wavelength of the perturbations remain constant. Modes that were once larger than the horizon eventually become absorbed into causal physics as the universe expands.

Our intuition of the horizon η and causal physics conflicts with observations of the CMB. Observations of the CMB on all modes have found it to be very isotropic with a constant temperature. This applies to modes that have very large wavelengths and have only entered the universe very recently, after recombination. These modes, which were causally disconnected until they entered the horizon, entered the horizon after recombination, by which time the photons of the CMB were free-streaming and no longer interacted. These large CMB modes seemingly could not have physically interacted with the rest of the CMB photons, and yet they have the same temperature and other physical properties. This is highly suspicious. Furthermore, cosmologists have observed CMB radiation from opposite sides of the universe that could not have been connected at the epoch of decoupling but still have the same temperature. How can this be?

It is because our picture of the expansion of the universe is not complete, and in the very early universe, points that are causally separated today were once connected. This logic is encompassed in the theory of inflation. To understand this theory requires a careful treatment of conformal time, so we rewrite the definition:

$$\eta = \int_0^t \frac{dt}{a(t)} = \int_0^a \frac{da'}{a'} \frac{1}{a'H(a')} \tag{85}$$

The key component here that we will focus on is the co-moving Hubble radius $(aH)^{-1}$. We briefly touched on the physical Hubble radius (H^{-1}) earlier this section. It has units of [s], so it is a distance in general relativity. It is roughly equal to the maximum distance a particle can travel over the course of the expansion time. The scale factor in the denominator of equation 84, however, places our quantity on the co-moving grid. The co-moving Hubble radius $(aH)^{-1}$ is roughly equal to the maximum co-moving distance a particle can travel in one expansion time. Therefore, the co-moving Hubble radius is another determinant of causality. Objects separated by distances greater than the commoving Hubble radius cannot currently communicate. Note, however, the subtle difference between the Hubble radius and conformal time as determinants of causality. If two objects are separated by a distance larger than the Hubble radius, they cannot *currently* communicate. If two objects are separated by a distance larger than η, they *could never* have communicated and must be causally disconnected. This insight provides the foundation for the theory of inflation.

Cosmological experiments have shown that, up to the earliest ages of the universe that we can probe, the Hubble radius has grown continuously with the scale factor. How universe behaved at the very earliest times, however, we cannot be sure. The energies of the cosmic plasma are too high to be probed experimentally. Thus, it is possible that at the very beginning of the universe, the co-moving Hubble radius started off very large and that its values decreased dramatically in the early epochs of the universe. If this assumption were true, then distances which may have been larger than the co-moving Hubble radius after inflation may once have been smaller than the co-moving Hubble radius before inflation. Thus, the particles separated by these distances were in causal contact during inflation and only fell out of contact after inflation. But the important fact remains that these particles once were able to exchange physical information. This is a viable solution to the uniform temperature of the CMB.

For the co-moving Hubble radius to decrease with time, (aH) must increase with time. This leads to the following equation

$$\frac{d}{dt}[aH] = \frac{d^2a}{dt^2} > 0 \tag{86}$$

This equation implies the scale factor must have been increasing at an increasing rate for the co-moving Hubble radius to decrease. This is the origin of the term inflation; the universe ballooned in size at a rapid rate so that the co-moving Hubble radius decreased during this epoch. It is easy to understand this result intuitively if we assume the physical Hubble radius is a constant during inflation (this is indeed assumed in most theories of inflation). At the absolute beginning of the universe when the scale factor was small, the co-moving Hubble radius was large. The physical Hubble radius encompassed much (if not all) of the co-moving coordinate grid of the universe. Most (if not all) particles were contained within the Hubble radius, thereby causally connected. Then the universe expanded at a fantastic rate, the scale factor blew up, and the co-moving Hubble radius became small. The physical Hubble radius does not change in size, but the coordinate grid was pulled apart so drastically by inflation that the physical Hubble radius now only encompasses a small portion of this grid. Most particles in the universe are no longer in physical contact, but they once were. Thus, inflation solves the horizon problem. The scales that today appear causally disconnected were once in causal contact before inflation.

It is also possible to understand this result in terms of conformal time. The universe before inflation was microscopically small, and all of its constituents were causally connected. As inflation drove the rapid expansion of the universe, the causally connected region also dramatically increased. The causally connected region is bounded by η, so η must have blown up due to inflation. η is still increasing today as the universe expands, but most of its value was accrued during inflation. Because of inflation, our conformal time horizon covers enormous distances. The distances we previously assumed to be causally disconnected are actually still contained within the conformal time horizon. This makes our old definition of conformal time useless in the Boltzmann-Einstein equations. We redefine it so that

$$\eta = \int_{t_e}^{t} \frac{dt'}{a(t)} \tag{87}$$

where t_e is the time at which inflation ends. This redefinition of η allows us to consider the conformal time horizon beginning after inflation. It will be helpful when we consider the evolution of perturbations to present times.

Inflation is a beautiful theory that solves the horizon problem. Even though objects may seem out of contact today because they are separated by distances greater than the co-moving horizon, inflation tells us that very early on when the universe was microscopically small, these distance were in causal contact. Inflation explains this phenomenon by requiring a period during which the scale factor increases at an increasing rate. This is an entirely new idea. The forms of energy that we discussed in section I - matter and radiation - cause the universe to expand at a decreasing rate. The second derivative of the scale factor is less than zero. Thus, some unknown form of energy must have driven inflation.

It is easy to show this form of energy must have had negative pressure. All we need are the two zero-order Einstein equations (equations 7 and 8). We can combine these two equations to isolate the double time derivative of the scale factor. It reads

$$\frac{d^2a/dt^2}{a} = -\frac{4\pi G}{3}(\rho + 3P) \tag{88}$$

Inflation requires the second time derivative of the scale factor to be positive. Thus, the left-hand side of the above equation is positive. For the right-hand side of the above equation to be positive,

$$P < -\frac{\rho}{3} \tag{89}$$

The energy density must always be positive, so inflation requires a form of energy whos pressure is negative. This requirement leads us to consider a scalar field as the unknown form of energy that drove inflation.

It is possible for a scalar field to have positive energy density and negative pressure. Furthermore, it is possible to show that a scalar field results in perturbations to the spacial curvature. Inflation driven by a scalar field may have set the initial conditions of Φ. Of course, none of this can has been verified imperially â inflation occurred at energies too high to probe with modern day devices. For the purposes of this paper, we assume that inflation was driven by a scalar field, and we will use the resultant value of Φ as the initial condition. A more thorough derivation requires scale field theory, a topic beyond the scope of this paper. I will merely quote the result and give a rough sketch of its justification.

Energy in the form of a scalar field causes Φ to have a non-zero variance as it emerges from inflation. During inflation, Φ couples to the energy density of the scalar field. The scalar field is a quantum mechanical object. The mean value of its energy distribution goes to zero during inflation, but due to the Heisenberg uncertainty principle, it has non-zero variance. Small quantum mechanical fluctuation exist in the scalar field energy. Φ, being coupled to this energy, picks up these small fluctuations. After the end of inflation as the scalar field vanishes, the modes of $\Phi(k)$ retain a non-zero variance. These values will be our sought after initial conditions.

The variance of the modes of Φ are usually encoded in a power spectrum $P_\Phi(k)$.

It is defined in the following way

$$\langle \Phi(k)\Phi^\dagger(k') \rangle \equiv (2\pi)^3 P_\Phi(k)\delta^3(k-k') \tag{90}$$

where the term in the brackets is the quantum mechanical variance. The Dirac-delta function enforces the independence of the variance of the different modes. The power spectrum of the gravitational perturbations at the end of inflation caused by the scalar fields is

$$P_\Phi(k) = \frac{8\pi}{9k^3}\frac{H^2}{\epsilon m_{pl}^2}\bigg|_{aH=k} \tag{91}$$

Note that the power spectrum is a function of the co-moving wavelength. The modes of the spacial curvature perturbation decouple from the scalar field when the co-moving wavelength of the mode becomes larger than the co-moving horizon. Remember that the co-moving horizon decreases during inflation, while the co-moving wavelength remains constant. When the modes cross the horizon affects the amplitude of their power. This accounts for the evaluation at $aH = k$.

The power spectrum is typically redefined in the following way

$$P_\Phi(k) \equiv \frac{50\pi^2}{9k^3}\left(\frac{k}{H_0}\right)^{n-1} \delta_H^2 \left(\frac{\Omega_m}{D_1(a=1)}\right)^2 \tag{92}$$

This definition is by convention, and changes nothing of the content of equation 91. It merely recodes it in a different way; it will become more clear why in the next section. δ_H represents the amplitude of each mode as it crosses the horizon. n is known as the spectral index, and controls the k dependence of the power spectrum. Different models of the universe predict different values for the spectral index, although most are very close to $n = 1$. D_1 is the growth function of the perturbation. We will define and elaborate on this in more detail in the next section, but it is conventional to include this in the definition for the power spectrum of Φ.

One final note on the power spectrum. By the definition in equation 90, the power spectrum has units $[m^3]$. To normalize the power spectrum, we multiply by a factor of k^3. If the spectral index is equal to one, which it is very close to in many theories, then $k^3 P_\Phi(k)$ is a constant, or scale-free. This is an important concept for if the power spectrum is scale-free, then the amplitude of the mode $\Phi(k)$ is independent of the time it exits (and re-enters) the co-moving horizon.

We have now successfully found the initial conditions for Φ. Armed with this knowledge, and the six Boltzmann-Einstein evolution equations, we can solve for the distribution of dark matter inhomogeneities as they appear today.

3 Distribution of Matter Over-Densities Today

In the previous section, we developed the formalism that allows us to deal with perturbations to the smooth universe. We derived six differential equations, known as the Boltzmann-Einstein equations, that govern the evolution of our six perturbation variables. We related the initial conditions of each to the initial condition of Φ. We then discussed inflation as the source of the non-zero initial condition for Φ, and quantified this initial condition in the form of a primordial power spectrum.

In this section, we will use the primordial power spectrum of Φ and the Boltzmann-Einstein equations to derive an analogous expression for the power spectrum of the dark matter perturbations in the universe today. This is the end goal that motivated our formalism. The power spectrum of the dark matter perturbation is the theoretical tool with which we can quantify the inhomogeneities of matter in the universe.

Once we have derived this expression, we will use the power spectrum to evaluate the variance in the over-densities of matter contained within a sphere of radius R. We will plot the variance as function of the scale R for different cosmological models of the universe. This plot will be bounded by empirical cosmological data. These bounds should provide a strong insight into which cosmological models most accurately represent our universe.

3.1 The Power Spectrum of Dark Matter Perturbations in the Late Universe

Before we embark on the difficult journey of solving the evolution equations for the power spectrum of matter ($P_\delta(k,a)$), it is helpful to have some physical insight into the processes that govern the distribution of matter. The two forces that affect dark matter are pressure and gravity. Gravity is an attractive force, sucking the matter into over-dense regions causing clumps. Pressure increases as density increases due to random thermal motion, and it causes matter to flow outward. Thus, if we were to sketch Newton's law for density perturbations it would look something like

$$\ddot{\delta} = [\text{Gravity - Pressure}]\delta \tag{93}$$

Gravity and pressure are the two competing forces that guide the evolution and distribution of dark matter in the universe. These equation will be contingent upon certain cosmological conditions, like the dominant form of energy in the universe, but it will be helpful to ground our formal results within this physically intuitive framework.

3.1.1 Method

It is now time to lay the framework for our derivation of the power spectrum of δ. It is defined in an analogous way to the power spectrum of Φ_p

$$\langle \delta(k)\delta^\dagger(k') \rangle \equiv (2\pi)^3 P_\delta(k)\delta^3(k-k') \tag{94}$$

Because the power spectra of δ and Φ_p are defined in the same way, we can find a relation between the power spectra if we can find a relation between the perturbations. Equation 71 provides the first step in deriving this relationship. It relates δ to Φ (this is not Φ primordial). Because we are interested in the values of their power spectra today, we can drop the radiation density term. Radiation is a negligible form of energy density today. A second simplification arises because the modes of interest today have long since entered co-moving horizon. As a result, $aH/K \ll 1$ and we can drop the last term in equation 71. This leaves

$$\delta(k,a) = \frac{k^2\Phi(k,a)}{4\pi G\rho_m a^2} = \frac{2k^2\Phi(k,a)a}{3\Omega_m H_0^2} \tag{95}$$

This only holds for $a > a_{late}$ where a_{late} denotes an epoch well into the matter dominated era ($a_{late} \gg a_{eq}$. Equation 95 relates present day values of δ and Φ. If we can find a relationship between $\Phi_{primordial}$ and Φ_{late}, then we can find the power spectrum of δ today from the known primordial power spectrum of Φ.

Finding Φ_{late} from $\Phi_{primordial}$ will be a thorny endeavor. It will involve understanding how each independent mode Φ_k evolves over time. Each of these modes is coupled to the other perturbations throughout its evolution. To complicate matters more, each mode re-enters the horizon at different times. After inflation ends, the horizon is very small. Most modes of the cosmic perturbations are super-horizon, $k\eta \ll 1$. As the horizon grows, smaller modes will enter the universe at much earlier times than larger modes. Thus, we must understand the super-horizon and sub-horizon evolution of each mode Φ_k and how to relate the two.

To reduce the complexity of this problem, we introduce two new functions: the transfer function ($T(k)$) and the growth function ($D_1(a)$). These two functions relate the primordial values of $\Phi(k)$ to its values today. The evolution of Φ can be broken into two stages. It turns out that once the perturbation modes are well in to the matter dominated era ($a > a_{late}$), they evolve independently of their mode. They only evolve as a function of the scale factor. Up until this point ($a < a_{late}$), the evolution of each mode Φ_k is dependent upon k. Nevertheless, the large-scale modes of Φ are super-horizon until the matter dominated era. While they are super-horizon, they remain roughly constant. The large-scale modes of Φ at a_{late} are roughly equal to the primordial values. We therefore define the transfer function by

$$T(k) \equiv \frac{\Phi(k, a_{late})}{\Phi\left(k_{large-scale}, a_{late}\right)} \tag{96}$$

and define the growth function by

$$\frac{D_1(a)}{a} \equiv \frac{\Phi(a)}{\Phi(a_{late})} \qquad (a > a_{late}) \tag{97}$$

With the transfer and growth functions defined in this way, we can relate $\Phi_p(k)$ to its values in late epoch $\Phi(k, a)$. The relation is given in the following equation

$$\Phi(k, a) = \frac{9}{10}\Phi_p(k)T(k)D_1(a) \qquad (a > a_{late}) \tag{98}$$

The transfer function propagates each mode of $\Phi_p(k)$ to its appropriate value at $a = a_{late}$. After this epoch, the perturbations evolve as a function the scale factor, independent of the mode. The growth function accordingly scales the perturbations during the matter dominated era and beyond ($a > a_l ate$). The factor of $9/10$ is present because the large-scale modes decrease by a factor of $9/10$ from their primordial values during their super-horizon evolution.

The transfer and growth functions equate the primordial gravitational perturbation to its value today. Equation 95 equates the gravitational perturbation to the dark matter over-density today. Combining these two equations, we can express the δ as a function of our initial condition from inflation. It is given by

$$\delta(k, a) = \frac{3k^2}{5\Omega_m {H_0}^2}\Phi_p(k)T(k)D_1(a) \qquad (a > a_{late}) \tag{99}$$

With this relationship between δ and Φ_p, we can find the power spectrum of the dark matter over-density perturbations. It will be function of the Φ_p power spectrum, the transfer function, and the growth function. Using the definition in equation 94 and replacing $\langle\Phi(k)\Phi^\dagger(k')\rangle$ with our primordial power spectrum of Φ, we find the power spectrum of the dark matter over-density is

$$P(k, a) = 2\pi^2\delta_H^2 \frac{k^n T^2(k)}{H_0^{n+3}} \left(\frac{D_1(a)}{D_1(a = 1)}\right)^2 \tag{100}$$

where we have input the analytical solution for the primordial power spectrum of Φ. From here, we need only derive analytical solutions for the transfer function and growth function to completely determine the power spectrum of the dark matter over-densities.

The transfer and growth functions provide much needed structure for our derivation of the dark matter power spectrum, but we have yet to mathematically justify their place in equation 98. We have no particular reason, for instance, to believe the evolution of Φ becomes independent k during the matter dominated era, or that large scale modes of Φ are roughly constant while they are super-horizon. In the following subsection, we will use the Boltzmann-Einstein equations to derive analytical expression for the transfer and growth functions, and in the process defend the basic arguments the led to their definitions and usefulness.

3.1.2 Transfer Function

Determining the analytic form of the transfer function will be a difficult business, like we discussed above. The transfer function encodes the entire k-dependence of the evolution of Φ until a_{late}. The perturbations stay mostly constant when they are super-horizon, as we would expect because they are outside the realm of causal physics. The main difficulty in finding the transfer function is determining the sub-horizon evolution of each mode Φ_k and connecting it with the super-horizon evolution. In fact, it is not possible to solve this riddle for all modes. Instead, we will investigate the behavior of large and small-scale modes. Once we have determined their k-dependence, we will spline together our results to arrive at an analytic transfer function that covers all k-values.

Large-scale modes remain super-horizon until they enter the universe in the matter dominated era. We have already said the super-horizon evolution of all modes is solvable. The most important information, then, is that the sub-horizon evolution of the large-scale modes takes place when the radiation energy density is negligible. We can use fact to drop the radiation terms in our Boltzmann-Einstein equations, allowing us to solve analytically for the evolution of Φ_{large}. Similarly, small-scale modes remain super-horizon for a short period of time and enter the universe in the radiation dominated era. Thus, we can drop the matter terms in our Boltzmann-Einstein equations to find the evolution of Φ_{small} during the radiation era. The small scale perturbations of Φ will experience decay due to the large pressure of radiation. This decay will allow us to analytically track the perturbations through the epochs of equality and radiation domination.

We cannot analytically solve for the k-dependence of intermediate modes precisely because we cannot utilize any of the above approximations. Intermediate modes enter the horizon around the time of matter-radiation equality. All perturbation terms will have an important effect. Without the ability to simplify our evolution equations in some way, they are unsolvable analytically. We will have to be content with splining together our results of large and small scale modes to determine the k-dependence of intermediate modes.

Now that we have outlined our logic for finding Φ, it is time to delve into the specifics of the Boltzmann-Einstein evolution equations. Note that there is not an overarching blueprint for how to solve these equations. Instead, we must make smart approximations and choose our equations wisely based upon the specific evolution we are studying.

We start by considering the evolution of super-horizon evolution of large-scale modes. In this limit, the approximation $k\eta \ll$ holds. We can drop all terms proportional to k, and the velocity and dipole perturbations decouple from the evolution equations. We can simplify our 3 first order evolution equations to two using the initial condition from equation 83. $\delta = 3\Theta_{r,0}$ is in fact a solution to the super-horizon evolution at all times. Plugging $\delta/3 = \Theta_{r,0}$ into equation 72, we are left with two coupled first order differential equations. They are

$$\dot{\delta} = -3\dot{\Phi}3H\left(H\Phi + \dot{\Phi}\right) = 4\pi G a^2 \rho_{dm}\delta\left(1 + \frac{4}{3y}\right) \tag{101}$$

where $y \equiv (\rho_{dm}/\rho_r) = (a/a_{eq})$. To solve a pair of couple first order differential equations, we must derive and then we can eliminate one variable. The price is a second order derivative. This procedure in terms of our new variable y gives

$$\Phi + \frac{21y^2 + 54y + 32}{2y(3y+4)(y+1)}\Phi + \frac{1}{y(3y+4)(y+1)}\Phi = 0 \tag{102}$$

This second order differential equation has an analytic solution. With the proper initial condition, the solution is given by

$$\Phi = \frac{\Phi(0)}{10y^3}\left(9y^3 + 2y^2 - 8y + 16(y+1)^{1/2} - 16\right) \tag{103}$$

This equation describes the large-scale super-horizon evolution of Φ. At small y, Φ simplifies to its primordial value ($\Phi = \Phi(0)$) as it should. At large y, the equation asymptotes to 9/10 the primordial value of Φ. This justifies our inclusion of the factor of 9/10 in equation and support the argument that large-scale modes are roughly constant as they cross the horizon.

To finish our discussion of the evolution of large-scale modes of Φ, we must determine how they evolve when they enter the horizon in the matter dominated era. In this limit, the radiation perturbations are unimportant. We would like to show that Φ is a constant in the matter dominated era. To do this, all we need to show is that $\dot{\Phi} = 0$ is a solution to the Boltzmann-Einstein equations. We know from our discussion of the super-horizon evolution of large modes that $\dot{\Phi} = 0$ is the initial condition. Therefore, if constant Φ is a solution to the evolution equations, it must be the unique solution to the evolution equations.

The Boltzmann-Einstein evolution equations in the no-radiation limit are

$$\dot{\delta} + ikv = 0 \tag{104}$$

$$\dot{v} + aHv = ik\Phi \tag{105}$$

$$k^2\Phi = \frac{3}{2}(aH)^2\left(\delta + \frac{3aHiv}{k}\right) \tag{106}$$

where we have chosen to use the algebraic form of Einstein's equations. We use the third of these equations to eliminate δ from the first equation. Then, we use the second equation to eliminate $\dot{v}$ from the resultant equation. This leaves:

$$\frac{k^2\dot{\Phi}}{(aH)^2} + \left(\frac{3iv}{2k} + \frac{\Phi}{aH}\right)\left[\frac{9(aH)^2}{2} + k^2\right] = 0 \tag{107}$$

This equation, along with equation 106, form our pair of coupled first order differential equations. Remember, we would like to show $\Phi = C$ is a valid solution to our system of differential equations. This is true if the second order decoupled equation is of the form $A\ddot{\Phi} + B\dot{\Phi} = 0$. We would like to show, therefore, the second order equation has no terms proportional to Φ. To do this, we differentiate equation 107, replace $\dot{v}$ terms using equation 105, and keep only the terms proportional to Φ. The terms proportional to Φ are given in the following expression:

$$(\Phi\,\text{terms}) \rightarrow -\left[\frac{3iaHv}{k} + \Phi\right](9(aH)^2 + k^2) \tag{108}$$

The term in the brackets is proportional to $\dot{\Phi}$ by equation 107. Therefore, there are terms proportional to Φ in the second order differential equation. $\Phi =$ constant is the unique solution that describes the evolution of large-scale Φ perturbations in the matter dominated era. This result agrees with our physical intuition. In the matter dominated era, there is no pressure. Therefore, there is no outward to dampen the potential and it remains constant.

Aside from the factor of 9/10 that comes during the super-horizon evolution, the large-scale modes of Φ remain constant and independent of k. Thus, the transfer function for large scale modes is $T(k) = 1$. This approximation holds to about 10% accuracy for modes such that $k < k_{eq}/3$, where $k_eq = a_{eq}H(a_{eq})$.

Finding transfer function at small-scale modes requires a different technique than we just used for large-scale modes. We cannot focus solely on the potential perturbations. Instead, we use the potential perturbations to find the evolution of the density perturbations. We will track the density perturbations through the radiation era and epoch of equality, and use our finding to derive a transfer function.

Small scale modes re-enter the horizon deep in the radiation era. The dark matter perturbations have no effect on the potential, but the evolution of the potential radiation still shapes the dark matter perturbations. To solve for δ at small scales in the radiation era requires two steps. First, we must find how Φ evolves as it couples with Θ_r. Then we solve for δ using Φ as a driving force in the differential equation.

Ignoring the dark matter perturbations in the radiation era, the evolution equations become

$$\dot{\Theta}_0 + k\Theta_1 = -\dot{\Phi} \tag{109}$$

$$\dot{\Theta}_1 - \frac{k}{3}\Theta_0 = -\frac{k}{3}\Psi \tag{110}$$

$$\Phi = \frac{16\pi G a^2 \rho_r}{k^2} \left[\Theta_{r,0} + \frac{3aH}{k} \Theta_{r,1} \right] \tag{111}$$

Again, we have chosen the algebraic form of Einstein's equation. The same procedure to solve this coupled system follows. We used the third equation to eliminate $\Theta_{r,0}$ from the two radiation equations. We are left with a pair of coupled first order equations. We differentiate and isolate one variable. The final second order differential equation is

$$\ddot{\Phi} + \frac{4}{\eta}\dot{\Phi} + \frac{k^2}{3}\Phi = 0 \tag{112}$$

whose solution is a spherical Bessel function of the first order. Thus,

$$\Phi = 3\Phi_p \left[\frac{\sin(k\eta/\sqrt{3}) - (k\eta/\sqrt{3})\cos(k\eta/\sqrt{3})}{(k\eta/\sqrt{3})^3} \right] \tag{113}$$

The above equation demonstrates that Φ decays in the radiation era. After it decays, it oscillates around zero. This behavior is expected given the high pressure in the radiation era. The high pressure causes $\Theta_{r,0}$ to oscillate. The expansion of the universe, however, dilutes its density. Therefore, we see the oscillating decay of Φ. Φ oscillates with $\Theta_{r,0}$, but it's amplitude decreases as η^{-2}.

To find δ, we treat Φ as a driving force in our coupled equations 74 and 75. First, we isolate δ by differentiating equation 74 and eliminating v and $\dot{v}$. This leaves

$$\ddot{\delta} + \frac{1}{\eta}\delta = -3\ddot{\Phi} - \frac{3}{\eta}\dot{\Phi} + k^2\Phi \tag{114}$$

The possible solutions to a linear inhomogeneous second order differential equation are a linear combination of the two homogeneous solutions and the particular solution. The particular solution can be found using Green's function. The homogeneous solutions are given by a constant term and a natural log of the conformal time $C_1 + C_2 ln(\eta)$. At early times, δ is constant and the particular solution is small. The coefficient of the logarithmic term must be zero, and constant term must equal the initial condition $C_1 = (3/2)\Phi_p$. The particular solution, found by integrating the source term with a Green's function, is proportional to $ln(k\eta)$. Thus, we expect the dark matter perturbation to take the form

$$\delta(k, \eta) = A\Phi_p \ln(Bk\eta) \tag{115}$$

The logarithm can be expanded to the sum of a constant term and a term proportional to $\ln(k\eta)$ as required by the above argument. Finding A and B requires the evaluation of a Green's function integral. The appropriate values are A = 9.mp6 and B = 0.44.

Equation 115 tells us that the dark matter perturbations grow even in the radiation dominated era, albeit at a logarithmic rate. Again, the pressure due to radiation slows the growth of the perturbations. Equation 115 also incorporates a k-dependence. This will be significant in determining the transfer function.

Before we can determine the transfer function for small-scale modes, however, we must track the evolution of δ through the epoch of equality and into the matter dominated era. We need the k-dependence of the small scale modes at a_{late}. We are saved by the fact that $\rho_{dm}\delta$ becomes greater than $\rho_r\Theta_{r,0}$ at some point before $\rho_r = \rho_{dm}$. This is because the radiation pressure suppresses $\Theta_{r,0}$ while δ grows logarithmically. At some point long before matter-radiation equality but far after the small-scale modes have crossed the horizon ($a_H < a < a_{eq}$), the gravitational potential stops coupling to $\Theta_{r,0}$ and starts coupling to δ. We will solve for the the evolution of δ in this limit, and match it to the evolution of δ deep in the radiation era given in equation 115.

To simplify our evolution equations, we make two approximations. First, we drop the radiation perturbations because they become suppressed. Second, we assume $aH/k \ll 1$ because the small-scale modes are well within the horizon at this point. These approximations only hold in the limit $a \gg a_H$, where a_H is the value of the scale factor when the small-scale modes crossed the horizon. In this limit, our three coupled equations become

$$\delta + \frac{ik}{aHy}v = -3\Phi \tag{116}$$

$$v' + \frac{v}{y} = \frac{ik}{aHy}\Phi \tag{117}$$

$$k^2\Phi = \frac{3y}{2y+2}(aH)^2\delta \tag{118}$$

where we have written the equation in terms of the parameter y we defined earlier. To solve this system of differential equations, we go through the same routine. We differentiate the first of the three equation, and eliminate $\dot{v}$. To arrive at a second order equation that isolates δ, we must assume that δ is much larger than Φ on sub-horizon scales. This approximation is justified because Φ, like $\Theta_{r,0}$ is suppressed in the radiation era. The second order differential equation is then

$$\delta'' + \frac{2+3y}{2y(y+1)}\delta' - \frac{3}{2y(y+1)}\delta = 0 \tag{119}$$

This equation is known as the Meszaros equation. It governs the evolution of the cold dark matter perturbations after radiation perturbations have been suppressed. There are two solutions to the Meszaros equation. They are given by

$$D_1 = y + 2/3 D_2 = (y+2/3)\ln\left[\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right] - 2\sqrt{1-y} \tag{120}$$

As y becomes large, D_1 scales as y while D_2 scales as $y^{-3/2}$. D_1 is the growing solution and D_2 is the decaying solution. Indeed, in the next section we will see that the large-y limit of D_1 is equal to the growth function during the matter dominated era. Neither of these solutions has a k-dependence.

The general solution to the Meszaros equation is a linear sum of its two solutions.

$$\delta(k, y) = C_1 D_1(y) + C_2 D_2(y) \qquad\qquad y \gg y_H \tag{121}$$

To determine the constants, we must match the above equation with equation 115, the evolution equation for δ deep in the radiation era. To appropriately match the equations, we must find an epoch during which both evolution equations are valid. The general solution to the Meszaros equation holds in the limit that the small-scale modes have become significantly sub-horizon ($y \gg y_H$). Equation 115 holds deep in the radiation-dominated era ($y \ll 1$). Therefore, the matching epoch occurs after the small-scale modes have become significantly sub-horizon but remain deep in the radiation era ($y_H \ll y_m \ll 1$). This is only possible because the gravitational perturbation couples to the dark matter perturbations before the epoch of equality. Matching the two evolution equations yields

$$A\Phi_p ln(Bk\eta) = C_1 D_1(y_m) + C_2 D_2(y_m) \tag{122}$$

$$\frac{A\Phi}{y_m} = C_1 D_1'(y_m) + C_2 D_2'(y_m) \tag{123}$$

The matching conditions will determine the constants C_1andC_2 in the general solution to the Meszaros equation. By matching this solution to the evolution of δ in the radiation era, we have patched together a function that describes the evolution of the small scale modes of δ from their entry into the horizon through to a_{late} epoch in the matter dominated era. We must now turn this evolution function into a transfer function from small-scale modes.

To find the transfer function, we are interested in the large-y limit of the general solution to the Meszaros equation. The D_2 term decays in this limit, so we are only interested in the first term. Using equation to solve for C_1, we find

$$C_1 = \frac{3A\Phi_p(k)}{2} \ln\left[\frac{4Ba_{eq}e^{-3}}{a_H}\right] \tag{124}$$

Plugging this constant into equation 121 and dropping the decaying term, we find

$$\delta(k, a) = \frac{3A\Phi_p(k)}{2} \ln\left[\frac{4Ba_{eq}e^{-3}}{a_H}\right] D_1(a) \qquad\qquad (a \gg a_e q) \tag{125}$$

This equation give the small-scale dark matter perturbations long after matter-radiation equality. Comparing this with equation 99, we derive an expression for the transfer function at small scales:

$$T(k) = 12\left(\frac{k}{k_{eq}}\right)\ln\left[\frac{k}{8k_{eq}}\right] \tag{126}$$

We have found expressions for transfer function at large and small-scales. We can spline these results together to determine an approximate function that describes the transfer function on all scales. The exact analytic solution is given by the BBKS transfer function. It is

$$T(x \equiv k/k_{eq}) = \frac{ln(1+0.171x)}{0.171x}\left[1 + 0.284x + (1.18x)^2 + (0.399x)^3 + (0.490x)^4\right]^{-1/4} \tag{127}$$

This exact equation agrees with both the large scale and small scale forms of the transfer functions that we found. On small scales, the BBKS transfer function goes as $\ln(k)/k^2$ and on large scales the transfer function goes to 1. Note that the logarithmic dependence of the growth function is entirely due to the logarithmic growth of the dark matter perturbations in the radiation dominated era.

3.1.3 Growth Function

Up until this point, we have entirely glossed over the growth function. We have shown that large-scale potential perturbations remain constant in the matter dominated era, and although we have not shown it, potential perturbations on all scales are constant in the matter dominated era. Why then define a growth function for $a > a_{late}$ if the gravitational potential is constant?

The growth function is really included only to account for the fact that the dark matter perturbations grow with the scale factor during the matter dominated era. We only defined it in terms of the gravitational potential so that we could relate its primordial values to its present day values. The growth function during the matter dominated era does not affect the gravitational potential.

We have in fact already derived the growth function. Equation 121 is the general solution to the Meszaros equation. Although we derived this equation for small-scale modes, this solution holds for all modes in the matter-dominated era. The approximations we made to derive the Meszaros equation were (1) the radiation perturbations were negligible and (2) the modes were contained well within the horizon. These approximations hold for all modes of interest in the matter dominated era. The growth function is the large-y (or large-a) limit of equation 121. The second term, which decays as the scale factor grows, drops out and the growing term asymptotes to $D_1 = a$. The prefactor, remember, is included in the transfer function, so we leave it out of the growth function. Thus, the form of the growth function in the matter dominated era is

$$D_1(a) = a \qquad\qquad (a > a_{late}) \tag{128}$$

If another form of energy were to dominate our universe, like dark energy, then the form of the growth function could change. This would require re-deriving the Meszaros equation allowing for other forms of energy. This growth function would affect all perturbations and could cause the gravitational potentials to no longer remain constant.

This concludes our discussion on the present day power spectrum of the dark matter perturbations. The dark matter power spectrum, given in equation 100, depends upon the primordial power spectrum of the gravitational potential, the transfer function, and the growth function. We have derived all of these. Equation 92 gives the primordial power spectrum of Φ, equation 127 gives the transfer function, and equation 128 gives the growth function. All that is left is to explore the implications this power spectrum has on the distribution of matter in the universe, and to compare these predictions to observed cosmological data.

3.2 Bounding the RMS Over-Densities of Different Models

The beauty and force of the formalism we have just developed lies in its ability to predict the inhomogeneities of matter in our universe. We can use the power spectrum of the dark matter over-densities to derive theoretical formulas that attempt to characterize the statistical distribution of matter. The statistical measurement that we focus on is the expected RMS over-density.

$$\sigma_R^2 \equiv \langle \delta_R^2(x) \rangle \tag{129}$$

$$\delta_R(x) = \int d^3x' \delta(x) W_R(x - x') \tag{130}$$

Here, $W_R(x)$ is the window function. For points within a distance of radius R center at a point x, the window function is equal to one. For all other points, the window function is equal to zero.

Phyiscally, the expected RMS over-density σ_R^2 predicts the variance in the over-densities contained within a sphere of radius throughout the universe. It gives us an idea of how much the over-densities, contained within a sphere, vary as we change the position of the sphere. A high RMS over-density values suggests that clumping is very prevalent while a low value suggests a relatively smooth distribution of matter.

To derive a theoretical formula for RMS over-density, we must Faurier transform equation 130. This leads to an integral in k-space over the dark matter power spec-

Figure 1

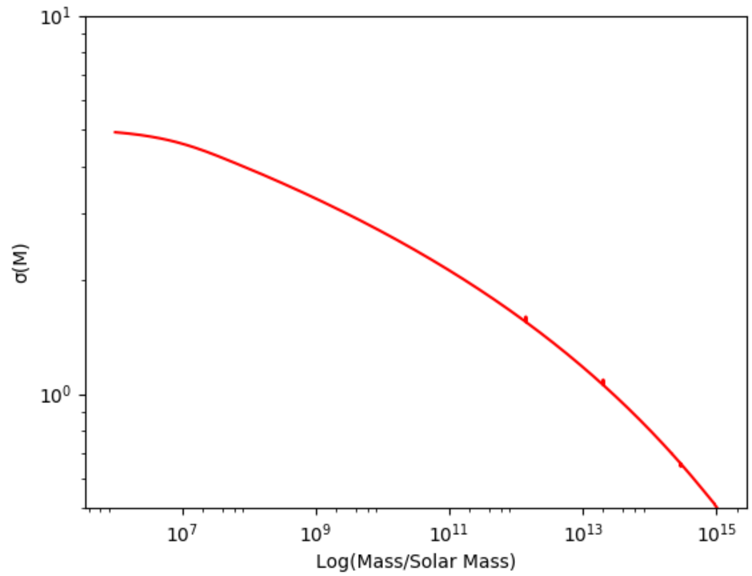

Figure 1: A plot of the variance of the matter over-densities as a function of the mass scale. On large scales, the variance is small and the universe appears smooth. On smaller scales, clumping becomes prevalent.

trum. It is given by the following equation

$$\sigma_R^2 = \int_0^\infty \frac{dk}{2\pi^2} k^2 P(k) W_R^2(k) \tag{131}$$

Here, we see that the power spectrum plays an essential role in determining this statistical measure. Each mode of δ does not contribute equally to the total over-densities in the universe. The power spectrum allows us to weight each mode of the over-densities appropriately.

Combining our solution for the dark matter over-density power spectrum with equation 131, we can plot the expected RMS over-density against the size of our sphere. Figure 1 shows the resulting curve. We have used a mass scale to determine the size of the sphere rather than its radius. Because we know Ω_m and ρ_{cr}, it is simple to switch between the mass contained within a sphere of radius R using $M = 4/3(\pi R^3)\Omega_m \rho_{cr}$.

To calculate the numerical values in Figure 1, we used conventionally accepted values for cosmological parameters (δ_H^2, H_0, and Ω_m, and n) found in equation 100. Note that the variance in the matter over-density decreases as the mass scale increases. This is consistent with our knowledge that the universe appears homogeneous on large scales.

The beauty of equations 100 and 131, however, is that they are completely general. Equation 100 determines the expected form of the dark matter power spectrum and

Figure 2

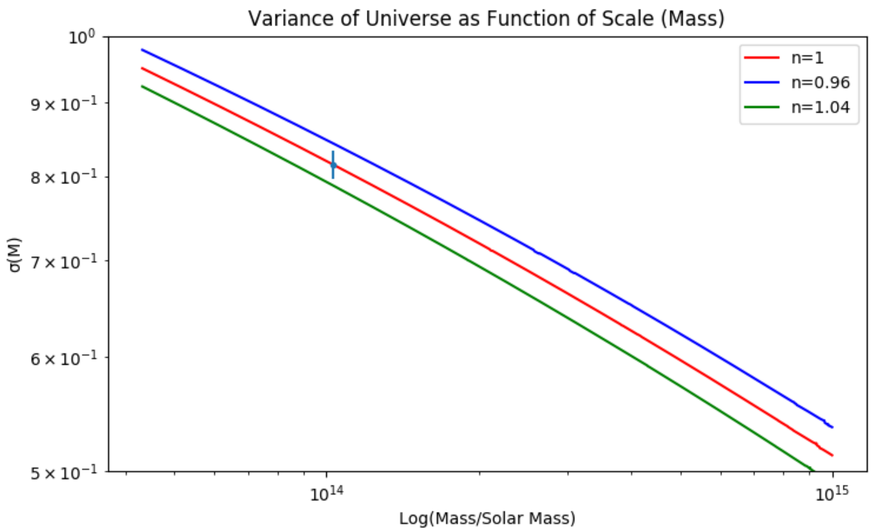

Figure 2: A plot of the variance in the matter over-densities for different values of the spectral index. The error bar represents an empirically-observed bound. For the cosmological values used, this plot suggest the spectral index equals 1.

equation 130 determines the form of the RMS over-density, but they quantify these values in terms of a few cosmological parameters. Different cosmological models predict different parameters, and we can use plots of the RMS over-density to compare these models.

Furthermore, the variance in the matter over-density of the universe is something we can determine observationally. We use observational bounds of the variance in the over-densities of the universe to help decipher between the validity of different cosmological models. We have done just this in Figure 2. Figure 2 shows three different RMS over-density curves, each with a slightly different value for the spectral index. Also plotted is a bound on the RMS over-density for a given mass scale. This bound was taken from a recent cosmological study that empirically determined the value of RMS over-density for a given mass scale. The bound suggests that, for the values of the cosmological parameters given above, $n = 1$ is the only reasonable value for the spectral index.

Figure 2 is an example of how the theoretical formula for the RMS over-density of matter in the universe acts as a bridge between observations today and the processes that governed the evolution of the universe far in its past. Remember that the spectral index was a property of inflation. The bound in Figure 2 can help us determine what exactly its value is. By measuring the distribution of matter in our universe today, we can explore the physics of the universe even in its earliest epochs.

4 Acknowledgements

For much of my thesis, I referenced Scott Dodelson's textbook titled *Modern Cosmology*. It's reference can be found here [1].

I would also like to thank my advisor, Professor Savvas Koushiappas, for the time, energy, and guidance he gave me while working on my thesis.

References

[1] Scott Dodelson. *Modern Cosmology*. Academic Press, 2003.