Anyons and Dualities in 2+1 Dimensions

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1 Introduction

All elementary particles known to us fall into two big classes: bosons and fermions. This classification is made according to how two identical particles can coexist in the same state. Electrons, quarks, and other fields of matter are fermion. They satisfy Pauli exclusion principle, which states that fermions can not simultaneously occupy the same state. Photons, gravitons, and so on are the fields that carry interaction. These are bosons. One can also form composite particles expanding the number of examples. Mathematically quantum mechanical wave function $(\mathbf{r}_1, \ldots, \mathbf{r}_N)$ of N identical particles satisfies

$$(\ldots, \mathbf{r}_i, \ldots, \mathbf{r}_j, \ldots) = \exp(i\ )\ \ (\ldots, \mathbf{r}_j, \ldots, \mathbf{r}_i, \ldots) \tag{1}$$

$= 0$ for bosons and for fermions.

Needless to say that finding a system of identical particles whose wave function has $\neq 0$, was considered a fascinating discovery. Such systems were found first theoretically and later experimentally.

According to [5] the starting point was Hall effect. Let us think of electrons on an infinite conducting strip of width w on the xy-plane. There is a uniform magnetic field B pointing in z-direction. If an electric current is flowing in the x-direction, the magnetic field forces the current to curl in the negative y-direction due to Lorentz force, so a Hall voltage V_H develops in the y-direction. Experimentally, though, there hasn't been found a "curling" of current inside such a strip. For the current to flow in a straight line, the Lorenz force originating from the magnetic field should be cancelled by the force originating from the gradient of the Hall voltage. Thus, one expects,

$$\frac{e}{c} vB = \frac{V_H}{w} \tag{2}$$

where v is the electron's velocity, e is the electron charge and c is the speed of light. Since the current is $I = vwne$, with n being the electron's density, we get

$$\frac{V_H}{I} = \frac{B}{nec} \tag{3}$$

This ratio of the Hall voltage to the current is known as the Hall resistance, denoted by R_H. Once the force that results from the Hall voltage cancels the force that results from the magnetic field, there will be no other effect of the magnetic field. Thus, the longitudinal voltage, the voltage drop parallel to the current, will be independent of the magnetic field. The ratio of this voltage to the current is the longitudinal resistance R_l.

In practice when the Hall effect is measured in high mobility two dimensional electronic systems at low temperatures, the Hall resistance is not linear in magnetic field, in contrast to what (3) suggests (see Fig 1 the diagonal graph). The graph shows horizontal steps. The length of a step depends on a sample, but the value of R_H at the steps (the height of

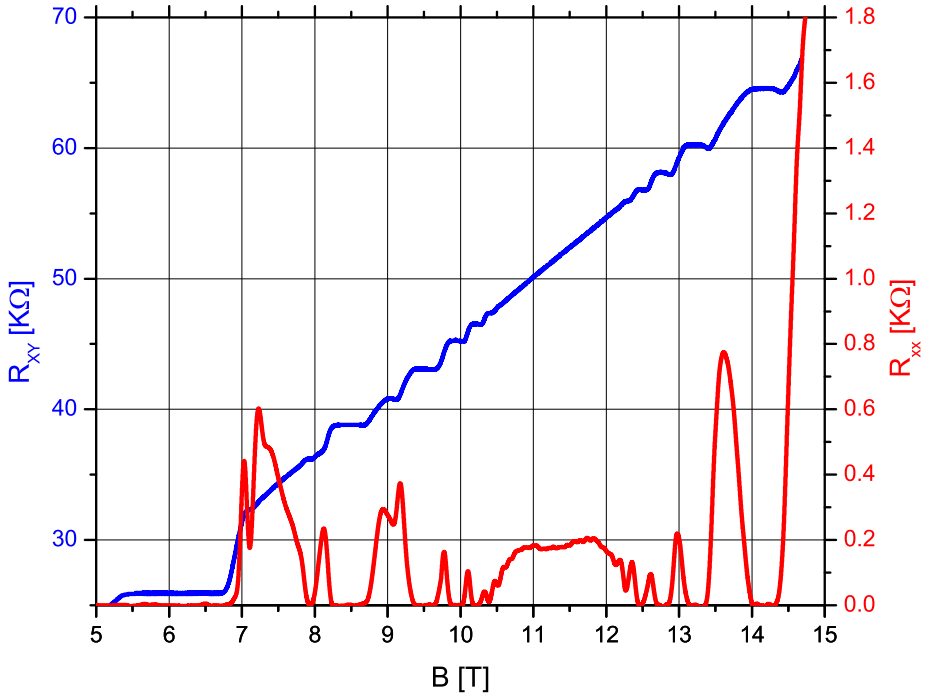


Figure 1: The quantum Hall effect. When the Hall resistance (R_{XY} or as we call it in the text R_H) is measured as a function of magnetic field, plateaus at quantized values are observed. In regions of the magnetic field where the Hall resistance is in a plateau, the longitudinal resistance (R_{XX} or as we call it in the text R_l) vanishes (sample grown by Vladimir Umansky and Measured by Merav Dolev, the Weizmann Institute of Science).

a step) is universal. At the steps,

$$R_H = \frac{h}{e^2}\frac{1}{-} \tag{4}$$

Here h/e^2 (h is the Planck constant) is the quantum of resistance. The dimensionless number has, in the observed steps, either integer values $= 1; 2; 3:::$ going up to several tens, or "simple" fractions p/q. In addition, at the steps $R_I = 0$ we are dealing with (anisotropic) superconductor!

Later Arovas and Schrieffer, ([1]), and Halperin, [4], put forward a theory that gave an explanation of this effect through the use of anyons. In this text we are going to discuss some aspects of their theory. Any theory of quantum Hall effect must have some minimal set of features. Conductances R_H^{-1} and R_I^{-1} are functions of temperature T. We have a nice picture with steps at $T \sim 0$. It is known from experiments that deviation of $R_H^{-1}(T)$ and $R_I^{-1}(T)$ from $R_H^{-1}(0)$ and $R_I^{-1}(0)$ has the order of magnitude $\exp(-T_0/T)$. The function $\exp(-T_0/T)$ goes to zero exponentially fast as $T \to 0$. Such thing is possible only in theories with spectral gap. Thus we have to show that there is no continuous family of eigenvalues of the Schrödinger operator of the theory that starts at zero (energy of vacuum). We will see in Section 3.1 that this is true in some approximation. Needless to say that particles have to be coupled with a vector potential that gives magnetic field B.

One should emphasize that anyons by no means are elementary particles. These are composed objects like Cooper pairs (which are made of electrons) in superconductors. They disappear when temperature goes up. Still under right conditions they can be manipulated as ordinary electrons. The difference is that their wave function satisfy rather strange symmetry relation (1). After looking at it, it becomes clear that the phase depends on the direction we rotate an anyon at $\mathbf{r}_i$ about an anyon at $\mathbf{r}_j$. In other words the phase remembers some information about the paths of particles. The double slit experiment shows the relation between the phase of wave function and interference of electrons. Something similar can be done in the context of anyons: if one anyon makes a full turn about another anyon their interference patterns are supposed to change. Indeed several interferometers for anyons have been proposed: In the Fabry-Perot interferometer a stream of anyons splits in two. One of the streams makes a loop and acquires a phase. In a series of beautiful experiments by Camino *et al.* (see e.g. [2]), devices of the Fabry-Perot type were fabricated, and were measured in the integer and fractional quantum Hall regime.

This paper has the following structure: Section 2 covers Chern-Simons action, its interpretation, and why it is utilized in describing anyons. It continues on introducing the action for anyons

This paper sticks to the following structure: Chern-Simons Action section covers Chern-Simons action, its interpretation, and why it is utilized in describing anyons. It continues on, introducing the action for anyons and manipulating it to find the equations of motion that produce some important results. We find the Hamiltonian, which is used to find the energy spectrum. In Random Phase Approximation we generalize the system so that instead of having finite amount of anyons, we have an infinite uniformly continuous

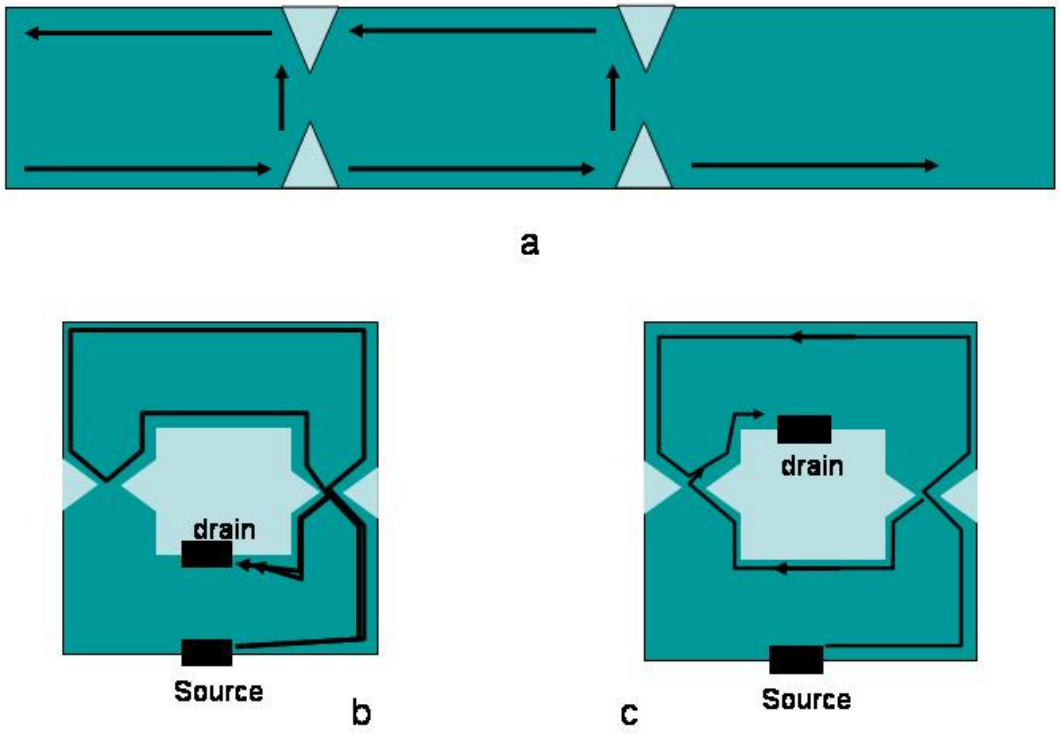


Figure 2: The Fabry-Perot (a,b) and Mach-Zehnder (c) interferometers. The second drawing is meant to emphasize the difference between the two interferometers. The interior edge is a part of the interference loop in the Mach-Zehnder interferometer, while it is not part of that loop in the Fabry-Perot interferometer. Furthermore, in the former only single tunnelling events take place, while the latter allows for multiple reflections and the formation of resonances.

distribution of anyons. We find the Hamiltonian for such a system and the corresponding energy spectrum. Then we start representing anyons as perturbation to boson and fermion and explain why choosing to represent anyons as perturbation of fermions is better. Next we do second quantization of the anyonic Hamiltonian. We derive an effective Coulomb interaction which is responsible for formation for the mass gap in the interacting theory. In the Vortices section we cover some dual theories. The main achievement of dualities is that they find the correspondence between statistics of quasiparticles and defect statistics in the same theories.

2 Chern-Simons action

Occasionally we will be using the language of differential forms. It will prove useful when change of variables is needed. The space R^3 has coordinates $r = (t; x; y)$. Later we will interpret it a 1+2-dimensional space-time. A triple of functions $(a_0(t; x; y); a_1(t; x; y); a_2(t; x; y))$ encodes a differential form $a = a_0 dt + a_1 dx + a_2 dy$. Curl and exterior differential are related by the formula

$$\begin{aligned} da &= (\vec{\nabla} \times (a_0; a_1; a_2)) \cdot (dxdy; dydt; dtdx) \\ df &= \vec{\nabla} f \cdot (dt; dx; dy) \\ d((a_0; a_1; a_2) \cdot (dxdy; dydt; dtdx)) &= \vec{\nabla} \cdot a \; dV(r) \end{aligned} \tag{5}$$

To avoid cluttering we denote $dxdydz$ by $dV(r)$. We assume that coefficients of the form a decay rapidly and the Chern-Simons functional

$$CS(a) = \frac{1}{2} \int_{\mathsf{R}^3} ada \tag{6}$$

is well defined. $d = dt\partial_t + dx\partial_x + dy\partial_y$ is the De Rham differential, ada stands for the product of forms. A more explicit formula for CS is $\frac{1}{2}\int_{\mathsf{R}^3} a \cdot (\vec{\nabla} \times a)\; dV(r)$. Let $f : \mathsf{R}^3 \to S^1 \subset \mathsf{C}^\times$ be a complex-valued function. We can use it to modify the form a

$$a \to a - i f^{-1} df \tag{7}$$

Observe that $d(f^{-1}df) = -f^{-2} df\, df - f^{-1} d^2 f = 0$ because $d^2 = 0$ and $df\, df = -df\, df = 0$. Let us see how the action changes under such transformation of a field:

$$\begin{aligned} &\frac{1}{2}\int_{\mathsf{R}^3} (a - i f^{-1} df) d(a - i f^{-1} df) = \\ &= \frac{1}{2}\int_{\mathsf{R}^3} (ad(a) - iad(f^{-1}df) - i f^{-1} df\, d(a) - (f^{-1}df) d(f^{-1}df)) \\ &= \frac{1}{2}\int_{\mathsf{R}^3} (ad(a) - i f^{-1} df\, d(a)) \end{aligned}$$

Let B^3 be a ball in $\mathbb{R}^3$ of radius R and S^2 its boundary sphere. Suppose that $f^{-1}df$ is nonzero only inside B^3. We use identity

$$f^{-1}df\, d(af^{-1}df) = -d(af^{-1}df) + d(f^{-1}df)af^{-1}df = -d(af^{-1}df)$$

and Stocks theorem $\int_{B^3} d(af^{-1}df) = \int_{S^2} af^{-1}df = 0$ to show that that action is invariant under such transformations. (7) are called gauge transformations.

Should we replace $\mathbb{R}^3$ by a more general manifold, e.g. torus (see (56) for appropriate boundary conditions), a variation of Chern-Simons action under general (big) gauge transformation takes integral values. To ensure that $\exp(i2\ kCS(a))$ is gauge invariant k must be an integer. This is condition of quantization of the coupling constant.

(8)

Let us compute equations of motion corresponding to CS:

$$\begin{aligned} CS(a) &= \frac{1}{2}\int_{B^3} a d\ a + \frac{1}{2}\int_{B^3}\ ada \\ &= \int_{B^3}\ ada - \frac{1}{2}\int_{S^2} a\ a \end{aligned}$$

We used that $a d\ a = -d(a\ a) + d(a)\ a$. Let us now assume that a is nonzero only inside B^3. Then variation will be equal to $\int_{B^3} da\ a$. As a is arbitrary we conclude that $da = 0$ in B^3 at critical a for CS. As the radius of B^3 can be chosen arbitrary large, $da = 0$ on $\mathbb{R}^3$. In components equation $da = 0$ is equivalent $\vec{\nabla} \times a = 0$.

It is important to mention that by Poincare lemma $da = 0$ implies that there is a function f such that $df = a$. In components it means that $a = \vec{\nabla} f$. This can be interpreted as follows: a is a result of a gauge transformation of a trivial solution $a = 0$ with a transformation $g = e^{if}$. (By definition gauge transformation is $a \to a - i\vec{\nabla}\log g$.) From this point of view all classical solutions of abelian CS theory are trivial on $\mathbb{R}^3$ or on a ball B^3 or any shape the ball can be deformed to. This is equivalent to the statement that CS theory has no local degrees of freedom.

It is no longer the case if we introduce some topology. For example, we can look for solution on $\mathbb{R}^3$ with t-axis removed. We know that $=\ !$ (82) satisfies $d\ = 0$. On the other hand if is an integer by Remark 1 we know that it is logarithmic gradient of $\frac{x+iy}{\sqrt{(x^2+y^2)}}$. Thus is gauge-equivalent to $_{+n}$. The set of real numbers up to integrals shifts is equal to a circle. It is possible to prove that any solution d on $\mathbb{R}^3$ with a line removed is gauge equivalent to with defined up to integral shift.

2.1 Symmetries of the action

Let $\tilde{r}(r) = (\tilde{t}(t;x;y);\tilde{x}(t;x;y);\tilde{y}(t;x;y))$ be an invertible change of variables. We can use $\tilde{r}(r)$ and $a_0dt + a_1dx + a_2dy = a(r)\cdot dr$ to construct a new form by pull-back construction:

$$\begin{aligned}\tilde{r}^*a\cdot dr &:= a(\tilde{r}(r))\cdot d(\tilde{r}(r))\\ &= a(\tilde{r}(r))\cdot(@\tilde{r}/@r)\cdot dr:\end{aligned}$$

It requires a simple computation to show that

$$(\tilde{r}^*a\cdot dr)d(\tilde{r}^*a\cdot dr) = (a\cdot(\vec{\nabla}\times a))(\tilde{r}(r))\det(@\tilde{r}/@r):$$

From this we conclude (with a help of change of variables)that

$$\begin{aligned}CS(a) &= \frac{1}{2}\int_{R^3} a\cdot\vec{\nabla}\times a\ dxdydt\\ &= \frac{1}{2}\int_{R^3}(a\cdot\vec{\nabla}\times a)(\tilde{r}(r))\det(@\tilde{r}/@r)dV(r)\\ &= \frac{1}{2}\int_{R^3}(\tilde{r}^*a\cdot dr)d(\tilde{r}^*a\cdot dr)\\ &= CS(\tilde{r}^*a)\end{aligned}$$

In other words such change of variables (or diffeomorphism) defines a symmetry of Chern-Simons theory.

2.2 Quantization of abelian Chern-Simons

I would like to apply ideas of Gaussian integrals to Chern-Simons functional. We need to modify Chern-Simons action (6) slightly: For this we fix N closed disjoint curves in R^3: $_i : [0;1] \to R^3$, $_i(0) = _i(1)$. Later we will identify non-closed curves with relativistic particle trajectories. will be the local time of an observer. We postulate that curves have no self-intersections. We define the modified action by the formula

$$CS\ (a) = kCS(a) + \sum_{i=1}^{N} k_i \int a\cdot\ '_i(\ )d\ ;\quad k;k_i\in R$$

If space-time has nontrivial topology (e.g. torus) the the sum in the action is not invariant with respect to arbitrary gauge transformation. The effect is similar to (8). In order to avoid it k_i must be integers.

One approach to quantization is through computation of path integrals. Statistical sum of the quantized theory is equal to

$$I = \int \exp(iCS\ (a))Da \tag{9}$$

This infinite-dimensional integral bears resemblance to finite integral (88). Indeed, the field a is analogous to x. The action $-iCS(a)$ is quadratic expression in a and is analogous to $b(x)/2$ (it is quadratic in x). The sum $i\sum_{i=1}^{N} k_i \int a \cdot {}_i'(\ )d$ is linear in a. It is similar to linear function $ip_j x^j$ (p_j is fixed). The idea is to derive an analogue of the formula (89).

In finite dimensions linear transformations are defined by means of matrices. Can this be done in infinite dimensions. For example the analogue of the identity matrix ${}_{ij}$ is the Dirac delta function $(x-y)$. The identity transformation on functions is defined by operator $f(x) = \int_{-\infty}^{\infty} (y-x)f(y)dy$. Matrix coefficients of the operator $@^k$, which is $(-1)^k\ {}^{(k)}(y-x)$, can extracted from $@^k f(x) = \int_{-\infty}^{\infty}(@_x^k\ (y-x))f(y)dy$. In the case of Chern-Simons theory CS can be written (it follows directly from (5)) as

$$
\begin{aligned}
&CS(a) = \\
&= \frac{1}{2}\int_{R^3} \vec{\nabla}\times a\cdot a\ dV(r) \\
&= \frac{1}{2}\int_{R^3} a\cdot \begin{pmatrix} 0 & -@_y & @_x \\ @_y & 0 & -@_t \\ -@_x & @_t & 0 \end{pmatrix}\cdot a\ dV(r) \\
&= \frac{1}{2}\int_{R^3} a(r')\cdot \begin{pmatrix} 0 & -@_{y'}\ {}^3(r-r') & @_{x'}\ {}^3(r-r') \\ @_{y'}\ {}^3(r-r') & 0 & -@_{t'}\ {}^3(r-r') \\ -@_{x'}\ {}^3(r-r') & @_{t'}\ {}^3(r-r') & 0 \end{pmatrix}\cdot a(r)\ dV(r)\ dV(r') \\
&= \frac{1}{2}\int_{R^3} a(r')\cdot\Delta(r-r')\cdot a(r)\ dV(r)\ dV(r');\quad r=(t;x;y);\ r'=(t';x';y')
\end{aligned}
\tag{10}
$$

In our computations with pure Chern-Simons theory we use Euclidean (non-Minkowski) dot-product in the definition of the curl, gradient, and divergence. We can do this because Chern-Simons action is independent of the choice of the dot-product and the final answer should not depend on our computational technique. What we have to do is to find an inverse to the "matrix" $\Delta(r-r')$ whose indices runs through the set $\{(r;i)|r\in R^3; i=1;2;3\}$. The analogue of the equation for the inverse matrix b^{-1}: $b_{ij}b^{-1jk} = {}_i^k$ is

$$
\begin{aligned}
\int_{R^3}\Delta(r-r')_j^i G(r'-r'')_k^j dr' &= \begin{pmatrix} {}^3(r-r'') & 0 & 0 \\ 0 & {}^3(r-r'') & 0 \\ 0 & 0 & {}^3(r-r'') \end{pmatrix} \\
&= {}_k^i\ {}^3(r-r'');\quad i;j;k=1;:::;3
\end{aligned}
\tag{11}
$$

Equations (11) on the columns $(G_1(r);G_2(r);G_3(r))$ of matrix G are

$$
(\vec{\nabla}\times G_1(r);\vec{\nabla}\times G_2(r);\vec{\nabla}\times G_3(r)) = -\begin{pmatrix} {}^3(r) & 0 & 0 \\ 0 & {}^3(r) & 0 \\ 0 & 0 & {}^3(r) \end{pmatrix} = D
$$

I am going to use matrix

$$A = \begin{pmatrix} \frac{1}{4\ |r|} & 0 & 0 \\ 0 & \frac{1}{4\ |r|} & 0 \\ 0 & 0 & \frac{1}{4\ |r|} \end{pmatrix}$$

made out of fundamental solutions of Laplace equation to construct an approximation to G. Fundamental solution satisfies $\nabla^2 \frac{1}{4\ |r|} = \ ^3(r)$. I set

$$G_i = \vec{\nabla} \times A_i: \tag{12}$$

More explicitly

$$G(r) = \frac{1}{4} \begin{pmatrix} 0 & \frac{y}{(t^2+x^2+y^2)^{3/2}} & -\frac{x}{(t^2+x^2+y^2)^{3/2}} \\ -\frac{y}{(t^2+x^2+y^2)^{3/2}} & 0 & \frac{t}{(t^2+x^2+y^2)^{3/2}} \\ \frac{x}{(t^2+x^2+y^2)^{3/2}} & -\frac{t}{(t^2+x^2+y^2)^{3/2}} & 0 \end{pmatrix}$$

Recall the identity

$$\vec{\nabla} \times (\vec{\nabla} \times a) = \vec{\nabla}(\vec{\nabla} \cdot a) - \nabla^2 a;$$

where ∇^2 is the Laplace operator on vectors, then

$$\vec{\nabla} \times G_i = \vec{\nabla}(\vec{\nabla} \cdot A_i) - \nabla^2 A_i = \vec{\nabla}(\vec{\nabla} \cdot A_i) + D_i: \tag{13}$$

Operator G is an infinite-dimensional analogue of the matrix (90). We have to check equation (91). In our setting after taking (13) into account it becomes

$$\begin{aligned} &\int_{R^3} dV(r') a(r') \cdot \left(\vec{\nabla}_{r'} \times \left(\int_{R^3} G(r-r') \cdot a'(r)\ dV(r) \right) \right) = \\ &= \int_{R^3} \int_{R^3} a \cdot D(r-r') \cdot a'(r)\ dV(r)\ dV(r') + \\ &\quad + \int_{R^3} \int_{R^3} a \cdot \vec{\nabla}_{r'} \left(\vec{\nabla}_{r'} \cdot A(r-r') \right) \cdot a'(r)\ dV(r)\ dV(r') \end{aligned}$$

After integration by parts it becomes

$$\begin{aligned} &\int_{R^3} a(r) \cdot a'(r)\ dV(r) + \int_{R^3} \int_{R^3} (\vec{\nabla}_{r'} \cdot a(r')) \cdot \left(\vec{\nabla}_{r'} \cdot A(r-r') \right) \cdot a'(r)\ dV(r)\ dV(r') = \\ &= \int_{R^3} a(r) \cdot a'(r)\ dV(r) \quad \text{if } a; a' \in \operatorname{Ker} \vec{\nabla} \cdot \end{aligned}$$

We found infinite-dimensional analogue of the matrix g_{ij}. Now we have to formulate the analogue of $b^{-1ij} p'_i p'_j = g^{ij} p'_i p'_j$. Recall that the analog of $ix^j p_j$ with x^j replaced by the field $(a_0; a_1; a_2)$ was the sum $A = \sum_{i=1}^{n} k_i \int a \cdot \ _i'(\ )d$ -the result of integration of a over the union of curves with multiplicities which we write additively $C = \sum k_i \ _i$. To make analogy

$ix^j p_j$ more apparent we want to think about A as a pairing $\langle a; C\rangle$. This reflects additivity property of A (very much similar to $ix^j p_j$ when x is fixed) with respect to $\ _i$:

$$\langle a; C\rangle + \langle a; C'\rangle = \langle a; C + C'\rangle$$

$b^{-1ij} p_i p_j$ is no longer linear in p, it is rather quadratic. We have to use G to write quadratic function in C.

Equation $p' \in \mathrm{Ker}\, b^{\perp}$ in our setup becomes an equation $\langle a; C\rangle = 0$ for $a \in Triv$. But this equation is trivially satisfies because

$$\sum_{i=1}^{N} k_i \oint a \cdot \ _i' d\ \ = \sum_{i=1}^{N} k_i g(\ _i(\ \ ))|_0^2\ \ = 0 \text{ for } a = \vec{\nabla} g$$

We define $L(\ _i;\ _j)$ as

$$L(\ _i;\ _j) = \int_0^2 \int_0^2 \ \ _i(\ \ )' \cdot G(\ _i(\ \ ) -\ _j(\ \ ))\ _j(\ \ )' d\ \ d \tag{14}$$

The function we are looking for is

$$-\frac{i}{k} \sum_{i;j=1}^{N} k_i k_j L(\ _i;\ _j): \tag{15}$$

The answer for the functional integral (9) is

$$I = \mathrm{const} \exp\left(-\frac{i}{2k} \sum_{i;j=1}^{N} k_i k_j L(\ _i;\ _j)\right)$$

2.3 Topological Interpretation of $L(\ _i;\ _j)$

Introduce a two-form

$$! = \frac{1}{4} \frac{tdxdy + ydtdx + xdydt+}{(t^2 + x^2 + y^2)^{3/2}}$$

and a vector

$$A_0 = \frac{1}{4\ \ (t^2 + x^2 + y^2)^{3/2}} (t; x; y) \tag{16}$$

In spherical coordinates

$$\begin{aligned} t &= \ \cos(\ \ ); \\ x &= \ \sin(\ \ )\cos(\ \ ); \\ y &= \ \sin(\ \ )\sin(\ \ ); \\ &\geq 0;\ 0 \leq \ \ \leq \ \ ;\ 0 \leq \ \ \leq 2 \end{aligned}$$

it becomes

$$\frac{1}{4}\sin(\ )d\ d\ : \tag{17}$$

Which is (up to $\frac{1}{4}$-factor) an element of the unit sphere. It takes a simple computation with *Mathematica* to show that under the map $_{i;\ j}(\ ;\ ) := (\ _i(\ ) - \ _j(\ ))/|\ _i(\ ) - \ _j(\ )|$ the pullback of $!$ is the integrand of (14). By construction $_{i;\ j}(\ ;\ )$ defines a map of a torus to unit two-dimensional sphere. The integral (14) computes the volume of the torus with respect to the volume elements pulled-back from the sphere. This pull-back construction of of the volume form has a nice geometric interpretation. Fix some triangulation of the unit sphere. We can approximate area of the sphere by the sum of areas of triangles in triangulation. Preimage of the vertices of triangulation under the map can be used to define triangulation of the torus (we think about the torus as a square $[0;2\ ]\times[0;2\ ]$ with identified opposite sides). maps triangle Δ_i on the torus to triangle (Δ_i). Of course areas (Δ_i) and Δ_i are not equal. They are different by the factor approximated by Jacobian $J(\ _i;\ _i) = \left|\begin{matrix}\frac{@}{@} & \frac{@}{@}\\ \frac{@}{@} & \frac{@}{@}\end{matrix}\right|(\ _i;\ _i), (\ _i;u_i)$ in Δ_i. Triangles (Δ_i) cover the sphere but the sum $\sum Area(\ (\Delta_i))$ is different the area of the unit sphere. Here is why.

By definition degree deg $_{i;\ j}$ of $_{i;\ j}$ is the number of solutions of the equation $_{i;\ j}(\ ;\ ) = (\ ;\ )$ for a generic coordinate $\ ;\ $ on the sphere taken with signs:

$$\deg\ _{i;\ j} := \sum_{_{i;\ j}(\ ;\ )=(\ ;\ )} \operatorname{sign} J(\ ;\ )$$

It turns out that deg doesn't depend on the particular choice of $\ ;\ $.

Because of the possible nontrivial degree collection of triangles $\{\ (\Delta_i)\}$ cover the sphere deg times. This is why $\sum Area(\ (\Delta_i)) = 4\ \deg\ $. On the other hand $\sum_i Area(\ (\Delta_i))$ converges as Δ_i becomes smaller to $\sum_i J(\ _i;u_i)Area(\Delta_i)$ which by definition of pullback converges to $\int_T\ ^*_{i;\ j}!$.

So

$$\int_T\ ^*_{i;\ j}! = \deg\ _{i;\ j}\int_{S^2}! = \deg\ _{i;\ j}\frac{4}{4}$$

Another interesting fact that this number of solutions doesn't depend on small deformations of . In fact degree doesn't change as long as the curves $_i;\ _j$ don't intersect.

Now we encounter an important quantum effect that happens if $i = j$. This term is certainly present in the sum (15). Quantization requires an additional data - a tiny deformation $_{i;}$ of $_i$ in normal direction:

As we change $_{i;}$ sweeps a band as it is shown on the picture above.

The quantity $L(\ _i;\ _{i;})$ is well defined. The numbers $L(\ _i;\ _j)$ and $L(\ _i;\ _{j;})$ are equal because the deformation is small and $_i;\ _j$ are sufficiently far away. We should be aware that $_{i;}$ is not unique-there are many ways to cable $_{i;}$ around $_i$.

There is a simple way to compute $L(\ _i;\ _j)$. For this we have to attach an oriented film S to $_i$ and count the number n of times (with signs) $_j$ intersects this film:

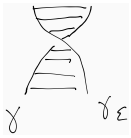


Figure 3: Deformation defines a band

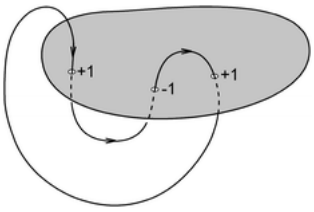


Figure 4: Computation of the linking number

This result has a nice interpretation in terms of magnetostatics. Imagine that the curve $_i$ represents position of a wire with a constant current I. By Biot-Savart Law it produces magnetic field $\mathbf{B}$:

$$\mathbf{B}(r) = \frac{_0 I}{4}\int_0^2 \quad _j(\ )' \times \frac{r - \ _j(\ )}{|r - \ _j(\ )|^3} d \ : \tag{18}$$

$_0$ is permeability of a free space. We can calculate line integral $\oint_{\ _i} \mathbf{B}(r)\cdot dr = \int_0^2 \ \mathbf{B}(\ _i(\ ))\cdot$ $_i(\ )' d$ by using Ampere's law:

$$\oint_{\ _i} \mathbf{B}(r)\cdot dr = \int_S \vec{\nabla}\times \mathbf{B}\cdot \mathbf{dS} = \ _0 \int_S \mathbf{J}\cdot \mathbf{dS} = \ _0 I_{encl} = \ _0 nI: \tag{19}$$

Here, I_{encl} means the total amount of current crossing the surface S. The number I_{encl}/I is the number of times n the wire crosses the surface S. On the other hand

$$\begin{aligned}\oint_{\ _i} \mathbf{B}(r)\cdot dr &= \oint_{\ _i}\oint_{\ _j} \frac{_0 I}{4}\frac{dr'\times(r-r')}{|r-r'|^3}\cdot dr \\ &= \frac{_0 I}{4}\oint_{\ _i}\oint_{\ _j} \frac{(r-r')\cdot(dr\times dr')}{|r-r'|^3};\end{aligned} \tag{20}$$

where the last step follows by applying the cyclic property of the triple product. Up to factor of $\ _0 I$ it is equal to (14). This proves the assertion.

Linking number number is s topological invariant of the system of the curves. If we deform configuration of curves keeping them disjoint free from self-intersection the new

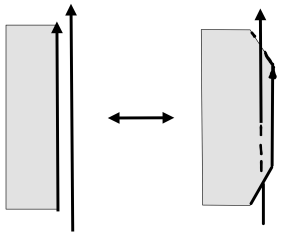


Figure 5: Modification of the linking number

configuration will have the same matrix $L_{ij} = L(\ _i;\ _j)$. This can be seen directly by looking at CS action. Indeed, as we know (see paragraph 2.1) the group of diffeomorphism (changes of variables) preserve the action. We can use such diffeomorphism to deform configuration of curves without affecting path integral. An example of a diffeomorphism is a flow along a vector field on $\mathbb{R}^3$ for a given duration of time.

Such invariance can be used for iterative computation of linking numbers. Imagine that we have a film attached to the curve $\ _j$ which we use for computation of linking number $L(\ _j;\ _l)$. We can align $\ _j;\ _l$ as it is shown on the left picture

Modification shown on the right picture increases linking number by one. The reverse procedure decreases the number by one. This modification is local and all other numbers $L(\ _m;\ _n); (m;n) \neq (j;l)$ stay the same. Denote the old configuration of curves by C and the new by C'. From this we see that the path integral $I(C')$ changes on phase: $I(C)\exp(\frac{ik_jk_l}{2k})$.

2.4 The action for anyons

In the more realistic model curves have to be dynamic objects-trajectories of relativistic $2+1$-dimensional particles. The action for $\gamma(\tau) = (t(\tau); x(\tau); y(\tau))$ is

$$S(\gamma) = mc \int \sqrt{c^2 t'^2 - x'^2 - y'^2}\, d\tau :$$

The total action

$$\begin{aligned} S(\gamma; a) &= \sum_{i=1}^{N} m_i c \int \sqrt{c^2 t_i'^2 - x_i'^2 - y_i'^2}\, d\tau + \left(\sum_{i=1}^{N} k_i a \cdot \gamma_i' \, d\tau + \frac{k}{2} \int_{\mathbb{R}^3} a \cdot \vec{\nabla} \times a \, dV(r) \right) \\ &= S(\gamma) + CS(a) \end{aligned}$$

The theory has certain Schrödinger operator H. The matrix coefficients of the evolution operator $\exp itH$ can be computed by means of path integral. Fix initial $r_i = (x_i; y_i; t_i)$ and final $\tilde{r}_i = (\tilde{x}_i; \tilde{y}_i; \tilde{t}_i)$ space-time coordinates. In addition $t_i = t$ $\tilde{t}_i = \tilde{t}$. Matrix coefficients are

$$\exp iH(r_1; \ldots; r_N; \tilde{r}_1; \ldots; \tilde{r}_N) = \int \exp(S(\gamma; a)) D\gamma \, Da \tag{21}$$

The integral is taken over configurations $\gamma_i(0) = r_i$; $\gamma_i(1) = \tilde{r}_i$. The gauge transformations of a are trivial at $r_i; \tilde{r}_i$. We can compute matrix coefficients in two steps. First we integrate out a. After that we do integration over γ_i. In the process of integrating out a we notice that

$$\int \exp(S(\gamma; a)) D\gamma \, Da = \int D\gamma \, \exp(S(\gamma)) \int \exp(CS(a)) Da$$

We have already know how to integrate $I_{braid} = \int \exp(CS(a)) Da$ the only modification that collection $\{\gamma_i\}$ defines a braid, rather then a knot. Familiar arguments work in this case also. I_{braid} is (up to a universal constant) equal to $\exp(-\frac{i}{2k} \sum_{i;j} k_i k_j L(\gamma_i; \gamma_j))$. The quantity $L(\gamma_i; \gamma_j)$ is no longer integral because $\gamma_{i;\gamma_j}(\tau; \tau')$ doesn't sweep the sphere. Still if we keep the ends of the braid fixed local modifications described on the Figure (5) change $L(\gamma_i; \gamma_j)$ on ± 1. It means that the effective action

$$\exp(S(\gamma) - \frac{i}{2k} \sum_{i;j=1}^{N} k_i k_j L(\gamma_i; \gamma_j)) \tag{22}$$

gets multiplied on the phase

$$\exp\left(\pm \frac{i}{2k} k_i k_j \right) : \tag{23}$$

Suppose that two trajectories γ_i and γ_j of particle A and B that meet a point. There are two ways to slightly perturb trajectories avoid collision-either A leaves B on the right or on the left. Such deformation has a negligible effect on the action $S(\gamma)$. As we saw the residue of the Chern-Simons term gives a significant contribution. This is why (22) is called the exponent of anyonic action.

The action $-mc\int\sqrt{c^2t'^2-x'^2-y'^2}d$ is invariant with respect to reparametrization $\to\tilde{}(\ )$:

$$\begin{aligned}
&-mc\int\sqrt{c^2t'^2(\tilde{}(\ ))(\tilde{}(\ )')^2-x'^2(\tilde{}(\ ))(\tilde{}(\ )')^2-y'^2(\tilde{}(\ ))(\tilde{}(\ )')^2}d\ =\\
&=-mc\int\sqrt{c^2t'^2(\tilde{}(\ ))(\tilde{}(\ )')^2-x'^2(\tilde{}(\ ))-y'^2(\tilde{}(\ ))}\tilde{}(\ )'d\ =\\
&=-mc\int\sqrt{c^2t'^2(\tilde{})-x'^2(\tilde{})-y'^2(\tilde{})}d\tilde{}
\end{aligned}$$

We want to use this invariance to get read of t'^2 in the Lagrangian and set it to one. For this we are going to use coordinate t as independent parameter on the curve. In this case $\frac{@t}{@t}=1$. $(x(t);y(t)):=(x(\ );y(\ ))$, where $t=t(\ )$.

After this the action becomes $-mc^2\int_{t_0}^{t_1}\sqrt{1-(x(\ )')^2/c^2-(y(\ )')^2/c^2}dt$, which is an equivalent form of $-mc\int_{\ 0}^{\ 1}\sqrt{c^2t'^2-x'^2-y'^2}d\ ;t_0=t(\ _0);t_1=t(\ _1)$.

We would like to study non-relativistic anyons. It means that

$$x'^2+y'^2\ll c^2: \tag{24}$$

In this case we can use Taylor expansion in relativistic Lagrangian which yields $-mc^2\sqrt{1-(x')^2/c^2-(y')^2/c^2}=-mc^2+m(x'^2+y'^2)/2+\cdots$. We are going to drop mc^2 term which won't affect the dynamics. The terms corresponding to $\cdots$ can also be dropped because of the assumption (24).

2.5 Integrating out Chern-Simons eld

Lagrangian becomes

$$\begin{aligned}
L(\ _i;a)&=\sum_{i=1}^{N}\left(m_i(x_i'^2+y_i'^2)/2+k_i(a_0+a_1x_i'(t)+a_2y_i'(t))\right)+\frac{k}{2}\int_{\mathbb{R}^2}a\cdot\vec{\nabla}\times adxdy\\
&=L(\ _i)+CS_2(\ _i;a)
\end{aligned} \tag{25}$$

In the formulas below $\ (t)\coloneqq(t;x(t);y(t))=(x_0(t);x_1(t);x_2(t))$ is a path in $\mathbb{R}^3$. Then we have that $\ '^2\coloneqq x'^2+y'^2$ and $a\cdot\ '\coloneqq a_0(t;x(t);y(t))1+a_1(t;x(t);y(t))x'(t)+a_2(x(t;x(t);y(t))y'(t)$. Equations of motion are

$$\begin{aligned}
&S(\ _i+\ \ _i;a)-S(\ _i;a)=\sum_{i=1}^{N}\int_{t_0}^{t_1}\left(m_i\ '_i\cdot\ \ '_i+k_i(@_ia)\cdot\ '\ x_i(t)+k_ia\cdot(\ \ )'\right)dt=\\
&=\sum_{i=1}^{N}\int_{t}^{t'}\left(-m_i\ ''_i\cdot\ \ _i+k_i((@_ia)\cdot\ '\ x_i-(a)'\cdot\ \ _i)\right)dt+(m_i\ '_i\cdot\ \ _i+k_ia\cdot\ \ _i)|_t^{t'}
\end{aligned} \tag{26}$$

$$\begin{aligned}
S(\ _i;a+\ a)-S(\ _i;a)=&\int_{t_1}^{t_2}\sum_{i=1}^{N}(k_i\ a\cdot\ '_i)dt+k\int_{B^3}\ a\cdot\vec{\nabla}\times a\ dV(r)-\\
&-\frac{k}{2}\int_{S^2}a\times\ a\cdot ndS
\end{aligned} \tag{27}$$

$$
\begin{aligned}
(@_{x_i} a)\cdot \quad' \; x_i - (a)'\cdot \quad &= (@_i a)\cdot \quad' \; x_i - ((@_{x_i} a)\cdot \quad)x_i' \\
&= \operatorname{curl} a \cdot (\quad \times \quad') \\
&= - \quad \cdot (\operatorname{curl} a \times \quad')
\end{aligned}
$$

If we assume that variation on t_1 and t_2 vanish we get

$$
-m_i \;\; _i'' - k_i(\operatorname{curl} a \times \;\; _i') = 0
$$

If a vanish on the curves $\;_i$ equation of motion (27) gives

$$
\vec{\nabla} \times a = 0 \quad x \in \mathbb{R}^3 \backslash \bigcup_{i=1}^{n} \;\; _i
$$

-function $\;_0$ concentrated at a point $0 = (0;0;0)$ is defined by equation

$$
\int f(t;x;y) \;\; _0(t;x;y) \; dV(r) = f(0;0;0):
$$

By analogy -function concentrated at a curve is defined by equation

$$
\int f(r) \quad (r) \; dV(r) := \int_{t_1}^{t_2} f(t;x(t);y(t))dt:
$$

In terms of curl and equation (27) can be written as

$$
k\vec{\nabla} \times a(r) = -\sum_{i=1}^{N} k_i \quad _i(r) \;\; _i'(t) \tag{28}
$$

which is the form Ampere's law for fictitious magnetic field a.

Let n_0 be a unit vector $(1;0;0)$. Fix time t_0 and choose a closed oriented contour C in the plane $(t_0;x;y)$, which bounds domain S. Equation (28) implies that

$$
\begin{aligned}
k \oint_C a(r)\cdot dr &= \int\int k\vec{\nabla} \times a(r)\cdot n dx dy \\
&= -\sum_{i=1}^{N} k_i \int \quad _i(r)(\;\; _i'(t_0)\cdot n_0) dx dy \\
&= -\sum_{i=1}^{N} k_i \int \quad _i(r) dx dy \\
&= -\sum_{i=1}^{N} k_i
\end{aligned}
$$

In the formula we used Green's formula for integrals. We have control over the sign $\pm$ in front of $\sum_{i=1}^{N} k_i$ by means of reversing time orientation or by changing orientation of the

contour C. We arrive to the formula $k\oint_C a(r)\cdot dr = -\sum_{i=1}^{N} k_i$ if a is a solution of equation of motion.

To make connection with formulas in [3] we have to set

$$k_i = e; k = \ ; m_i = m \text{ and reverse direction of time compared to ours as in [3].} \qquad (29)$$

The last formula simplifies to $\Phi = \oint_C a(r)\cdot dr = eN$, (c.f. formula 2.3 in [3]).The phase (23) becomes $\exp(ie^2/2\ )$ (c.f. formula 2.4 in [3]).

Again, our goal is to analyze functional integral

$$\int \exp(S(\ _i; a))D\ _i Da_0 Da_1 Da_2$$

with boundary conditions on $\ _i$ as in integral (21). As before we are going to integrate out a. We want make some simplifications first.

In paragraph about Gaussian integral we discussed what happens with Fourier transform of the Gaussian integral when coefficient b goes to zero or infinity. From this we deduce that -function $(x-x_0)$ has an integral presentation $(x-x_0) = \frac{1}{2}\int \exp(i(x-x_0)p)dp$. It admits an obvious multidimensional generalization $^n(x-x_0) = \frac{1}{(2\ )^n}\int \exp(i(x_j-x_{j0})p^j)\,dV(p)$. The integral $\frac{1}{(2\ )^n}\int f(x;x_0)\exp(i(x_j-x_{j0})p^j)\,dV(p)\,dV(x)\,dV(x_0)$ is equal to $\int f(x_0;x_0)\,dV(x_0)$. Let us replace $x_j - x_{j0}$ with a more general $K_j(x;x_0)$. $K_j(x;x_0) = 0$ defines implicit relation between x and x_0 which can be solved $x_j = M_j(x_0)$. As before

$$\frac{1}{(2\ )^n}\int f(x;x_0)\exp(iK_j(x;x_0)p^j)\,dV(p)\,dV(x)\,dV(x_0) \qquad (30)$$

reduces to $\int f(M(x_0);x_0)\,dV(x_0)$. This could be useful in computation of path integral. For example consider action $S = \int (i(w'(t)-y(t))p(t) - y^2(t))dt$.The path integral (with appropriate boundary conditions for fields $w(t);p(t);y(t)$) $\int \exp(S)DwDyDt$ has a formal analogy with (30). Variable x corresponds to $y(t)$, x_0 - to $w(t)$ Array of functions $\{K_j(x;x_0)\}$ corresponds to $(y(t);w(t)) \to w'(t) - y(t)$. Function $f(x;x_0)$ corresponds to $\exp(\int(-y^2(t))dt)$. We conclude that up to some standard constant corresponding $\frac{1}{(2\ )^n}$

$$\int \exp(S)DwDyDp = \int \exp(\int(-w'^2(t))dt)Dw$$

There is a simple, but less convincing method, to arrive to the same result. Euler-Lagrange equation for this Lagrangian corresponding to variation with respect to p is $w'(t)-y(t) = 0$. It is independent of p because t-derivatives of p never appear. It defines a constraint, which allows to reduce our initial action to $\int(-w'^2(t))dt$. The argument with path integrals shows that after making this reduction we loose no information.

We would like to apply this idea to Lagrangian (25). We interpret expression (25) as a Lagrangian of a quantum mechanics with infinite degrees of freedom. Besides particles represented mathematically at time t_0 by two-dimensional vectors $(x_i(t_0);y_i(t_0))$ we also

have fictitious magnetic field $(a_0(t_0;x;y);a_1(t_0;x;y);a_2(t_0;x;y))$. We interpret it as a vector in the space of triples of functions on $\mathbb{R}^2$. Configuration space of such objects is infinite-dimensional.

We would like to look at the partial functional integral

$$\int \exp(S(\ _i;a))Da_0 = \exp(S(\ _i)) \int \exp(CS(\ _i;a))Da_0:$$

from this point of view. First we rewrite $a \cdot \vec{\nabla} \times a$ as a sum of two terms: one that contains t-derivatives, the other that doesn't. $x;y$ integrations gives

$$\begin{aligned}
&\int a \cdot \vec{\nabla} \times a dxdy = \\
&= \int_{\mathbb{R}^2} (a_2 \partial_t a_1 - a_1 \partial_t a_2) dxdy + \int_{\mathbb{R}^2} (a_1 \partial_y a_0 - a_0 \partial_y a_1 - a_2 \partial_x a_0 + a_0 \partial_x a_2) dxdy = \\
&= \int_{\mathbb{R}^2} (a_2 \partial_t a_1 - a_1 \partial_t a_2) dxdy + 2 \int_{\mathbb{R}^2} a_0 (\partial_x a_2 - \partial_y a_1) dxdy
\end{aligned}$$

We used integration by parts once. Lagrangian (25) transforms to

$$\begin{aligned}
L(\ _i;a) = &\sum_{i=1}^{N} m_i (x_i'^2 + y_i'^2)/2 + \\
&+ \int_{\mathbb{R}^2} \sum_{i=1}^{N} (k_i a_0 \quad _i(x;y)) + k a_0 (\partial_x a_2 - \partial_y a_1) dxdy + \\
&+ \sum_{i=1}^{N} k_i (a_1 x_i'(t) + a_2 y_i'(t)) + \frac{k}{2} \int_{\mathbb{R}^2} (a_2 \partial_t a_1 - a_1 \partial_t a_2) dxdy
\end{aligned} \tag{31}$$

Note that we transformed the term $a_0(t;x(t);y(t))$ in ((25)) to equivalent form $\int_{\mathbb{R}^2} a_0(t;x;y)\ ^2(x - x(t);y - y(t))dxdy$. We are using notations $^2(x - x(t);y - y(t)) =: (x;y)$.

We would like to point to analogy of of the middle term $\int_{\mathbb{R}^2} a_0(\sum_{i=1}^{N}(k_i \quad _i(x;y)) + k(\partial_x a_2 - \partial_y a_1))dxdy$ in (31) and the term $(w'(t) - y(t))p(t)$ from the example. Component a_0 is analogous to p. Its time derivatives never appear in the Lagrangian as it was explained can be integrated out leaving us with the constraint

$$\sum_{i=1}^{N} (k_i \quad _i(x;y)) + k(\partial_x a_2 - \partial_y a_1) = 0 \tag{32}$$

In this equation curves $_i$ serve as parameters. We are going to solve this equation at a given time $t = t_0$. Solution of this equation is obviously non unique. If $(a_1(t_0;x;y);a_2(t_0;x;y))$ is a solution then $(a_1(t_0;x;y) + \partial_x f(t_0;x;y);a_2(t_0;x;y)) + \partial_y f(t_0;x;y)$ is also a solution. This is due to the gauge symmetries of the initial problem. To eliminate non uniqueness we impose gauge-fixing condition $\partial_x a_1 + \partial_y a_2 = 0$.

From Section B we know that solution of $@_x a_2(x;y) - @_y a_1(x;y) = {}^2(x;y)$ is given by $! = \frac{1}{2}(\frac{-y}{x^2+y^2}; \frac{x}{x^2+y^2})$. Denote vector $(x;y)$ by r. The operator J has matrix

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{33}$$

$!$ can be written as $\frac{J\mathsf{r}}{2\ |\mathsf{r}|^2}$. By superimposing solution corresponding different $\ _i$ we get

$$(a_{1;i}; a_{2;i})(x;y) = -\sum_{j=1}^{N} k_i/k\,!\,(x - x_j(t_0); y - y_j(t_0)) = -\frac{k_i}{2\ k}\sum_{j=1}^{N} k_j \frac{J(\mathsf{r} - \mathsf{r}_j)}{|\mathsf{r} - \mathsf{r}_j|^2}$$

In order to simplify formulas we will use an abbreviation

$$\begin{aligned} \mathsf{a}_i(\mathsf{r}) &:= \frac{k_i}{2\ k}\sum_{j=1}^{N} k_j \frac{J(\mathsf{r} - \mathsf{r}_j)}{|\mathsf{r} - \mathsf{r}_j|^2} \\ \mathsf{a}_i &:= \frac{k_i}{2\ k}\sum_{j=1;j\neq i}^{N} k_j \frac{J(\mathsf{r}_i - \mathsf{r}_j)}{|\mathsf{r}_i - \mathsf{r}_j|^2} \\ \mathsf{a}(\mathsf{r}) &:= \frac{e}{2\ k}\sum_{j=1}^{N} \frac{J(\mathsf{r} - \mathsf{r}_j)}{|\mathsf{r} - \mathsf{r}_j|^2} \end{aligned} \tag{34}$$

Upon substitution into (31) Lagrangian becomes

$$\begin{aligned} L(\ _i) := L(\ _i; \mathsf{a}) &= \sum_{i=1}^{N} m_i|\mathsf{r}_i'|^2/2 - \frac{1}{2\ k}\sum_{1\le i\neq j\le N} k_i k_j \frac{J(\mathsf{r}_i - \mathsf{r}_j)\cdot \mathsf{r}_i'}{|\mathsf{r}_i - \mathsf{r}_j|^2} + \\ &+ \frac{1}{8\ {}^2k}\sum_{1\le i\neq j\le N} k_i k_j @_u F(\mathsf{r}_i(u); \mathsf{r}_j(t))|_{u=t} = \\ &= \sum_{i=1}^{N} m_i|\mathsf{r}_i'|^2/2 - \mathsf{a}_i\cdot \mathsf{r}_i' + \\ &+ \frac{1}{8\ {}^2k}\sum_{1\le i\neq j\le N} k_i k_j @_u F(\mathsf{r}_i(u); \mathsf{r}_j(t))|_{u=t} \end{aligned} \tag{35}$$

The function $F(\mathsf{r}_i; \mathsf{r}_j)$ is defined by the integral

$$\int_{\mathbb{R}^2} \frac{\langle \mathsf{r} - \mathsf{r}_i; \mathsf{r} - \mathsf{r}_j\rangle}{|\mathsf{r} - \mathsf{r}_i|^2|\mathsf{r} - \mathsf{r}_j|^2}\, dV(\mathsf{r}): \tag{36}$$

$\langle \mathsf{r}_i; \mathsf{r}_j\rangle$ stands for determinant $\det\left|\begin{smallmatrix} x_i & x_j \\ y_i & y_j \end{smallmatrix}\right|$. We used that $\langle J\mathsf{r}_i; J\mathsf{r}_j\rangle = \langle \mathsf{r}_i; \mathsf{r}_j\rangle$. It is hard to evaluate this integral directly. Instead we will use its symmetries to show that it is a constant function. Fix a vector q, then

$$F(\mathsf{r}_i + \mathsf{q}; \mathsf{r}_j + \mathsf{q}) = \int_{\mathbb{R}^2} \frac{\langle \mathsf{r} - \mathsf{r}_i - \mathsf{q}; \mathsf{r} - \mathsf{r}_j - \mathsf{q}\rangle}{|\mathsf{r} - \mathsf{r}_i - \mathsf{q}|^2|\mathsf{r} - \mathsf{r}_j - \mathsf{q}|^2}\, dV(\mathsf{r})$$

After a change of variables $\mathbf{r} - \mathbf{q} = \tilde{\mathbf{r}}$ we recover the original integral. Thus $F(\mathbf{r}_i + \mathbf{q}; \mathbf{r}_j + \mathbf{q}) = F(\mathbf{r}_i; \mathbf{r}_j)$ and $F(\mathbf{r}_i; \mathbf{r}_j) = F(\mathbf{r}_i - \mathbf{r}_j; 0)$. If we apply rotation matrix $\begin{pmatrix} \cos(\ ) & \sin(\ ) \\ -\sin(\ ) & \cos(\ ) \end{pmatrix}$ simultaneously to $\mathbf{r}_i; \mathbf{r}_j$ the integral won't change either - we just have to do a change of variables in the integral given by the same matrix. It means that $F(\mathbf{r}_i - \mathbf{r}_j; 0)$ depends only on $|\mathbf{r}_i - \mathbf{r}_j|$. To see that it is independent of the length we do rescaling of arguments by the factor of c:

$$F(c\mathbf{r}_i; c\mathbf{r}_j) = \int_{\mathbb{R}^2} \frac{\langle \mathbf{r} - c\mathbf{r}_i; \mathbf{r} - c\mathbf{r}_j \rangle}{|\mathbf{r} - c\mathbf{r}_i|^2 |\mathbf{r} - c\mathbf{r}_j|^2}\, dV(\mathbf{r}) = \int_{\mathbb{R}^2} \frac{\langle c\tilde{\mathbf{r}} - c\mathbf{r}_i; c\tilde{\mathbf{r}} - c\mathbf{r}_j \rangle}{|c\tilde{\mathbf{r}} - c\mathbf{r}_i|^2 |c\tilde{\mathbf{r}} - c\mathbf{r}_j|^2} c^2\, dV(\tilde{\mathbf{r}}) = F(\mathbf{r}_i; \mathbf{r}_j)$$

We derive that F is a constant C. Finally $C = F(\mathbf{r}_i; \mathbf{r}_j) = -F(\mathbf{r}_j; \mathbf{r}_i) = -C$ and $C = 0$. Our manipulations with the integral were a bit formal. The reason is that though the integral converges, it diverges absolutely. This conclusion agrees with explicit calculation with *Mathematica*. The final form of the Lagrangian is

$$L(\ _i) = \sum_{i=1}^{N} m_i |\mathbf{r}_i'|^2/2 - \mathbf{a}_i \cdot \mathbf{r}_i' \tag{37}$$

2.6 Derivation of Hamiltonian

We derive the Hamiltonian by means of Legendre transform of the Lagrangian: $H(p_i; q_i) = \sum_i p^i q_i' - L(q_i'; q_i)$. This amounts to solving $p_i = \frac{\partial L}{\partial q_i'}$ for q_i' and re expressing $\sum_i p^i q_i' - L(q_i'; q_i)$ in terms p_i and q_i only.

In our case $\mathbf{p}_i = m_i \mathbf{r}_i' - \mathbf{a}_i(\mathbf{r}_i)$ or $\mathbf{r}_i' = \frac{1}{m_i}(\mathbf{p}_i + \mathbf{a}_i)$.

$$H = \sum_{i=1}^{N} \mathbf{p}_i \cdot \left(\frac{\mathbf{p}_i + \mathbf{a}_i}{m_i} \right) - \frac{m_i}{2} \left| \frac{\mathbf{p}_i + \mathbf{a}_i}{m_i} \right|^2 + \mathbf{a}_i \cdot \left(\frac{\mathbf{p}_i + \mathbf{a}_i}{m_i} \right)$$

After expansion we get

$$H = \sum_{i=1}^{N} \frac{|\mathbf{p}_i|^2}{2m_i} + \frac{\mathbf{a}_i \cdot p_i}{m_i} + \frac{|\mathbf{a}_i|^2}{2m_i} = \sum_{i=1}^{n} \frac{1}{2m_i} |\mathbf{p}_i + \mathbf{a}_i|^2 \tag{38}$$

It agrees with the formula 2.9 in [3] under assumption (29).

As usual Schrödinger operator can be obtained by replacing momenta variables $\mathbf{p}_i = (p_{x;i}; p_{y;i})$ in the above formula by

$$\hat{\mathbf{p}}_i := (ih\partial_{x_i}; ih\partial_{y_i}) \tag{39}$$

Here is Schrödinger operator written explicitly:

$$\hat{H} = \sum_{i=1}^{N} \frac{1}{2m_i} |\hat{\mathsf{p}}_i + \mathsf{a}_i|^2 =$$
$$= \sum_{i=1}^{n} \frac{1}{2m_i} (-h^2 @_{x_i}^2 - h^2 @_{y_i}^2 + ih @_{x_i} \circ a_{1i} + ih @_{y_i} \circ a_{2i} + iha_{1i} @_{x_i} + iha_{2i} @_{y_i} + a_{1i}^2 + a_{2i}^2) =$$
$$= \sum_{i=1}^{N} \frac{1}{2m_i} (-h^2 @_{x_i}^2 - h^2 @_{y_i}^2 + 2iha_{1i} @_{x_i} + 2iha_{2i} @_{y_i} + a_{1i}^2 + a_{2i}^2 + ih @_{x_i} a_{1i} + ih @_{y_i} a_{2i})$$

Schrödinger operator acts on wave functions $(\mathsf{r}_1; \dots; \mathsf{r}_N)$. One way to define an equivalent Schrödinger operator is to conjugate it with a function $(\mathsf{r}_1; \dots; \mathsf{r}_N)$. $^{-1} \circ \hat{H} \circ = \sum_{i=1}^{n} \frac{1}{2m_i} \left| \ ^{-1} \circ (\hat{\mathsf{p}}_i + \mathsf{a}_i) \circ \ \right|^2$. Both components of a_i (thought as operators) commute with . Let us analyze x-x component of $^{-1} \circ \hat{\mathsf{p}}_i \circ$:

$$^{-1} \circ ih @_{x_i} \circ \ = ih @_{x_i} + ih \frac{@ _{x_i}}{}$$

Denote $(ih \frac{@_{x_i}}{}; ih \frac{@_{x_i}}{})$ by $^{-1} \hat{\mathsf{p}}_i \ = ih \ ^{-1} \frac{@}{@ \mathsf{r}_i}$. We obtain that

$$^{-1} \circ \hat{H} \circ \ = \sum_{i=1}^{n} \frac{1}{2m_i} \left| (\hat{\mathsf{p}}_i + \mathsf{a}_i + \ ^{-1} \hat{\mathsf{p}}_i \ ) \right|^2$$

We can completely eliminate a_i and simplify $\hat{H}$ if we solve equation $\mathsf{a}_i + \ ^{-1} \hat{\mathsf{p}}_i \ = \mathsf{a}_i + \hat{\mathsf{p}}_i \log \ = 0$ or more explicitly

$$\frac{k_i}{2 \ k} \sum_{1 \le j \le N; j \ne i} k_j \frac{\mathsf{r}_i - \mathsf{r}_j}{|\mathsf{r}_i - \mathsf{r}_j|^2} + ih \frac{@ \log}{@ \mathsf{r}_i} = 0 \tag{40}$$

In order to find such function we identify $\mathsf{r}_j = (x_j; y_j)$ with a complex number $z_j = x_j + iy_j$. We define the function by the formula

$$= \frac{\prod_{1 \le i < j \le n} (z_i - z_j)^{\frac{k_i k_j}{2 \ hk}}}{\left| \prod_{1 \le i < j \le n} (z_i - z_j)^{\frac{k_i k_j}{2 \ hk}} \right|} \tag{41}$$

We will be writing $\hat{H}_{free} = \sum_{i=1}^{n} \frac{1}{2m_i} |\hat{\mathsf{p}}_i|^2$ for Schrödinger operator corresponding to a collection of N noninteracting particles. To summarize: $^{-1} \circ \hat{H} \circ \ = \hat{H}_{free}$ or $\ \circ \hat{H}_{free} \circ \ ^{-1} = \hat{H}$.

Say we want to solve a stationary equation $\hat{H} \ (\mathsf{r}_1; \dots; \mathsf{r}_N) = \ (\mathsf{r}_1; \dots; \mathsf{r}_N)$. It is equivalent to $\ \circ \hat{H}_{free} \circ \ ^{-1} \ = \ $ or $\hat{H}_{free}(\ ^{-1} \ ) = \ ^{-1}$. From this we conclude that instead of studying Schrödinger operator $\hat{H}$ we can work with $\hat{H}_{free}$. The operator $\hat{H}$ acts on the space of functions $(\mathsf{r}_1; \dots; \mathsf{r}_N)$, $\hat{H}$ - on

$$(\mathsf{r}_1; \dots; \mathsf{r}_N) \ (\mathsf{r}_1; \dots; \mathsf{r}_N): \tag{42}$$

One could argue naively that the function $(\mathbf{r}_1;\dots;\mathbf{r}_N)$ can be absorbed into $(\mathbf{r}_1;\dots;\mathbf{r}_N)$ and the theory would become trivial. The catch is that $(\mathbf{r}_1;\dots;\mathbf{r}_N)$ is a multivalued function. If $\mathbf{r}_j$ makes a full turn around $\mathbf{r}_l$ the function multiplies of the phase factor $\exp(\pm i\frac{k_j k_l}{\hbar k})$. This is the square of the phase (23) (corresponding to a half of the full turn) in the units where $\hbar = 1$.

2.7 Discussion of spectrum in case of zero density

Even though Schrödinger operator $\hat{H}_{free}$ is very simple it acts on wave functions that are defined on a pretty complex space $\widetilde{\mathbf{C}}_n$. $\widetilde{\mathbf{C}}_n$ was discussed in Section A.3.

Still it not hard to determine the spectrum $\hat{H}_{free}$. Let A be a transformation of $\mathbf{R}^2\times\dots\times\mathbf{R}^2$ that rescales $\mathbf{r}_i$: $A\ \mathbf{r}_i = \mathbf{r}_i$ on a positive real number. Scaling transforms wave functions by the formula $A\ (\mathbf{r}_1;\dots;\mathbf{r}_N) := (\ ^{-1}\mathbf{r}_1;\dots;\ ^{-1}\mathbf{r}_N)$. The function stays invariant under scaling: numerator and denominator in (41) pick up factors $\prod_{1\le i<j\le n}\ ^{-\frac{k_i k_j}{2\ \hbar k}}$. Also $A^{-1}\circ\hat{\mathbf{p}}_i\circ A\ =\ ^{-1}\hat{\mathbf{p}}_i$. Indeed, let us take a look at x component of $\hat{\mathbf{p}}_i$.

$$\begin{aligned}A^{-1}\circ i\hbar\partial_{x_i}\circ A\ \ (\dots;x_i;\dots) &= i\hbar A^{-1}(\partial_{x_i}(\ \ (\dots;\ ^{-1}x_i;\dots)))\\ &=\ ^{-1}i\hbar A^{-1}(\partial_{x_i}\ \ )(\dots;\ ^{-1}x_i;\dots))\\ &=\ ^{-1}i\hbar\partial_{x_i}\ \ (\dots;\ ^{-1}\ x_i;\dots)\\ &=\ ^{-1}i\hbar\partial_{x_i}\ \ (\dots;x_i;\dots)\end{aligned}$$

We conclude that $A^{-1}\circ\hat{H}_{free}\circ A\ =\sum_{i=1}^n\frac{1}{2m_i}\left|A^{-1}\circ\hat{\mathbf{p}}_i\circ A\ \right|^2=\sum_{i=1}^n\frac{^{-2}}{2m_i}|\hat{\mathbf{p}}_i|^2=\ ^{-2}\hat{H}_{free}$. Suppose we found a function such that

$$\hat{H}_{free}\ \ ^{-1}=\ \ ^{-1}. \tag{43}$$

We can apply A^{-1} to both sides of equation (43): $A^{-1}\hat{H}_{free}\ \ =\ A^{-1}\ \ $. It leads us to

$$A^{-1}(\ \ ^{-1})=A^{-1}\hat{H}_{free}A\ \ (A^{-1}\ \ )\ ^{-1}=\ ^{-2}\hat{H}_{free}(A^{-1}\ \ )\ ^{-1}$$

In the above formula we used that is A -invariant. The function $_{;}:=A^{-1}\ \ =$ is a solution of

$$\hat{H}_{free}\ \ _{;}\ ^{-1}=\ ^2\ \ _{;}\ ^{-1}$$

As we can vary continuously in the interval $0<\ <\infty$ the eigenvalues ^{2} cover the set of positive real numbers without holes.

We conclude that in infinite two-dimensional area anyonic system has a continuous energy spectrum for any finite number of particles. In this case the density of particles is equal to zero because of infiniteness of the area. In the next section we will see that the spectrum properties change drastically when the density becomes nonzero.

3 Random phase approximation

Law of large numbers, proven in the course of probability theory, states that if $\xi_i; i = 1;:::;N$ are independent equally distributed random variables which take real values and f is a real function, then

$$\mathbb{P}\left(\left|\frac{1}{N}\sum_{i=1}^{N} f(\xi_i)) - \mathbb{E}(f)\right| > \varepsilon\right) \to 0 \text{ as } N \to \infty$$

In the above formula $\mathbb{P}$ stands for probability. We assume for simplicity that $\mathbb{P}(\xi_i < y) = \mathbb{P}(\xi_j < y) = \int_{-\infty}^{y} p(x)dx$. The quantity $\mathbb{E}(f)$, called mathematical expectation, is equal to $\int_{-\infty}^{\infty} f(x)p(x)dx$. Theorem works under assumption that $\int_{-\infty}^{\infty} |f(x)|p(x)dx$ is convergent. The law used in the Monte Carlo method for practical estimation of the integral $\int_{-\infty}^{\infty} f(x)p(x)dx$ by averaging $f(\xi_i))$ with ξ_i produced by computer. There is a multidimensional generalization of the theorem. $\xi_i = (\xi_{1;i};:::;\xi_{k;i})$ becomes a vector-valued random variables and $p(x_1;:::;x_k)$ is defined through the formula $\mathbb{P}(\xi_i < (y_1;:::;y_k)) = \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_k} p(x_1;:::;x_k)dx_1 \cdots dx_k$.

We would like to look at the function (34) that appear in the formulas (37,38) for Lagrangian and Hamiltonian from such statistical point of view. First we assume that $k = N$. If the dynamics of r_j is sufficiently chaotic we can safely substitute r_j by random, equally distributed variables. If, in addition, assumptions (29) hold, the sum (34) can be safely approximated by

$$\mathsf{a}(\mathsf{r}) = \frac{e^2}{2} \sum_{j=1}^{N} \frac{J(\mathsf{r} - \mathsf{r}_j)}{|\mathsf{r} - \mathsf{r}_j|^2} \sim \frac{e^2}{2} \int_{\mathsf{R}^2} \frac{J(\mathsf{r} - \mathsf{r}')}{|\mathsf{r} - \mathsf{r}'|^2} p(\mathsf{r}')\, dV(\mathsf{r}') \tag{44}$$

Probability $p(\mathsf{r}')$ is proportional to the density $\rho(\mathsf{r}')$ of particles. Let us assume that $p(\mathsf{r}')$ converges to a uniform distribution (whatever it means) so that $\rho(\mathsf{r}')$ becomes a constant ρ_-.

In formulas (85,86) we provided some (not very presise) justifications that

$$\frac{e^2}{2} \int_{\mathsf{R}^2} \frac{J(\mathsf{r} - \mathsf{r}')}{|\mathsf{r} - \mathsf{r}'|^2}\, dV(r') = \frac{e^2}{2}(-y;x) = \frac{e^2}{2} J\mathsf{r} \tag{45}$$

that supports a convergence

$$\frac{e^2}{2} \int_{\mathsf{R}^2} \frac{J(\mathsf{r} - \mathsf{r}')}{|\mathsf{r} - \mathsf{r}'|^2}\, \rho(\mathsf{r}')\, dV(r') \to \frac{e^{2\rho_-}}{2} J\mathsf{r} \tag{46}$$

We also discussed issues related to absence of convergence of this integral. Introduce a notation

$$\bar{\mathsf{a}} := \frac{e^{2\rho_-}}{2} J\mathsf{r} \tag{47}$$

In [6] the effect of multiplication on operator J is achieved by cross-product multiplication with a constant vector $\mathbf{B}$ in the direction perpendicular to $x;y$-plane (c.f. formula 2.9 in [6]).

Some comments are appropriate. To achieve desired randomness one should probably work with anyon billiard or deal with anyons on a torus T^2, which could be simpler. Then one should slowly, to preserve randomness, send size of a torus to infinity. This requires additional work. One has to find the appropriate version of the function $\mathbf{a}_{i;T^2}$ by solving (32) under periodic boundary conditions. After that one has to do the analogue of integral (45) for $\mathbf{a}_{i;T^2}$, which now makes more sense because torus T^2 has a finite area and $\mathbf{r}_i \in T^2$ have a chance to have a random (ergodic) dynamics. Hopefully after sending the area of the torus to infinity, torus will converge to the plane $\mathbf{R}^2$ and we will recover the function $\frac{e^2}{2}J\mathbf{r}$.

In this limit Hamiltonian becomes an infinite sum

$$H_{red} = \sum_{i=1}^{\infty} \frac{1}{2m_i}|\hat{\mathbf{p}}_i + \frac{e^2\,-}{2}J\mathbf{r}_i|^2 \tag{48}$$

of decoupled expressions (c.f. formula 2.11 in [6]).

3.1 Spectrum of $\frac{1}{2m}|\hat{\mathbf{p}} + \frac{e^2}{2}J\mathbf{r}|^2$

The operator $\hat{H}_{red} = \frac{1}{2m}|\hat{\mathbf{p}} + \frac{e^2}{2}J\mathbf{r}|^2$ is one of the i^{th} terms that makes the Hamiltonian (48). Tt would be interesting to find its spectrum. The argument which involves the function won't work here. Equalities $-\frac{e^{2-}}{2}y + ih\frac{@\log}{@x} = 0; \frac{e^{2-}}{2}x + ih\frac{@\log}{@y} = 0$ similar to (40)is not possible because $\frac{@-\frac{e^{2-}}{2}y}{@y} \neq \frac{@\frac{e^{2-}}{2}x}{@x}$. Operator can be written more explicitly

$$\begin{aligned} 2m\hat{H}_{red} &= (ih@_x - \frac{e^{2\,-}}{2}y) \circ (ih@_x - \frac{e^{2\,-}}{2}y) + (ih@_y + \frac{e^{2\,-}}{2}x) \circ (ih@_y + \frac{e^{2\,-}}{2}x) \\ &= -h^2(@_x^2 + @_y^2) + \frac{e^{4\,-2}}{4}(x^2 + y^2) + ie^{2\,-}h(-y@_x + x@_y) \end{aligned}$$

Our present goal is to find the spectrum of $\hat{H}_{red}$.

The last formula shows that $\hat{H}_{red}$ is a sum of the energy operator of a simple harmonic oscillator $\hat{H}_{harm} = -\frac{h^2}{2m}(@_x^2 + @_y^2) + \frac{e^{4\;2}}{8m}(x^2 + y^2)$ and operator $\hat{H}_{rot} = \frac{ie^2}{2m}h(-y@_x + x@_y)$ of infinitesimal rotation. Laplacian and potential in $\hat{H}_{harm}$ are rotation-invariant. Thus $\hat{H}_{harm}$ and $\hat{H}_{rot}$ commute. A theorem of linear algebra states that a pair of self-adjoint commuting operators have a common eigenbasis. We are going to find it and from this find the eigenvalues of $\hat{H}_{red}$.

It is well-know that one-dimensional quantum harmonic oscillator with energy operator $-\frac{h^2}{2m}@_x^2 + \frac{m\,!^{\,2}}{2}x^2$ has an eigenbasis

$$_n(x) = \frac{1}{\sqrt{2^n\,n!}} \cdot \left(\frac{m\,!}{h}\right)^{1/4} \cdot e^{-\frac{m\,!\,x^2}{2h}} \cdot H_n\left(\sqrt{\frac{m\,!}{h}}x\right); \qquad n = 0;1;2;::::$$

with energy levels $(\frac{1}{2}+n)\hbar\omega$. $H_n(x)$ are Hermite polynomials polynomials:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right):$$

$_n(x)$ is a result of orthogonalization of the set $\{\ \tilde{}_n(x) = x^n e^{-\frac{m\omega x^2}{2\hbar}}\}$ with respect to the inner product $(f,g) = \int_{-\infty}^{\infty} f(x)\bar{g}(x)dx$.

We translate this to our variables: the eigenbasis for $\hat{H}_{harm}$ is orthogonalization of the set $x^a y^b \exp(\frac{e^2}{4\hbar}(x^2+y^2))$. It is convenient to replace variables x,y by their complex linear combination $z = x+iy, \bar{z} = x-iy$. Then

$$H_{rot} z = -\frac{e^2 B}{2m} z, \quad H_{rot}\bar{z} = \frac{e^2 B}{2m}\bar{z}$$

Functions $z^a \bar{z}^b \exp(\frac{e^2 B}{4\hbar}|z|^2)$ are eigenvectors of H_{rot} with eigenvalue $\frac{e^2 B}{2m}(b-a)$. Let us fix $c = b-a$. By the above comment orthogonalization of the set $\{z^a \bar{z}^{c+a} \exp(\frac{e^2 B}{4\hbar}(|z|^2)\}$ leads to a set of eigenvectors $_{a;a+c}(z,\bar{z})$ of $\hat{H}_{harm}$ with eigenvalues $(1/2+a+1/2+c+a)\hbar\frac{e^2 B}{2m}$. We see that

$$\begin{aligned}
\hat{H}_{red}\ _{a;a+c}(z,\bar{z}) &= \hat{H}_{harm}\ _{a;a+c}(z,\bar{z}) + \hat{H}_{rot}\ _{a;a+c}(z,\bar{z}) \\
&= (1+2a+c)\hbar\frac{e^2 B}{2m}\ _{a;a+c}(z,\bar{z}) + \frac{e^2 B}{2m}c\ _{a;a+c}(z,\bar{z}) \\
&= (1+2(a+c))\hbar\frac{e^2 B}{2m}\ _{a;a+c}(z,\bar{z})
\end{aligned}$$

To summarize: the eigenvalues $_{a;b}$ of $\hat{H}_{red}$ are labelled by two non negative integers a,b. As the corresponding eigenvalue $(1/2+b)\hbar\frac{e^2 B}{m}$ depends only on b we have infinite degeneracy of each energy level (c.f. [6] beginning of the page 2754 and Appendix A).

3.2 Anyons as deformations of bosons and fermions

In Section 2.6 we learned that the theory of anyons can be formulated in a Hilbert space whose vectors are products $(\mathbf{r}_1,\ldots,\mathbf{r}_N)\ (\mathbf{r}_1,\ldots,\mathbf{r}_N)$. The function is given by the formula (41) with $k_i = e$. To emphasize e-dependence we will be writing $_e$ If we would like to work with undistinguishable anyons we we have to impose a condition that the product $_e$ picks up a phase $\exp(i\frac{e^2}{2k\hbar})$ after counter clockwise rotation of $\mathbf{r}_i$ about $\mathbf{r}_j$ on angle . We can achieve this by taking symmetric in $\mathbf{r}_i$. There is another possibility: we can take e' such that $\frac{e'^2}{2k\hbar} = \frac{e^2}{2k\hbar} -$. If we tale $\tilde{}$ to be anti-symmetric the product $\tilde{}_{e'}$ will produce the same phase $\exp(i\frac{e^2}{2k\hbar})$. We see that we can describe anyon either by means of a bosonic wave function or by means of fermionic wave function. The only difference is that we have to adjust the charge e.

Recall that a Cooper pairs of electrons is counterintuitive structure that exists in crystals and studied in the theory of superconductivity. Electrons can't stay long close to each other in the empty space because of the repulsing electric force caused by their negative charges. In crystals with a large number of positively charged ions in lattice nodes another effect can take over repulsion. As electron almost freely moves along the lattice it attracts heavy positively charged ions. This way it creates a trail of matter with increased density of positive charges behind it. Of course, soon at any given point ions restore their initial position and the density of positive charges will go back to normal. Still, as a wave the density surge (phonon) will follow the electron. Other electrons will get attracted to the surge. If we had been unaware of the wave of phonons we would have observed a rather strange phenomenon: electrons chaise electrons. If looked from afar the pair of such electrons (Cooper pair) form a quasi-particle that has properties of a boson. Electrons in the Cooper are separated by phonon and occupy a sizable part of the space (100 nm). It explain why Cooper pair behave as boson. If we have two Cooper pairs in the same state Pauly principle for individual electrons tells us nothing as these electrons are spread over a large part of the space.

One can hope to reproduce a phenomenon like Cooper pairs in the context of anyons. We have to answer a question-how many anyons does it take to forma a boson. With fermions the answer is known-it is two. It can be seen in the language of wave functions: given an antisymmetric $(\mathsf{r}_1;\dots;\mathsf{r}_{2N})$ we can reinterpret it as a wave function $\widetilde{(}\mathsf{q}_1;\dots;\mathsf{q}_N)$, where $\mathsf{q}_i = (\mathsf{r}_{2i+1};\mathsf{r}_{2i+2})$ is a space coordinate for the composite particle. It is obvious that $\widetilde{}$ is a symmetric function.

We can repeat this argument for anyons. Let us assume that in phase (23) the exponent is $i\frac{2}{n}$. If we rotate a point with coordinate r_i about the point with coordinate r_j the wave function will pick up a phase $\exp(i\frac{}{n})$. Suppose $n = 2m^2$. If we repeat this procedure for the group of variables $A = \{\mathsf{r}_1;\dots;\mathsf{r}_m\}$ $B = \{\mathsf{r}_{m+1};\dots;\mathsf{r}_{2m}\}$ in the function (42) by swapping the first and the second group. The total phase will be $\exp(i\frac{2\ m^2}{n}) = 1$. As in the case of fermions we can interpret (42) as a wave function of N/m bosons. Presence of such bosons can be a hint for superconductivity.

We, unfortunately, have a strange phenomenon. As n in the phase becomes large the phase approaches to one. We recover bosonic theory. As we saw this nearly bosonic theory does contain composite particles that look like bosons. The number of constituents, which is proportional to $\sqrt{n}$,growth with n, instead of decreasing. As a result composite particles do not continuously evolve to actual bosons as $n \to \infty$.

To avoid this difficulty we choose a point of view that anyons are deformations of fermions. In this case the phase could be chosen 2 $(1-\frac{1}{n})$. As $n \to \infty$ the theory degenerates into a gas of fermions which doesn't have superconducting properties. It agrees with the previous observation that composed bosons are scarce because a large number $\sim \sqrt{n}$ constituents which growth with n.

3.3 Anyonic Hamiltonian in second quantization formalism

The reader might wish to consult Appendix D for quick review of the formalism. The point of working in second quantization formalism is that Hamiltonian of the multi particle system can be written in concise form with the help of operators $\Psi(\mathbf{r})$ and $\Psi^\dagger(\mathbf{r})$. Now $\mathbf{r}$ again is a two-dimensional vector $\mathbf{r} = (x; y)$.

The simplest illustration is a free multi-particle fermionic system. In this case Schrödinger operator acts by the formula $\sum_{i=1}^{N} \Delta_{\mathbf{r}_i} \ (\mathbf{r}_1; \dots; \mathbf{r}_N)$. It is easy to see that it coincides with the operator $\int \Psi^\dagger(\mathbf{r}) \Delta_{\mathbf{r}} \Psi(\mathbf{r}) \, dV(\mathbf{r})$. Indeed

$$\begin{aligned}
&\int \Psi^\dagger(\mathbf{r}) \Delta_r \Psi(\mathbf{r}) \, dV(\mathbf{r}) \ \ (\mathbf{r}_1; \dots; \mathbf{r}_N) = \\
&= \sum_{i=0}^{N-1} (-1)^{(N-1)i} \int \int \ (\mathbf{r}_{\ i(1)} - \mathbf{r})(\Delta_{\mathbf{r}} \ (\mathbf{r}' - \mathbf{r})) \ \ (\mathbf{r}'; \mathbf{r}_{\ i(2)}; \dots; \mathbf{r}_{\ i(N)}) \, dV(\mathbf{r}) \, dV(\mathbf{r}') \\
&= \sum_{i=0}^{N-1} (-1)^{(N-1)i} \int \ (\mathbf{r}_{\ i(1)} - \mathbf{r}) \Delta_{\mathbf{r}} \ \ (\mathbf{r}; \mathbf{r}_{\ i(2)}; \dots; \mathbf{r}_{\ i(N)}) \, dV(\mathbf{r}) \\
&= (-1)^{(N-1)0} \Delta_{\mathbf{r}_1} \ \ (\mathbf{r}_1; \dots; \mathbf{r}_N) + (-1)^{(N-1)1} \Delta_{\mathbf{r}_2} \ \ (\mathbf{r}_2; \dots; \mathbf{r}_N; \mathbf{r}_1) + \cdots \\
&= \sum_{i=1}^{N} \Delta_{\mathbf{r}_i} \ \ (\mathbf{r}_1; \dots; \mathbf{r}_N)
\end{aligned}$$

The same argument shows that if $U(\mathbf{r}'; \mathbf{r})$ is a family of potentials that depends on $\mathbf{r}'$

$$\int \Psi^\dagger(\mathbf{r}) U(\mathbf{r}'; \mathbf{r}) \Psi(\mathbf{r}) \, dV(\mathbf{r}) \ \ (\mathbf{r}_1; \dots; \mathbf{r}_N) = \sum_{i=1}^{N} U(\mathbf{r}'; \mathbf{r}_i) \ \ (\mathbf{r}_1; \dots; \mathbf{r}_N) \tag{49}$$

In particular if $U = 1$ we get a particle density operator

$$\hat{}(\mathbf{r}) = \Psi^\dagger(\mathbf{r}) \Psi(\mathbf{r}) \tag{50}$$

The integral $\int \hat{}(\mathbf{r}) \, dV(\mathbf{r})$ scales $\ _N$ of the factor of N. The quantity $(\hat{} \ ; \ \ )$ is the average number of particles in the normalized state (92).

If we freeze $\mathbf{r}_i$ in the sum $\sum_{i=1}^{N} U(\mathbf{r}'; \mathbf{r}_i)$ we can apply the above identity to $\mathbf{r}'$:

$$\int \Psi^\dagger(\mathbf{r}') \sum_{i=1}^{N} U(\mathbf{r}'; \mathbf{r}_i) \Psi(\mathbf{r}') \, dV(\mathbf{r}) \ \ (\mathbf{r}_1; \dots; \mathbf{r}_N) = \sum_{j=1}^{N} \sum_{i=1}^{N} U(\mathbf{r}_j; \mathbf{r}_i) \ \ (\mathbf{r}_1; \dots; \mathbf{r}_N)$$

It almost explains the identity

$$\begin{aligned}
&\int \int \Psi^\dagger(\mathbf{r}') \Psi^\dagger(\mathbf{r}) U(\mathbf{r}'; \mathbf{r}) \Psi(\mathbf{r}) \Psi(\mathbf{r}') \, dV(\mathbf{r}) \, dV(\mathbf{r}') \ \ (\mathbf{r}_1; \dots; \mathbf{r}_N) = \\
&= \sum_{1 \le i \ne j \le N} U(\mathbf{r}_j; \mathbf{r}_i) \ \ (\mathbf{r}_1; \dots; \mathbf{r}_N)
\end{aligned}$$

The terms $U(\mathbf{r}_i;\mathbf{r}_i)$ drop out because of skew-symmetry of that is used in the actual calculation of the above integral.

The vector-valued momentum operator $\hat{\mathbf{p}}$ is $(ih\partial_x; ih\partial_y)$ (39). By (49) the operator

$$\hat{\mathbf{a}}(\mathbf{r}) = \int \Psi^\dagger(\mathbf{r})\frac{e^2}{2}\frac{J(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2}\Psi(\mathbf{r})\, dV(\mathbf{r}')$$

with fixed $\mathbf{r}$ acts of $_N$ by multiplication of $\mathbf{a}(\mathbf{r})$(34), all k_i are set to e.

After all these preliminaries we see that operator (38) with $m_i = m$ acting on $_N$ can be written as

$$\int \Psi^\dagger(\mathbf{r})(\mathbf{p}+\hat{\mathbf{a}}(\mathbf{r}))\cdot(\hat{\mathbf{p}}+\hat{\mathbf{a}}(\mathbf{r}))\Psi(\mathbf{r})\, dV(\mathbf{r}) \quad _N$$

The difference $(H - H_{red})$ $_N$(38)(48), corresponding to interaction, with $m_i = m$ is equal to

$$\begin{aligned} H_I \quad _N &= \sum_{i=1}^{N}(2(\mathbf{a}_i-\bar{\mathbf{a}})\cdot\mathbf{p}_i + |\mathbf{a}_i|^2 - |\bar{\mathbf{a}}|^2) \quad _N \\ &= \sum_{i=1}^{N}(2(\mathbf{p}_i+\bar{\mathbf{a}})\cdot(\mathbf{a}_i-\bar{\mathbf{a}}) + |\mathbf{a}_i-\bar{\mathbf{a}}|^2) \quad _N \end{aligned} \tag{51}$$

As we rearranged terms we used that $\mathbf{p}_i\cdot\mathbf{a}_i - \mathbf{a}_i\cdot\mathbf{p}_i = ih\vec{\nabla}\cdot_i \mathbf{a}_i = 0; \mathbf{p}\cdot\bar{\mathbf{a}} - \bar{\mathbf{a}}\cdot\mathbf{p} = ih\vec{\nabla}\cdot\bar{\mathbf{a}} = 0$
In the second quantized language

$$H_I = \frac{1}{2m}\int \Psi^\dagger(\mathbf{r})(\mathbf{p}+\bar{\mathbf{a}}(\mathbf{r}))\cdot(\hat{\mathbf{a}}(\mathbf{r})-\bar{\mathbf{a}}(\mathbf{r})) + (\hat{\mathbf{a}}(\mathbf{r})-\bar{\mathbf{a}}(\mathbf{r}))\cdot(\hat{\mathbf{a}}(\mathbf{r})-\bar{\mathbf{a}}(\mathbf{r})))\Psi(\mathbf{r})\, dV(\mathbf{r}) \tag{52}$$

Fix $\mathbf{r}$. We use (45) (with all provisions related to divergence of the integral) and (47) to rewrite $\hat{\mathbf{a}}(\mathbf{r}) - \bar{\mathbf{a}}$ in more convenient for our purposes form:

$$\hat{\mathbf{a}}(\mathbf{r}) - \bar{\mathbf{a}}(\mathbf{r}) = \frac{e^2}{2\ k}\int \frac{J(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2}(\Psi^\dagger(\mathbf{r}')\Psi(\mathbf{r}') - \ )\, dV(\mathbf{r}')$$

The constant is the density of particles. Then the operator H_I (52) can be written as

$$\begin{aligned} H_I =& \frac{1}{2m}\frac{e^2}{2\ k}\int\int \Psi^\dagger(\mathbf{r})(\mathbf{p}+\bar{\mathbf{a}}(\mathbf{r}))\cdot\frac{J(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2}(\Psi^\dagger(\mathbf{r}')\Psi(\mathbf{r}') - \ )\Psi(\mathbf{r})\, dV(\mathbf{r})\, dV(\mathbf{r}') + \\ &+ \frac{1}{2m}\frac{e^2}{2\ k}\frac{e^2}{2\ k}\int \Psi^\dagger(\mathbf{r})(\Psi^\dagger(\mathbf{r}')\Psi(\mathbf{r}') - \ )\frac{J(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2}\cdot\frac{J(\mathbf{r}-\mathbf{r}'')}{|\mathbf{r}-\mathbf{r}''|^2}\times \\ &\times(\Psi^\dagger(\mathbf{r}'')\Psi(\mathbf{r}'') - \ )\Psi(\mathbf{r})\, dV(\mathbf{r})\, dV(\mathbf{r}')\, dV(\mathbf{r}'') = \\ =& H_1 + H_2 \end{aligned}$$

We rewrite H_1 as

$$H_1 = \frac{e^2}{4m\ \ k}\int\int \Psi^\dagger(\mathsf{r})(\mathsf{p}+\bar{\mathsf{a}}(\mathsf{r}))\Psi(\mathsf{r})\cdot\frac{J(\mathsf{r}-\mathsf{r}')}{|\mathsf{r}-\mathsf{r}'|^2}(\Psi^\dagger(\mathsf{r}')\Psi(\mathsf{r}')-\ \ )\ dV(\mathsf{r})\ dV(\mathsf{r}')-$$
$$-\frac{e^2}{4m\ \ k}\int\int \Psi^\dagger(\mathsf{r})(\mathsf{p}+\bar{\mathsf{a}}(\mathsf{r}))\cdot\frac{J(\mathsf{r}-\mathsf{r}')}{|\mathsf{r}-\mathsf{r}'|^2}\ \ (\mathsf{r}-\mathsf{r}')\Psi(\mathsf{r}')\ dV(\mathsf{r})\ dV(\mathsf{r}')$$

The second term is obtained by transporting $\Psi(\mathsf{r})$ across $\Psi^\dagger(\mathsf{r}')\Psi(\mathsf{r}')$ by using relations (96). This term is infinite and is dropped in [3] (equation 5.12). Indeed that $\int \frac{J(\mathsf{r})}{|\mathsf{r}|^2}\ \ (\mathsf{r})\ \ (\mathsf{r})\ dV(\mathsf{r}) = \frac{J(0)}{|0|^2}\ \ (0)$ makes sense only for scalar test-function $\ \ (\mathsf{r})$ that has zero of order two at 0. We set

$$\hat{j}(\mathsf{r}) := \frac{1}{m}\Psi^\dagger(\mathsf{r})(\mathsf{p}+\bar{\mathsf{a}}(\mathsf{r}))\Psi(\mathsf{r})$$

$\hat{j}$ is a two-dimensional array of operators. In this notation a simplified version of H_1 becomes

$$H_1^{simp} = \frac{e^2}{4\ \ k}\int\int \hat{j}(\mathsf{r})\cdot\frac{J(\mathsf{r}-\mathsf{r}')}{|\mathsf{r}-\mathsf{r}'|^2}(\hat{\ }(\mathsf{r}')-\ \ )\ dV(\mathsf{r})\ dV(\mathsf{r}')$$

If we do similar manipulations with H_2 and get

$$H_2 = \frac{1}{2m}\frac{e^2}{2\ \ k}\frac{e^2}{2\ \ k}\int \Psi^\dagger(\mathsf{r})\Psi(\mathsf{r})(\Psi^\dagger(\mathsf{r}')\Psi(\mathsf{r}')-\ \ )\frac{J(\mathsf{r}-\mathsf{r}')}{|\mathsf{r}-\mathsf{r}'|^2}\cdot\frac{J(\mathsf{r}-\mathsf{r}'')}{|\mathsf{r}-\mathsf{r}''|^2}\times$$
$$\times(\Psi^\dagger(\mathsf{r}'')\Psi(\mathsf{r}'')-\ \ )\ dV(\mathsf{r})\ dV(\mathsf{r}')\ dV(\mathsf{r}'')$$
$$-\frac{1}{2m}\frac{e^2}{2\ \ k}\frac{e^2}{2\ \ k}\int \Psi^\dagger(\mathsf{r})\Psi(\mathsf{r})(\Psi^\dagger(\mathsf{r}')\Psi(\mathsf{r}')-\ \ )\frac{J(\mathsf{r}-\mathsf{r}')}{|\mathsf{r}-\mathsf{r}'|^2}\cdot\frac{J(\mathsf{r}-\mathsf{r}'')}{|\mathsf{r}-\mathsf{r}''|^2}\times$$
$$\times\ \ (\mathsf{r}''-\mathsf{r})\Psi(\mathsf{r}'')\ dV(\mathsf{r})\ dV(\mathsf{r}')\ dV(\mathsf{r}'')$$
$$-\frac{1}{2m}\frac{e^2}{2\ \ k}\frac{e^2}{2\ \ k}\int \Psi^\dagger(\mathsf{r})\Psi(\mathsf{r})\ \ (\mathsf{r}'-\mathsf{r})\Psi(\mathsf{r}')\frac{J(\mathsf{r}-\mathsf{r}')}{|\mathsf{r}-\mathsf{r}'|^2}\cdot\frac{J(\mathsf{r}-\mathsf{r}'')}{|\mathsf{r}-\mathsf{r}''|^2}\times$$
$$\times(\Psi^\dagger(\mathsf{r}'')\Psi(\mathsf{r}'')-\ \ )\ dV(\mathsf{r})\ dV(\mathsf{r}')\ dV(\mathsf{r}'')$$

Finally

$$H_2^{simp} = \frac{e^4}{8\ \ ^2mk^2}\int \Psi^\dagger(\mathsf{r})\Psi(\mathsf{r})(\hat{\ }(\mathsf{r}')-\ \ )\frac{J(\mathsf{r}-\mathsf{r}')}{|\mathsf{r}-\mathsf{r}'|^2}\cdot\frac{J(\mathsf{r}-\mathsf{r}'')}{|\mathsf{r}-\mathsf{r}''|^2}\times$$
$$\times(\hat{\ }(\mathsf{r}'')-\ \ )\ dV(\mathsf{r})\ dV(\mathsf{r}')\ dV(\mathsf{r}'') =$$
$$= \frac{e^4}{8\ \ ^2mk^2}\int(\hat{\ }(\mathsf{r})-\ \ )(\hat{\ }(\mathsf{r}')-\ \ )\frac{J(\mathsf{r}-\mathsf{r}')}{|\mathsf{r}-\mathsf{r}'|^2}\cdot\frac{J(\mathsf{r}-\mathsf{r}'')}{|\mathsf{r}-\mathsf{r}''|^2}\times$$
$$\times(\hat{\ }(\mathsf{r}'')-\ \ )\ dV(\mathsf{r})\ dV(\mathsf{r}')\ dV(\mathsf{r}'')+$$
$$+\frac{e^4}{8\ \ ^2mk^2}\int(\hat{\ }(\mathsf{r}')-\ \ )\frac{J(\mathsf{r}-\mathsf{r}')}{|\mathsf{r}-\mathsf{r}'|^2}\cdot\frac{J(\mathsf{r}-\mathsf{r}'')}{|\mathsf{r}-\mathsf{r}''|^2}\times$$
$$\times(\hat{\ }(\mathsf{r}'')-\ \ )\ dV(\mathsf{r})\ dV(\mathsf{r}')\ dV(\mathsf{r}'')$$

Unfortunately the arguments that we used in evaluation of (36) can not be used in the case of integral

$$G(\mathbf{r}';\mathbf{r}'') := \int \frac{J(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2}\cdot\frac{J(\mathbf{r}-\mathbf{r}'')}{|\mathbf{r}-\mathbf{r}''|^2}\,dV(\mathbf{r}) = \int \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2}\cdot\frac{(\mathbf{r}-\mathbf{r}'')}{|\mathbf{r}-\mathbf{r}''|^2}\,dV(\mathbf{r}): \qquad (53)$$

The reason is that the integral diverges. Here is the precise result obtained with *Mathematica*.Define

$$G(\Lambda;\mathbf{r}';\mathbf{r}'') = \int_{-}\int_{-\infty}^{\infty} dx \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2}\cdot\frac{(\mathbf{r}-\mathbf{r}'')}{|\mathbf{r}-\mathbf{r}''|^2}\,dV(\mathbf{r})$$

Then

$$G(\Lambda;\mathbf{r}';\mathbf{r}'') \sim 2 \quad \log\Lambda - \quad \log\left|\mathbf{r}'-\mathbf{r}''\right|^2 + 2 \quad \log 2 + O(1/\Lambda)$$

As we are free to add or subtract any constant value from the Hamiltonian this way changing the reference point for zero energy, the constant $2 \quad \log\Lambda + 2 \quad \log 2$ can be absorbed by this arbitrary constant. This way we get renormalized effective potential

$$G_{ren}(\mathbf{r}';\mathbf{r}'') = -2 \quad \log\left|\mathbf{r}'-\mathbf{r}''\right|$$

which agrees with formula (5.14) in [3].

This formula can be extracted directly from the potential in (51)

$$\sum_{i=1}^{N}|\mathbf{a}_i-\bar{\mathbf{a}}|^2 = \sum_{i=1}^{N}\mathbf{a}_i\cdot\mathbf{a}_i - 2\mathbf{a}_i\cdot\bar{\mathbf{a}} + |\bar{\mathbf{a}}|^2$$

We are interested in the first term of this sum

$$\begin{aligned}\sum_{i=1}^{N}\mathbf{a}_i\cdot\mathbf{a}_i &= \frac{e^4}{4\quad^2k^2}\sum_{i=1;j\neq j;k}^{N}\frac{J(\mathbf{r}_i-\mathbf{r}_j)\cdot J(\mathbf{r}_i-\mathbf{r}_k)}{|\mathbf{r}_i-\mathbf{r}_j|^2|\mathbf{r}_i-\mathbf{r}_k|^2}\\ &= \frac{e^4}{4\quad^2k^2}\sum_{i=1;j\neq i;k}^{N}\frac{(\mathbf{r}_i-\mathbf{r}_j)\cdot(\mathbf{r}_i-\mathbf{r}_k)}{|\mathbf{r}_i-\mathbf{r}_j|^2|\mathbf{r}_i-\mathbf{r}_k|^2}\\ &= \sum_{j;k=1}^{N}H(\mathbf{r}_j;\mathbf{r}_k);\\ H(\mathbf{r}';\mathbf{r}'') &= \frac{e^4}{4\quad^2k^2}\sum_{i=1}^{N}\frac{(\mathbf{r}_i-\mathbf{r}')\cdot(\mathbf{r}_i-\mathbf{r}'')}{|\mathbf{r}_i-\mathbf{r}'|^2|\mathbf{r}_i-\mathbf{r}''|^2}\end{aligned}$$

Again if we invoke randomness assumptions on $\mathbf{r}_i$ the sum can be approximated by the integral (53).

We just derived effective Coulomb interaction. It generates effective long-range attraction between oppositely signed charges and repulsion between equally signed one. This in turn is responsible for formation for the mass gap in the interacting theory.

4 Vortices

When a superconductor cools to its superconducting state any external magnetic field is expelled from the body of the sample. This is called the Meissner effect. The heuristic reason for this is that "magnetism and superconductivity are natural enemies": Cooper pairs, mentioned briefly in Section 3.2, are formed by electrons with their spins anti-parallel, while an outside magnetic field aligns the electron spins parallel to one another. If the energy of the magnetic field is not sufficient to bring the superconductor to non superconducting state, we observe expulsion of magnetic field from the sample. If the energy passes a certain threshold the sample acquire electric resistance.

In fact there is a third possibility - impurities are generated in the body of the sample where magnetic field is not zero. These impurities are one-dimensional objects. They are called vortices. The geometric shape L_v that underlies a vortex v is a line or a closed loop. Vector potential, A, that generates the magnetic field is smooth outside L_v, but (in our approximation) has a singularity on it. $B = \vec{\nabla} \times A$ is proportional to delta-function $_L$ with support on L. The vector-valued coefficient of proportionality is called vorticity of v.

Vortices also appear in the context of fluid dynamics and aerodynamics. The most common vortex is a dust-devil. The quantity similar to A in this case is velocity of the stream of particles of dust. Vorticity $B(x)$ is the angular speed of rotation of a small sample near the point x.

We are interested in vortices that appear in two-dimensional films that were used for the study of quantum Hall effect. As the dimension has been reduced by one, vortices in this context are point-like impurities. Vortices move around and have their own dynamics. In this respect they are similar to quasi-particles like Cooper pairs. One substantial difference between a vortex and a Cooper pair is that negative charge density drops at the vortex. In this sense it is similar to a hole in a semi-conductor.

Statistical properties of dynamics of vortices as functions of external magnetic field can be extracted from correlators of a field theory similar to anyons. The remarkable fact is that the quantum theory of vortices is dual to theory of anyons.

4.1 Example of a simplest pair of dual theories

First we have to explain what dual means (see Witten's lecture in [7] for details). Instead of giving a rigorous definition let us go over a simple pair of dual theories, which are sigma-models with values on a circle.

Lets look at a classical situation first, where we have a field $(x;y)$ that satisfy Laplace equation $\Delta = \vec{\nabla}\cdot\vec{\nabla} = 0$. The vector $(@_x\ ;@_y\ )$ is denoted by A. Of course $-@_yA_1+@_xA_2 = 0$ because A is a gradient vector field that comes from potential . We use J (33) to define $B = J\vec{\nabla} = (-@_y\ ;@_x\ )$. Remarkably B is also gradient vector field: $-@_yB_1 + @_xB_2 = -@_y(-@_y\ ) + @_x(@_x\ ) = 0$, since satisfies the Laplace equation. We conclude that B is a gradient $\vec{\nabla}$ of some potential . From this we see that the theory of Laplace equation

in two dimensions has two equivalent descriptions: one in terms of the field , the other in terms of the dual field . This is not very surprising because if we set $z = x + iy$ then the function of complex argument $f(z) = (z) + i (z)$ becomes holomorphic. The above relation between ; is a Cauchy-Riemann equation for f.

The quantum picture is much more surprising. We assume that the field is quasi-periodic

$$(x+1;y) = (x;y) + 2 Rn_1; \qquad (x;y+1) = (x;y) + 2 Rn_2; \quad R > 0;\ n_1;n_2 \in Z \tag{54}$$

We define an action by the formula $S( ) = \frac{1}{4}\int_0^1\int_0^1(@_x)^2 + (@_y)^2 dxdy$. We would like to think about a square $[0;1]\times[0;1]$ with identified opposite sides as a torus T^2 and a circle of radius R as an interval $[0;2 R]$ with identified ends. Condition (54) is equivalent to the statement that defines a map of T^2 to a S^1 of radius R.

It is not convenient to keep R inside of quasi-periodicity condition (54). To get rid of it we rescale by R. This way we get

$$(x+1;y) = (x;y) + 2 n_1; \qquad (x;y+1) = (x;y) + 2 n_2: \tag{55}$$

Once we fixed points $r_i; i = 1;:::;N$ we can define correlators by means of functional integral

$$\langle \exp(ik_1 _1)\cdots\exp(ik_N _N)\rangle := \sum_{n_1;n_2\in Z}\int \exp(ik_1 (r_1))\cdots\exp(ik_N (r_N))\exp(-S( ))\,D$$

$$S( ) = \frac{R^2}{4}\int_0^1\int_0^1(@_x)^2 + (@_y)^2 dxdy$$

We define almost the same theory whose field is . It satisfies the same quasi-periodicity conditions (55). The Lagrangian is $S( ) = \frac{1}{4R^2}\int_0^1\int_0^1(@_x)^2 + (@_y)^2 dxdy$. The first fact is that $\langle\rangle = \langle\rangle$ (vacuum expectation values of both theories coincide). This is not quite obvious because the coupling constant in the first action is $\sim 1/R^2$ whereas in the other it is $\sim R^2$. It is an indication that theories on circles of radius R and $1/R$ coincide. In order to formulate what correlator in theory corresponds to $\langle\exp(ik_1 _1)\cdots\exp(ik_N _N)\rangle$ we fix polar coordinates about points r_i. We are going to use special field for computation of the correlator. At a point r_i, which we assume is equal to $(0;0)$ the function $(;) \sim k_i$ as $\to 0$. This way $\exp(i)$ is a function with a singularity at r_i. When makes one loop about r_i $\exp(i)$ makes k_i turns about zero. We denote by $\exp(i)_{k_i}$ the procedure that modifies the set of all in the outlined above way. Then

$$\langle \exp(ik_1 _1)\cdots\exp(ik_N _N)\rangle = \langle \exp(i)_{k_1}\cdots\exp(i)_{k_N}\rangle$$

This is a statement of T-duality in the simplest form. We see that operator corresponding to observable $\exp(ik_1 (r))$ gets transformed into disorder operator $\exp(i)_{k_i}$. It is bizarre, indeed, to see an example of observable like $\exp(i)_{k_i}$ that doesn't take any value on the field.

4.2 Particle-vortex duality in 3 dimensions

This type of duality is based in generalization of ideas of T-duality to three dimensions.

We had been studying in Section 2.4 a gas of N particles $_i$ three-dimensional space-time. Later we sent N to infinity and passed to a model with continuous particle density. It is convenient to start and formulate the theory directly in terms of particle density . The field-theoretic Lagrangian in this case will be $L(;a) = \frac{m}{2}(\vec{\nabla} + ea)^2 + \frac{k}{2}a \cdot \vec{\nabla} \times a$. In it we are supposed to use Minkowskian dot-product. Instead, we Wick rotate all the fields and end up with Euclidean dot-product.

In fact there is whole universe of field theories whose Lagrangian is close to the written presented above and which exhibit duality. In contrast with T-duality even basic examples are not self-dual. It means that doesn't preserve the field content. On one side of duality all theories contain a scalar field , which is responsible for the density of particles. On the other side of duality a theory contains a vector potential A. In some occasions $\vec{\nabla} \times A$ is supported in a space-time trajectory and A can be interpreted as a vortex. In general, however, it is a continuous superposition of vortices and its tornado-shape is smeared. Dual theories can contain more fields like Chern-Simons field and interactions (potentials).

Let us see the pattern of duality in case of free fields in classical theory. Lagrangian of the vector (abelian gauge) field A is $|\vec{\nabla} \times A|^2$. Equation of motion is $\vec{\nabla} \times \vec{\nabla} \times A = 0$. I would like to extract the dual field from this equation. To do this I denote $B = \vec{\nabla} \times A$. Condition $\vec{\nabla} \times B = 0$ is equivalent (in three dimensions) that B is a gradient vector field. We see that (at lest locally) there is a field such that $B = \vec{\nabla}$. Note since $B = \vec{\nabla} \times A$ then $\vec{\nabla} \cdot B = \vec{\nabla} \cdot \vec{\nabla} \times A = 0$. But $0 = \vec{\nabla} \cdot B = \vec{\nabla} \cdot \vec{\nabla} = \nabla^2$. We conclude that satisfies three-dimensional Laplace equation, which is an equation of motion for Lagrangian $|\vec{\nabla} |^2$.

Duality holds on the quantum level. In order to formulate it we restrict ourself with $(t;x;y)$ that satisfy quasi-periodicity conditions (55) for all three variables without integrality assumptions on $n_1;n_2;n_3$.

> Vector field A satisfies two quasi-periodicity conditions:
> $\vec{\nabla} \times A(t + _1;x + _2;y + _3) = \vec{\nabla} \times A(t;x;y); \quad _i = 0;\pm 1$
> Fluxes $n_{tx};n_{xy};n_{yt}$ of $\vec{\nabla} \times A$ through faces of the cube $[0;1] \times [0;1] \times [0;1]$
> (opposite faces are identified) are integers. (56)

There is equality of functional integrals

$$\int \exp\left(-\frac{\Lambda}{4}\int |\vec{\nabla} |^2 dtdxdy\right) D = \\ = \sum_{n_{tx};n_{xy};n_{yt}\in Z} \int \exp\left(-\frac{1}{4 \Lambda}\int |\vec{\nabla} \times A|^2 dtdxdy\right) DA \tag{57}$$

where Λ is a coupling constant. Computations of correlators can also be carried through. In particular if is a curve and we impose on vectors $\{A\}$ condition that A is a sum $A_0 + A_1$

of a vortex A_0 with vorticity k along and a field A_1 with smooth $\vec{\nabla} \times A_1$ along . Then

$$\int \exp\left(\int k \; d \; - \frac{\Lambda}{4} \int |\vec{\nabla} \; |^2 dtdxdy\right) D \; =$$
$$= \int \exp\left(-\frac{1}{4 \; \Lambda} \int |\vec{\nabla} \times (A_0 + A_1)|^2 dtdxdy\right) DA_1$$

Here is another example of a simple duality discussed in [8]. One of the theories is the so called XY model.

$$S(\; ; A) = \int d^3x \; |(@ \; - iA \;) \; |^2 - V(\;) \tag{58}$$

Its dynamical field is and A is a parameter. It is dual to the the theory with action

$$S = \int d^3x \; |(@ \; - ia \;)\Phi|^2 - \tilde{V}(\Phi) + \frac{1}{2} \qquad A \; @ \; a \tag{59}$$

This is the Abelian-Higgs model. Its dynamical fields are a scalar Φ. a is also a parameter which is a background gauge field.

Its coupling to the currents in the two theories reveals that the particle density of in (58) is equated to the flux density $f/2 \; = da/2$ in 59. This is the essence of particle-vortex duality. This is an explanation of particle-vortex duality according to [8].

Similar duality exists for Dirac fermion with the action

$$S(\; ; A) = \int d^3x \; i^- \quad (@ \; - iA \;)$$

It is believed that it is dual to Quantum Electrodynamics in three dimension with one fermionic field Ψ, and one dynamical abelian gauge field a . As before A is a parameter.

$$S = \int d^3x \; i\bar{\Psi} \quad (@ \; - ia \;)\Psi + \frac{1}{4} \qquad A \; @ \; a$$

Inspired by these formulas the authors of [8] proposed a duality that manifests in equalities of partition function

$$Z_{fermion}(A) = \int \exp\left(i \int (^- \quad (@ \; - A \;) \; - \frac{1}{4} A \cdot \vec{\nabla} \times A) dtdxdy\right) D \; =$$
$$\int \exp\left(\int i(|(@ \; - a \;) \; |^2 + \frac{1}{2} a \cdot \vec{\nabla} \times a + \frac{1}{2} a \cdot \vec{\nabla} \times A) dtdxdy\right) D \; \; Da =$$
$$Z_{scalar+flux}$$

which could be interpreted as three dimensional bosonization because it relates the theory of fermion to bosons and a . In the above formula is a complex fermion, A is a background vector (abelian gauge field), is a scalar field, a is an abelian gauge field. The

term $\frac{1}{4}A\cdot\vec{\nabla}\times A$ (half of Chern-Simons action) was included in $Z_{fermion}(A)$ in order to compensate gauge anomaly. This anomaly typically appears in the process of computation of Pfaffian of Dirac operator from the kinetic term of fermion Lagrangian. The field a appears in $Z_{scalar+flux}$ in order to emulate objects whose correlators have fermionic properties with respect to permutations. We obtain it in a fashion familiar to example (57). Instead of integrating over the smooth fields a at points $(t_i x_i y_i)$ we assume that a has singularities at these points. Let us assume for simplicity that $(t_i; x_i; y_i) = (0; 0; 0)$. Then $\vec{\nabla}\times a$ approaches to (16) as $(t; x; y) \to (0; 0; 0)$. Let M_i be modification rule at $(t_i; x_i; y_i)$ of the described kind. M stands for monopole. stands for evaluation of at $(t_i; x_i; y_i)$. Conjecturally 3-dimansional fermion-vortex duality(and bosonisation) shows up in identity

$$\int \ {}_1\cdots \ {}_N \exp\left(i\int(\ ^{-}\quad(@\ -A\ )\ -\frac{1}{4}A\cdot\vec{\nabla}\times A)dtdxdy\right)D\ =$$
$$\int M_1\cdots M_N \exp\left(\int i(|(@\ -a\ )\ |^2+\frac{1}{2}a\cdot\vec{\nabla}\times a+\frac{1}{2}a\cdot\vec{\nabla}\times A)dtdxdy\right)D\ \ Da$$

valid for arbitrary A. Note that the vortices (or monopoles) that appear here are instantaneous in time.

5 Topological insulators and superconductors

5.1 Back to fermions

Three dimensional action $\int d^3x\ i^{-}\quad(@\ -ia\ )$ for fermions that has appeared in Section 4.2 in relation to fermion-vortex duality can be augmented by four-dimensional action:

$$S=\int d^{2+1}x\left[i^{-}\quad(@\ -iA\ )\ \right]-\frac{1}{4e^2}\int d^{3+1}x\,F^2\ : \tag{60}$$

In the action above the Dirac fermion is supported on 2+1 dimensional surface in 3+1 dimensional space-time. The gauge field A lives on the 3+1 dimensional space-time and fermions interact with A through the restriction on the surface.

Son in [10] came up with an idea of how effective action of the dual theory might look like. According to him it is

$$S_{\mathrm{e}}\ =\int d^{2+1}x\left(i^{-}\quad(@\ +2ia\ )\ +\frac{1}{2}\qquad A\ @\ a\ \right)-\frac{1}{4e^2}\int d^{3+1}xF^2\ +:::: \tag{61}$$

In the above action is a Dirac composite fermion quasiparticle, a is an emergent gauge field and A is an external electromagnetic field with field strength $F\ =@\ A\ -@\ A$. Son showed that this theory has a close relation to theory of anyons with $=\frac{1}{2}=\frac{e^2}{2\ kh}$.

Seiberg and Witten proposed the low energy effective action for the composite fermion to be

$$S_e = \int d^{2+1}x \left(i^{-} \quad (@ \quad + 2ia \quad + iA \quad) \quad - \frac{1}{2} \qquad A \ @ \ a \ - \frac{1}{4} \qquad A \ @ \ A \ \right) \tag{62}$$

$$- \frac{1}{4e^2} \int d^{3+1}x F^2 \quad + \cdots :$$

If the particle and vortex scalars are made up of two fermions, and the fermionic actions (interacting with electromagnetism) are (60) and (62), the corresponding boundary scalar actions for the composite scalars, must be

$$S_{\mathrm{TI}} = \int d^{2+1}x \left[-\frac{1}{2} |(@ \quad - iqA \quad)\Phi|^2 - V(|\Phi|^2) - \frac{1}{4} F^2 \quad + \cdots \right] ; \tag{63}$$

where $q = 2$, for the composite scalar of two Dirac fermions, and for the low energy effective (boundary) action of the composite (and electromagnetically neutral) scalar of two composite fermions,

$$S_{\mathrm{TSC}} = \int d^{2+1}x \left[-\frac{1}{2} |(@ \quad + 2ia \quad)\tilde{\Phi}|^2 - V(|\tilde{\Phi}|^2) + \frac{1}{2} \qquad a \ @ \ A \ - \frac{1}{4} F^2 \quad + \cdots \right] : \tag{64}$$

The relation between these two actions is exactly the particle-vortex duality. Here is how [8] explains it. First we write complex Φ as a product of absolute value of $|\ |e^{i}$. Then we solve the equation $\vec{\nabla} V = 0$. We choose v to be one of the solutions and replace in the above action by v. This way we get a simplified action. We assume that absolute value of is fixed at a value v that minimizes the potential V, we get the first order action,

$$S_P = \int d^{2+1}x \left[\frac{1}{2v^2} \ ^2 - \quad (@ \quad - qA \quad) \right] ; \tag{65}$$

in terms of an auxiliary field . In order to exhibit particle-vortex duality we decompose into a periodic part and a singular part that is responsible for winding:

$$= \ _{\mathrm{smooth}} + \ _{\mathrm{vortex}} : \tag{66}$$

We integrate out $_{\mathrm{smooth}}$ and we get a constraint on :

$$= \qquad @ \ a \ : \tag{67}$$

If we substitute back into equation (65) we get:

$$S_P = \int d^{2+1}x \left[-\frac{1}{4v^2} f^2 \ + \qquad @ \ a \ (@ \quad _{\mathrm{vortex}} - qA \quad) \right]$$

$$= \int d^{2+1}x \left[-\frac{1}{4v^2} f^2 \quad + 2 \quad a \ j_{\mathrm{vortex}} - A \ J \ \right] ; \tag{68}$$

where the vortex current is

$$j_{\text{vortex}} = \frac{1}{2} \quad @\ @\ \ _{\text{vortex}}\ ; \tag{69}$$

and the current is $J\ = q\quad @\ a$. We have a description in terms of To complete the description, which is now in terms of vortices, coupled to the new gauge field a , as evidenced by the presence of the vortex current in the action, one needs to introduce another (vortex) scalar field $\tilde{\Phi}$ that couples directly to a . Finally then, the description in the particle-vortex dual theory is via the action

$$S_{\text{V}} = \int d^{2+1}x \left[-\frac{1}{4v^2} f^2\ \ - \frac{1}{2} |(@\ \ - i2\ \ a\ \ )\tilde{\Phi}|^2 - V(|\tilde{\Phi}|) - A\ \ (q\quad @\ a\ \ ) \right] ; \tag{70}$$

which is nothing but the action (64) after a rescaling of a by 2 . This establishes our claim then, that (63) and (64) are particle-vortex dual.

Appendix

A Quantum symmetries

Recall the setup of quantum mechanics. Ingredients that define a system is a Hilbert space of states L and a Schrödinger operator H. An element of the Hilbert space is called a states or a wave function. Given a states and a complete set of states $\{\ _1; \ldots;\ \ _n \ldots;\}$(mathematically they form an orthogonal basis of L) we can compute probability p_n to find in the state described by $\ _n$ as

$$p_n = \frac{|\langle\ \ ;\ \ _n\rangle|^2}{\langle\ \ ;\ \ \rangle\langle\ \ _n;\ \ _n\rangle}. \tag{71}$$

From this formula it is clear that the probability doesn't change if we multiply and each of $\ _n$ on nonzero complex numbers. This is why it is reasonable not to distinguish and c as they carry the same physical information. We will refer to defined up to a factor as a physical state.

Let G be a group. We say that G acts on the Hilbert space L if for each $g \in G$ there is a unitary operator $T(g) : L \to L$, which means $\langle T(g)\ \ _1; T(g)\ \ _2\rangle = \langle\ \ _1;\ \ _2\rangle$ for any $\ _1;\ \ _2 \in L$. The operator $T(g)$ must commute with the Hamiltonian H: $T(g)H = HT(g)$. This is the condition that $T(g)$ is the symmetry of the system. Physically unitarity means that $T(g)$ preserves probabilities computed by (71). One more condition on $T(g)$ is the compatibility with the group law:

$$T(g_1 g_2) = T(g_1)T(g_2). \tag{72}$$

We say that G acts projectively on the Hilbert space L if (72) is replaced by

$$T(g_1g_2) = c_{g_1;g_2}T(g_1)T(g_2) \quad c_{g_1;g_2} \in \mathbb{C}^\times \tag{73}$$

$\mathbb{C}^\times$ stands for the set of nonzero complex numbers. Observe that though $T(g_1g_2)$ and $T(g_1)T(g_2)$ are distinct, they differ by a multiplicative factor. They define the same physical state. Thus (73) is the condition of compatibility with the group law, written in the language of physical states. The numbers $c_{g_1;g_2}$ are not arbitrary:

$$T(g_1g_2g_3) = c_{g_1;g_2g_3}T(g_1)T(g_2g_3) = c_{g_1;g_2g_3}c_{g_2;g_3}T(g_1)T(g_2)T(g_3) =$$

$$T(g_1g_2g_3) = c_{g_1g_2;g_3}T(g_1g_2)T(g_3) = c_{g_1g_2;g_3}c_{g_1;g_2}T(g_1)T(g_2)T(g_3)$$

So

$$c_{g_1;g_2g_3}c_{g_2;g_3} = c_{g_1g_2;g_3}c_{g_1;g_2} \tag{74}$$

We say that subgroup $H \subset G$ is a symmetry of the state if $T(h) = (h)\ ;h \in H$. Here (h) are nonzero complex numbers. The numbers (h) has to be compatible with $c_{h_1;h_2}$:

$$T(h_1h_2) = (h_1h_2) = c_{h_1;h_2}T(h_1)T(h_2) = c_{h_1;h_2}\ (h_1)\ (h_2)$$

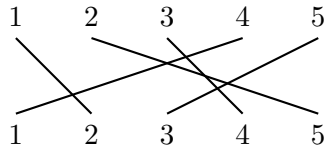
or

$$c_{h_1;h_2} = (h_1h_2)\ (h_1)^{-1}\ (h_2)^{-1}; h_1; h_2 \in H$$

We will denote that space of states with symmetry group H and character by $L^{H;}$.

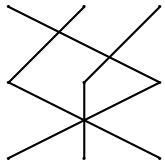
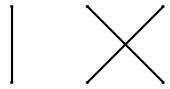
Example Symmetric group S_n on n letters permutes elements of $[1::n]$. A transposition $\begin{pmatrix} 1 & 2 & ::: & n-1 & n \\ i_1 & i_2 & ::: & i_{n-1} & i_n \end{pmatrix}$ will be shorthanded to $(i_1; i_2; :::; i_{n-1}; i_n)$. For example, we denote $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 1 & 3 \end{pmatrix}$ by $(2; 5; 4; 1; 3)$. Elementary transpositions $_i; i = 1; :::; n-1$ are $(1; :::; i+1; i; :::; n)$. One can visualize a permutation by a diagram.For example:

$(2; 5; 4; 1; 3) \quad \Rightarrow$



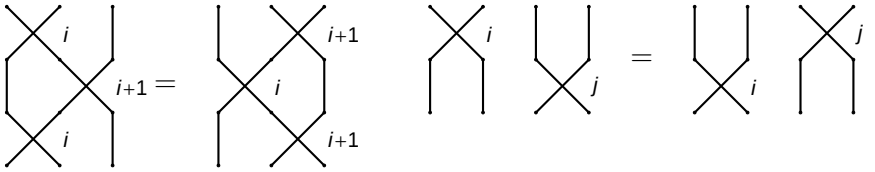
(75)

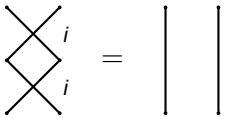
The labelling on the diagrams are obviously redundant and will be dropped.The composition of permutation corresponds to concatenation of diagrams, left permutation corresponds to the top diagram: Permutation $_i$ correspond to the diagram

$(3;2;1)(3;1;2) = (1;3;2) \Rightarrow$  $=$ 

$\Rightarrow$ 

The group S_n is generated by $_i$ in a sense that any permutation is a product of $_i$. Elements $_i$ satisfy Coxeter relations $_i\ _{i+1}\ _i = _{i+1}\ _i\ _{i+1}$, $_i\ _j = _j\ _i; |i-j| \geq 2$ and $_i^2 = 1$.





There is a theorem that states that all other relations between $_i$ follow from Coxeter relations.

Given a Hilbert space L we can form a tensor product $L \otimes \cdots \otimes L = L^{\otimes n}$. Its elements are linear combinations of $\Psi_{i_1;\ldots;i_n} = _{i_1} \otimes \cdots \otimes _{i_n}$. The inner product is defined by the rule

$$\langle \Psi_{i_1;\ldots;i_n}; \Psi_{j_1;\ldots;j_n} \rangle = \langle _{i_1}; _{j_1} \rangle \cdots \langle _{i_n}; _{j_n} \rangle$$

The hamiltonian is H is the sum of elementary Hamiltonians $H_i = 1 \otimes \cdots \otimes \underset{i}{H} \otimes \cdots \otimes 1$ Symmetric group acts by the formula

$$T(\ )\Psi_{i_1;\ldots;i_n} = \Psi_{(i_1);\ldots;(i_n)}$$

Here is how this specializes to the case of a free particle on two-dimensional space $\mathbb{R}^2$. Wave-function is a complex-valued function in two-variables $(x;y)$. The Hamiltonian H is $-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$. The space L is $\{ (x;y) | \int\int | (x;y)|^2 dxdy < \infty \}$. The space $L^{\otimes n}$ is the space of square-integrable functions in $x_1;y_1;\ldots;x_n;y_n$. $\Psi_{i_1;\ldots;i_n}$ is the product $_{i_1}(x_1;y_1)\ _{i_2}(x_2;y_2)\cdots\ _{i_n}(x_n;y_n)$. The Hamiltonian $H_i = -\frac{\partial^2}{\partial x_i^2} - \frac{\partial^2}{\partial y_i^2}$. The group S_n acts by the formula

$$T(\ )\Psi(x_1;y_1;\ldots;x_n;y_n) = \Psi(x_{\ 1};y_{\ 1};\ldots;x_{\ n};y_{\ n})$$

A.1 Symmetric states

Suppose we are interested in the states which are symmetric with respect to the whole group S_n. In order order to classify such states we have to fix the function $\chi(g)$. In our case the constants $c_{g;g'}$ are equal to one. Thus $\chi(g)$ must satisfy

$$\chi(gg') = \chi(g)\,\chi(g')$$

Such functions are called characters. S_n supports only two characters: there is a trivial character $\chi(g) = 1$ and there is the sign character:

$$sign(\sigma) = \frac{P(x_{\sigma(1)};\dots;x_{\sigma(1)})}{P(x_1;\dots;x_n)};\quad P(x_1;\dots;x_n) = \prod_{1\le i<j\le n}(x_i - x_j)$$

If $\sigma = \sigma_{i_1}\cdots\sigma_{i_k}$, then $sign(\sigma) = (-1)^k$. There is a standard terminology related to this construction. A state $\Psi_{i_1;\dots;i_n}$ satisfies Bose statistics if

$$T(\sigma)\Psi_{i_1;\dots;i_n} = \Psi_{\sigma(i_1);\dots;\sigma(i_n)} = \Psi_{i_1;\dots;i_n}$$

A state $\Psi_{i_1;\dots;i_n}$ satisfies Fermi statistics if

$$T(\sigma)\Psi_{i_1;\dots;i_n} = \Psi_{\sigma(i_1);\dots;\sigma(i_n)} = sign(\sigma)\Psi_{i_1;\dots;i_n}$$

How to set $c_{g;g'}$ to one There is a mathematical trick that allows to get rid of the constants $c_{g;g'}$ in the formula (73). The price you have to pay is the increase in the size of the group. Here is the construction: the group $\widetilde{G}$ is the set of pairs $\{(\alpha;g)|\ \alpha\in\mathbb{C}^\times; g\in G\}$. with multiplication rule

$$(\alpha;g)(\alpha';g') := (\alpha\alpha' c^{-1}_{g;g'};gg')$$

Equation (74) guarantees that the group law satisfies associativity. Its unit is $(1;1)$ and the inverse to $(\alpha;g)$ is $(\alpha^{-1}c_{g;g^{-1}};g^{-1})$. Define a representation of $\widetilde{G}$ by the rule $T(\alpha;g) = \alpha T(g)$. We see that in

$$T(\alpha;g)T(\alpha';g') = T(\alpha;g)(\alpha' T(g')) = \alpha\alpha' T(g)T(g')$$

$$\begin{aligned} T((\alpha;g)(\alpha';g')) &= T((\alpha\alpha' c^{-1}_{g;g'};gg')) \\ &= \alpha\alpha' c^{-1}_{g;g'}T(gg') \\ &= \alpha\alpha' c^{-1}_{g;g'}c_{g;g'}T(g)T(g') \\ &= \alpha\alpha' T(g)T(g') \end{aligned}$$

$c_{g;g'}$ cancels out. In the following we will always work (if necessary) with the group $\widetilde{G}$.

Free particle on a circle-another example of system with symmetries The Hilbert space of this system $L(\mathbf{R};2\ \text{-periodic})$consists of 2 -periodic functions on the real line $\mathbf{R}$ with coordinate . the Hamiltonian is $-\frac{d^2}{d\ ^2}$. Another model for this space is the space of functions $L(S^1)$ on $S^1 = \{z \in \mathbf{C}||z|^2 = 1\}$. The isomorphism is defined by the formula $f(z) \to f(e^i\ )$.

We can think about 2 periodic functions on $\mathbf{R}$ as states symmetric with respect to the group of integers $2\ \mathbf{Z} = \{2\ n\}$:

$$L(\mathbf{R};2\ \text{-periodic}) = L(\mathbf{R})^{2\ \mathbf{Z}}$$

The character in this case is trivial. Unitary characters ($|\ (g)|^2 = 1$) of the group $2\ \mathbf{Z}$ have the form

$$(2\ n) = e^{2\ in}\ ;\quad \in \mathbf{R}$$

An interesting case is $= 1/k$. States symmetric with respect to this character are quasi-periodic.

$$(T(g)f)(\ ) = f(\ +2\ ) = e^{2\ i/k}f(\ );g = 2$$

Such states are genuinely periodic with respect to translation on $2\ k$ units:

$$L(\mathbf{R})^{2\ \mathbf{Z};\ k} \subset L(\mathbf{R})^{2\ k\mathbf{Z};1}$$

I would like to identify $L(\mathbf{R})^{2\ k\mathbf{Z};1}$ with functions on the circle S^1 by the map $\to e^{i\ /k} = z$. Multiplication on $e^{2\ i/k}$ defines a rotation. Under the map $\to e^{i\ /k}$ this rotation corresponds to the shift $\to\ +2\ $. As $(e^{2\ i/k})^k = 1$ the element $e^{2\ i/k}$ generates a finite cyclic group $\mathbf{Z}_n$. We identify

$$L(\mathbf{R})^{2\ \mathbf{Z};\ k} = L(S^1)^{\mathbf{Z}_n;\ k} = \{g(z)||z| = 1;g(e^{2\ i/k}z) = e^{2\ i/k}g(z)\}$$

The spaces of functions $L(\mathbf{R})^{2\ \mathbf{Z}}$, $L(\mathbf{R})^{2\ k\mathbf{Z}}$ are based on the circles S^1_a, S^1_b, but these are different circles: We have map $S^1_b \to S^1_a$. $p : e^{i\ } \to e^{ki\ }$. In terms of the complex coordinate z it is given by the formula $p : z \to z^k$. If you take a function $f \in L(\mathbf{R})^{2\ \mathbf{Z}}$ and compose it with p, the result will be $2\ /k$-periodic and lie in $L(\mathbf{R})^{2\ k\mathbf{Z}}$. The map has a property that any point $z' \in p^{-1}(z)$ contains k pre-images. These are $z^{1/k};e^{2\ i/k}z^{1/k};:::;e^{2\ i(k-1)/k}z^{1/k}$. This is an example of a covering map.

Another example of a covering map is $\to e^{i\ }$. Pre-image of $z \in S^1$ consists of point $-i\log z + 2\ n$. Observe that $e^{2\ i(l)/k}$ and $2\ n$ in formulas for covering maps have already appeared as symmetries.

A.2 Topological construction of coverings and group actions.

In the last example we saw that if we have a space X with a group action and a character of the group we can generate $L(X)^{G;}$ of states that have G as a symmetry group. In this section we discuss how topology can help to construct such actions.

Let X be a subspace in $\mathbf{R}^n$ or $\mathbf{C}^n$ defined by equations or inequalities.

Examples of spaces A circle $S^1 = \{(x;y) \in \mathbb{R}^2 | x^2 + y^2 = 1\} = \{z \in \mathbb{C} | |z|^2 = 1\}$. Another example is the configuration space

$$C_n = \{z = (z_1;\dots;z_n) | z \notin \cup_{1\le i<j\le n} \Delta_{i;j}\}; \quad \Delta_{i;j} = \{z = (z_1;\dots;z_n) | z_i = z_j\}$$

$$C_1 = C; C_2 = \{(z_1;z_2) | z_1 \neq z_2\}; C_3 = \{(z_1;z_2;z_3) | z_1 \neq z_2; z_1 \neq z_3; z_2 \neq z_3\}$$

A path in the space is a continuous function $f : [0;1] \to X$. A closed path is an f such that $f(0) = f(1)$. For example $e^{2\ it}; 0 \le t \le 1$ defines a closed path in the circle.

We say that two paths $f_1 : [0;1] \to X$ and $f_2 : [0;1] \to X$ are *homotopic* if there is a continuous function $g(t;q); 0 \le t;q \le 1$ such that $g(t;0) = f_1(\ ); g(t;1) = f_2(\ )$. Fix a point $x_0 \in X$. The set of homotopy classes of paths that start and end at x_0 is a group $\ _1(X;x_0)$. The group law is defined by concatenation of paths:

$$f_1 * f_1(\ ) = \begin{cases} f_1(2t) & \text{if } 0 \le t \le 1/2 \\ f_2(2t-1) & \text{if } 1/2 \le t \le 1 \end{cases}$$

We define the universal covering space as a set of pair

$$\widetilde{X} := \{x \in X; g : [0;1] \to X | f(0) = x_0; g(1) = x; g(\ ) \text{ is defined up to a homotopy}\}:$$

The group $\ _1(X;x_0)$ acts on $\widetilde{X}$: $f \times (x;g) = (x; f * g)$. This way we get a space with a group action. The is forgetful map $p(x;g) = x$

The universal cover of S^1 Fix a point $x_0 = 1$ a path from $z = e^{2\ i} \in S^1$ to 1 up to a homotopy is characterized by its winding about $0 \in \mathbb{C}$. An examples of nonhomotopic paths is $f(\ ) = e^{2\ i(\ +n)t}; 0 \le t \le 1$. Pair $0 \le \ < 1; n \in \mathbb{Z}$ defines a point $\ + n$ on the universal cover which must be equal to the real line $\mathbb{R}$. Preimage $p^{-1}(1) = \{n\} = \mathbb{Z}$ by construction coincides with the fundamental group $\ _1(S^1;1)$.

Quotient There are some interesting modifications of the universal cover $\widetilde{X}$. Let $\ :\ _1(X;x_0) \to G$ be a homomorphism of groups We define X_G as the quotient space $\ _1(X;x_0)\backslash\widetilde{X} \times G$. The element $h \in \ _1(X;x_0)$ acts by the formula

$$h(\tilde{x};g) = h\tilde{x};\ \ (h)g$$

If $\ $ is onto then

$$X_G = \text{Ker } \ \backslash\widetilde{X}:$$

A.3 Braid group

We begin by taking a layer $L = \{(x;y;z) \in \mathbb{R}^3 | 0 \le x_3 \le 1\}$, and in it we place n strands of string, subject to the following conditions:

1. No part of any strand lies outside the layer.

2. Each strand begins on the top face of the layer, and ends on the bottom face.

3. No two strands intersect.

4. As we traverse any strand from the top face, we are always moving downwards. This means that no strand has any horizontal segment, or any segment that loops up.

Given an n-braid (in the layer) we say it is equivalent to another n-braid $'$ if the strands of can be perturbed to the strands of $'$ without doing any of the following:

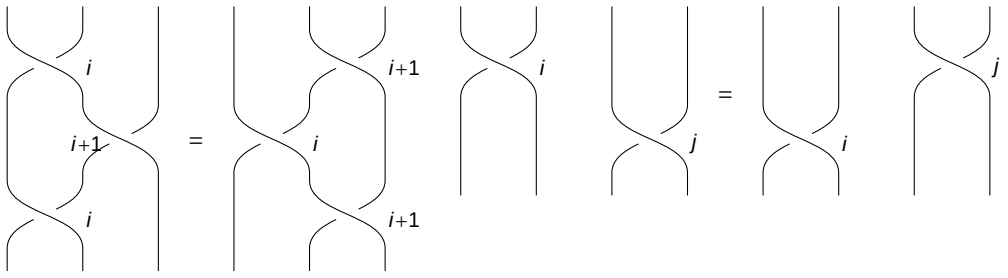
1. Moving any part of any strand out of the layer.

2. Cutting any strand.

3. Moving any endpoint of any strand.

We can also multiply two n-braids and $'$ by joining the bottom of to the top of $'$. By doing this we create a new n-braid which we shall denote by $'$ (we assume that the set of $(x;y)$ coordinates of the upper ends of stands coincide with the similar set for lower ends). In the end of this operation we will have to appropriate shift and rescaling in z-direction. Here is a picture of a braid on 5 strands.

This picture is a two-dimensional image (xz-projection) of the strand. It bares an obvious similarity with the diagram describing a permutation (75). Though both diagrams are flat the difference is that the second carries information about under and over crossing of strands. The braid group B_n on n strands can be defined as an abstract group by relations similar to Coxeter relations. The generators are $_i; i = 1;:::;n$. Relations are $_i\ _{i+1}\ _i = \ _{i+1}\ _i\ _{i+1};\ _i\ _j = \ _j\ _i$.



Analytically a braid is defined by a collection of function $(x_i(\ );y_i(\ );z(\ ) = 1 - t);i = 1;:::;n\ f_i : [0;1] \to L \subset R^3$. A function $(x_1(\ );y_1(\ );:::;x_n(\ );y_n(\ )) = (Z_1(\ );:::;Z_n(\ ))$ represents a path in $R^2 \times \cdots \times R^2$. By definition of the braid $Z_i(\ ) \neq Z_j(\ )$. Thus $(Z_1(\ );:::;Z_n(\ ))$ defines a closed path in C_n/S_n and an element of its fundamental group $_1(C_n/S_n;x_0)$. In fact this correspondence identifies $_1(C_n/S_n;x_0)$ and B_n.

We see that the universal covering $\widetilde{C_n/S_n} = \widetilde{C_n}$ is equipped with the action of the braid group. There is a forgetful homomorphism $m : B_n \to S_n$, $\mathrm{Ker}\, m = {}_1(C_n; x_0)$. By definition of universal cover $\widetilde{C_n}/\mathrm{Ker}\, m = \widetilde{C_n}$. There are some other interesting choices of homomorphisms. Consider a group G_n generated by g_i subject to relations $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}; g_i g_j = g_j g_i; g_i^2 = g_j^2 = a$. Let be a homomorphism $B_n \to G_n$ such that $({}_i) = g_i$. The quotient $K_n := \widetilde{C_n}/\mathrm{Ker}$ is equipped with the action of G_n. In particular the group Z generated by z acts on K_n. We have $K_n/Z = C_n$.

The space K_n has an explicit description. The function $A : C^\times \to S^1 \subset C^\times$ is defined by the formula $z \to z/|z|$. The space K_n is a subspace in the product $C_n \times R$:

$$\begin{aligned} K_n = \{ & (z_1; \dots; z_n; \ ) | (z_1; \dots; z_n) \in C_n; \ \in R; \\ & P(z_1; \dots; z_n) = e^{i} \ \} \end{aligned} \tag{76}$$

where

$$P = A\Big(\prod_{1 \le i < j \le n} (z_i - z_j)^2 \Big) \tag{77}$$

An element $a^l \in Z$ acts by $(z_1; \dots; z_n; \ ) \to (z_1; \dots; z_n; \ + 4 \ l)$. In order to define the action of G_n on K_n we observe that P is defined on C_n/S_n. We associate with any element $= (Z_1(\ ); \dots; Z_n(\ )) \in {}_1(C_n/S_n) = B_n$ an integer $l(\ )/(2 \ )$: the number of times $P(Z_1(\ ); \dots; Z_n(\ ))$ winds about zero. In particular $l(\ _i)/(2 \ ) = 1$ Then

$$g_i(z_1; \dots; z_i; z_{i+1}; \dots; z_n; \ ) = (z_1; \dots; z_{i+1}; z_i; \dots; z_n; \ + 2 \ ) \tag{78}$$

Any function $(z_1; \dots; z_n; \ )$ can be interpreted as a multi-valued function of $(z_1; \dots; z_n) \to \{ \ (z_1; \dots; z_n; \ _0 + 2 \ l)\}$. Here ${}_0(z_1; \dots; z_n)$ is one of the solutions of $P(z_1; \dots; z_n) = e^{i}$.

Let us consider an example of such function $\to E \ (\ ) = e^{i(1-\ )}$. This is a quasi-periodic function

$$E \ (\ + 2 \ ) = e^{2 \ i(1-\ )} E \ (\ ): \tag{79}$$

As a multivalued function on C_n it is equal to $P(z_1; \dots; z_n)^{1-}$. Equations (78,79) imply that $g_i P(z_1; \dots; z_n)^{1-} = e^{2 \ i(1-\ )} P(z_1; \dots; z_n)^{1-}$ More general anyonic multivalued wave function ${}_{1-}(z_1; \dots; z_n)$ must come from single-valued function on K_n and satisfy

$$g_i \ {}_{1-}(z_1; \dots; z_n) = e^{2 \ i(1-\ )} \ {}_{1-}(z_1; \dots; z_n)$$

Note that the fraction of such two functions is a symmetric function in $(z_1; \dots; z_n)$. Thus

$${}_{1-}(z_1; \dots; z_n) = \ (z_1; \dots; z_n) P(z_1; \dots; z_n)^{1-} \tag{80}$$

and $(z_1; \dots; z_n)$ is symmetric

B On solutions of equation $d! =$

Let R be the vector with coordinates $(x;y)$. In this section we verify that the function the vector–valued function $(a(x;y);b(x;y)) = \frac{B\times R}{R^2}$ is a solution of a pair of differential equation:

$$@_y a - @_x b = -2 \quad ^2(x;y); \quad @_x a + @_y b = 0 \tag{81}$$

$^2(x;y) = (x)\ (y)$ is the two-dimensional -function.

It is convenient to think about $(a;b)$ as components of a differential 1 - form $! = adx + bdy$. Equation that involves -function becomes

$$d! = (dx@_x + dy@_y)(adx + bdy) = -(@_y a - @_x b)dxdy = dxdy$$

Let us verify this identity in polar coordinates. In rectangular coordinates we have

$$B = (0;0;1); R = (x;y;0); F = \frac{R}{R^2}; ! = (B \times F)\cdot dR$$

In this formula $\times$ stands for the vector product. We have

$$! = \frac{xdy - ydx}{x^2 + y^2} \tag{82}$$

In polar coordinates $x = r\cos(\ ); y = r\sin(\ )$ the differentials transform as

$$dx = dr\cos \ - d\ r\sin \ ; \quad dy = d\ r\cos \ + dr\sin$$

We omit straightforward computation which gives

$$! = d$$

As the de Rham differential doesn't depend on system of coordinates we conclude that $d! = (dr@_r + d\ @\ )d \ = 0$. It remains to check that $d!(0;0) = \infty$ and we have a true delta-function. To check this we are going to use the definition of delta-function:

$$\int\int_{R^2} f(x;y)\ (x;y)dxdy = f(0;0)$$

In the formula f is an arbitrary smooth function which is zero outside the disk D of large radius. We have to check that

$$2\ f(0;0) = \int\int_{R^2} f d! = \int\int_D f d!$$

By Stocks theorem

$$\int\int_D d(!f) = \int_{S^1} !f = 0 \tag{83}$$

if the radius of the disk D is sufficiently large (in this case the function f restricted on its boundary will be zero). By using an identity

$$f(d!) = d(f!) - (df)!$$

and (83) we conclude that

$$\begin{aligned}\int\int_D f\,d! &= \int\int_D -df! = -\int\int_D (dr\partial_r + d\ \partial\ f)d \\ &= -\int\int_D \partial_r f\,dr d\ -\int\int_D (\partial_r f(r\cos(\ );r\sin(\ ))dr d \\ &= -\int_0^2 f(r\cos(\ );r\sin(\ ))|_0^R \\ &= \int_0^2 f(0;0) = 2\ f(0;0)\end{aligned}$$

Divergence $\partial_x a + \partial_y b$ is equal to $(2xy)/(x^2+y^2)^2 - ((2xy)/(x^2+y^2)^2) = 0$

Here is a procedure that enables us to construct a more general solution of the same pair of equations. Recall that de Rham operator d satisfies $d^2 = 0$. Pick a function $g(x;y)$. Then $! + dg$ still satisfies $d(! + dg) = \ + d^2 g = \ $. Divergence $\nabla(! + dg) = 0 + \nabla(dg)$ is equal to Δg. Δ is a Laplace operator. A function g is harmonic if $\Delta g = 0$. Any harmonic function g in two dimensions is a sum $\mathrm{Re}(s_1(z)+s_1(\bar{z}))$, where $s_1(z); s_1(z)$ are holomorphic functions. In fact the general solution of (81) is $adx + bdy = ! + \partial_x g dx + \partial_y g dy$ where g is harmonic.

Suppose we would like to find a solution of a more general equation

$$\partial_y a - \partial_x b = -2\ h(x;y); \partial_x a + \partial_y b = 0: \tag{84}$$

We can use $!$ for this purposes. As $h(R) = \int\int_{R^2} {}^2(R-R')h(R')dx'dy'$ we can write

$$\begin{aligned}2\ h dxdy =& 2\ \int\int_{R^2} {}^2(R-R')h(R')dx'dy'dxdy \\ =& (dx\partial_x + dy\partial_y)\int\int_{R^2} !(R-R')h(R')dx'dy'\end{aligned} \tag{85}$$

The one-form that encodes the functions a and b is $\int\int_{R^2} !(R-R')h(R')dx'dy'$. One has to be careful with this construction - if, for example, the function h is constant, the integral diverges. It converges if h decay rapidly.

We can, however, turn the theorem into a definition. This means that if we find a solution $d!' = hdxdy$ (up to ambiguities discussed above) we can postulate it to be the value of the integral. For example if $h = 1$ then $!' = (-ydx + xdy)/2$ satisfies

$$d!' = 1dxdy: \tag{86}$$

In addition $\nabla !' = 0$. $!'$ is a unique solution of (86) up to addition of a gradient of harmonic function and a good candidate for definition of right hand side of (85) with $h = 1$.

Remark 1 *Observe that $id\ $ is a logarithmic gradient of the function e^i. In rectangular coordinates $e^i\ $ is equal to $\frac{x+iy}{\sqrt{(x^2+y^2)}}$. In complex notations $e^i\ = \frac{z}{|z|}$, where $z = x + iy$.*

C Gaussian integrals

Pick a symmetric $n \times n$ matrix a_{ij}. It defines a function $a(x) = \sum_{i,j=1}^{n} a_{ij} x^i x^j$ on $\mathbb{R}^n$. Let assume that $a(x) \geq 0$ and $a(x) = 0$ only if $x = (x^1, \ldots, x^n)$ is zero. We would like to compute an integral $\int_{\mathbb{R}^n} \exp(-a(x)/2) dx$. This can be done through the series of steps.

$$c = \int_{-\infty}^{\infty} \exp(-x^2) dx = \sqrt{\ }$$

Indeed

$$c^2 = \int_{-\infty}^{\infty} \exp(-x^2) dx \int_{-\infty}^{\infty} \exp(-y^2) dy = \int_{\mathbb{R}^2} \exp(-x^2 - y^2) dx dy$$
$$= \int_0^2 \int_0^{\infty} \exp(-\ ^2)\ d\ = -2\ \exp(-\ ^2)/2|_0^{\infty} = -2\ (-1/2) =$$

With a help of change of variables $x = \sqrt{2/b} y$ we transform the integral

$$\int_{-\infty}^{\infty} \exp\left(-\frac{bx^2}{2}\right) dx = \int_{-\infty}^{\infty} \exp\left(-y^2\right) \sqrt{2/b} dy = \sqrt{\frac{2}{b}}$$

We now know that if b_{ij} is a diagonal matrix $diag(b_1, \ldots, b_n)$ then

$$\int_{\mathbb{R}^n} \exp(-b(x)/2) dx = \prod_{i=1}^{n} \int_{-\infty}^{\infty} \exp\left(-\frac{b_i x_i^2}{2}\right) dx_i = \prod_{i=1}^{n} \sqrt{\frac{2}{b_i}} = \frac{\sqrt{2}^n}{\sqrt{\det b}}$$

To integrate a general quadratic function we can use diagonalization technique. Indeed first we find the spectrum $\ _1, \ldots, \ _n$ of b by solving $P_B(\ ) = \det(b_{ij} - \ _{ij}) = 0$. There is an orthogonal matrix c_{ij} whose columns are made of normalized eigenvectors of b. The change of variables

$$x_i = \sum_{j=1}^{n} c_{ij} y_j \tag{87}$$

transforms $b(x) \to b(c(y)) = \sum_{i=1}^{n} \ _i y_i^2$. The matrix c satisfies $1 = \det(\mathrm{id}) = \det(cc^t) = (\det c)^2$. Thus $\det c = \pm 1$. It is convenient to work with matrices c whose determinant is one. If $\det c = -1$ we can always compose it with a matrix $diag(1, \ldots, 1, -1)$. The new matrix c' will still satisfy $b(x) \to b(c'(y)) = \sum_{i=1}^{n} \ _i y_i^2$ but $\det(c') = (-1)(-1) = 1$. After the change of variables (87)

$$\int_{\mathbb{R}^n} \exp(-b(x)/2) dx = \int_{\mathbb{R}^n} \exp(-b(cy)/2) \det c dy = \prod_{i=1}^{n} \sqrt{\frac{2}{\ _i}} = \frac{\sqrt{2}^n}{\sqrt{\det b}}$$

We used that $1 \det b1 = \det c^t bc = \det(diag(\ _1;:::;\ _n)) = \prod_{i=1}^n\ _i$. Now it is easy to compute Fourier transform of $\exp(-b(x)/2)$. Introduce an abbreviation $p \cdot x = \sum_{k=1}^n p_k x^k$

$$\begin{aligned}\int_{R^n} \exp(-b(x)/2 - ip \cdot x)dx &= \int_{R^n} \exp(-b(x + ia^{-1}p)/2 - b^{-1}(p)/2)dx \\ &= \exp(-b^{-1}(p)/2) \int_{R^n + ib^{-1}p} \exp(-b(y)/2)dy \\ &= \exp(-b^{-1}(p)/2) \int_{R^n} \exp(-b(y)/2)dy \\ &= \exp(-b^{-1}(p)/2) \frac{\sqrt{2}^n}{\sqrt{\det b}}\end{aligned} \tag{88}$$

In the proof I used identities

$$\begin{aligned}b(x + ia^{-1}p) =& b_{kl}(x^k + ia^{-1;ks}p_s)(x^l + ia^{-1;lt}p_t) \\ =& b_{kl}x^k x^l + ib_{kl}x^k b^{-1;lt}p_t + ib_{kl}b^{-1;ks}p_s x^l - b_{kl}b^{-1;ks}p_s b^{-1;lt}p_t \\ =& b_{kl}x^k x^l + 2ix^k p_k - b^{-1;st}p_s p_t \text{ and } b_{kl}b^{-1;ks} = \ ^l_s; \quad b_{kl}b^{-1;ks}b^{-1;lt} = b^{-1;st}\end{aligned}$$

and the fact that analytic $\exp(-b(y)/2)$ decay rapidly at infinity in the complex domain and $\exp(-b(y)/2)$ has no singularities. This enables me to deform the domain of integration from $R^n + ib^{-1}p$ to R^n. Inverse Fourier transform is

$$\frac{1}{(2\ )^n} \int_{R^n} \exp(-b^{-1}(p)/2 + ipx) \frac{\sqrt{2}^n}{\sqrt{\det b}} dp = \exp(-b(x)/2)$$

In particular in one dimension

$$\frac{1}{\sqrt{2\ b}} \int_{-\infty}^{\infty} \exp(-p^2/2b + ipx)dp = \exp(-bx^2/2)$$

From this we see that as $b \to 0$ the Fourier transform of $\exp(-bx^2/2)$ approaches to Dirac -function. This is a general fact: in the case $b(x) \geq 0$ but $\det b = 0$ Fourier transform of $\exp(-b(x)/2)$ becomes a product of -functions in the direction of null-space $\mathrm{Ker}\, b$ of b. In the following we are going to decompose R^n into $\mathrm{Ker}\, b + \mathrm{Ker}^{\perp} b$. Fourier transform of degenerate b will be

$$\text{const}\ \ (p) \exp(-b^{-1}(p')/2) \quad p \in \mathrm{Ker}\, b; p' \in \mathrm{Ker}^{\perp} b: \tag{89}$$

For convergence of Fourier transform we needed condition $b(x) \geq 0$. The final answer does not need this condition for its formulation. Thus we postulate 89 to be Fourier transform for nonpositive $b(x)$.

Choose a basis $\{e_1;:::;e_k\}$ for $\mathrm{Ker}^{\perp} b$ and $\{e_{k+1};:::;e_n\}$ for $\mathrm{Ker}\, b$. In the basis $\{e_1;:::;e_n\}$ the matrix of b has the block form $\left(\begin{smallmatrix} B & 0 \\ 0 & 0 \end{smallmatrix}\right)$. b^{-1} can be extended to an operator with matrix $\left(\begin{smallmatrix} B^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right)$. In fact we can also use

$$g = \begin{pmatrix} B^{-1} & C \\ C^t & D \end{pmatrix} \tag{90}$$

in computations of 89 as long as $p' \in \mathrm{Ker}^{\perp} b$, for the blocks $C; C^t; D$ will make no contributions. Note that $b \cdot g = \left(\begin{smallmatrix} 1 & BC \\ 0 & 0 \end{smallmatrix}\right)$. To verify the las equation it suffice to check

$$p'_1 \cdot b \cdot g \cdot p'_2 = p'_1 \cdot p'_2 \tag{91}$$

D Elements of second quantization formalism

Second quantization formalism was designed to describe quantum mechanics of systems with variable number of particles. Even though the number of anyons N remain constant in our theory it is convenient to use this formalism. As we are interested in fermionic system (see previous section why) we are going to briefly remind the fermionic version of the formalism.

We assume that the Hilbert space of one particle is the space of functions $\{\ (\mathsf{q}) | \mathsf{q} \in \mathsf{R}^n;\ \int |\ (\mathsf{q})|^2\ d\mathsf{V}(\mathsf{q}) < \infty\}$ A wave function in multi-particle Hilbert space is

$$= \{\ _0;\ _1(\mathsf{q}_1);\ _2(\mathsf{q}_1; \mathsf{q}_2); \dots\} \tag{92}$$

All $_n$ is anti-symmetric functions. By definition

$$(\ ;\ ) = |\ _0|^2 + \sum_{n=1}^{\infty} \int |\ _n|^2\ d\mathsf{V}(\mathsf{q}_1) \dots\ d\mathsf{V}(\mathsf{q}_n)$$

Fix a function (q). We use to define two operators $\Psi(\ ); \Psi^{\dagger}(\ )$. Creation operator $\Psi^{\dagger}(\ )$ takes the function $_k$ and create a wave-function in $k+1$ variables by the formula

$$(\Psi^{\dagger}(\ )\ _k)(\mathsf{q}_1; \dots; \mathsf{q}_{k+1}) := \frac{1}{\sqrt{k+1}} \sum_{i=0}^{k} (-1)^{ik^-} (\mathsf{q}\ _{i(1)})\ _k(\mathsf{q}\ _{i(2)}; \dots; \mathsf{q}\ _{i(k+1)})$$

is a cyclic permutation $(1; 2 \dots; k+1)$. The other operator decreases the number of variable (this is why it is called annihilation operator) acts by the formula

$$(\Psi(\ )\ _k)(\mathsf{q}_1; \dots; \mathsf{q}_{k-1}) := \sqrt{k} \int_{\mathsf{R}^n} (\mathsf{q})\ _k(\mathsf{q}; \mathsf{q}_1; \dots; \mathsf{q}_{k-1})\ d\mathsf{V}(\mathsf{q})$$

$\Psi(\ )$ acts on $_0$ by zero. Operators $\Psi(\ )$ for different anti-commute

$$\Psi(\ _1)\Psi(\ _2) + \Psi(\ _1)\Psi(\ _2) = 0. \tag{93}$$

Indeed

$$\Psi(\ _1)\Psi(\ _2)\ _k = \sqrt{k}\sqrt{k-1} \int_{\mathsf{R}^n} \int_{\mathsf{R}^n}\ _1(\mathsf{q})\ _2(\mathsf{q}')\ _k(\mathsf{q}'; \mathsf{q}; \mathsf{q}_1; \dots; \mathsf{q}_{k-2})\ d\mathsf{V}(\mathsf{q})\ d\mathsf{V}(\mathsf{q}') =$$
$$-\sqrt{k}\sqrt{k-1} \int_{\mathsf{R}^n} \int_{\mathsf{R}^n}\ _1(\mathsf{q})\ _2(\mathsf{q}')\ _k(\mathsf{q}; \mathsf{q}'; \mathsf{q}_1; \dots; \mathsf{q}_{k-2})\ d\mathsf{V}(\mathsf{q})\ d\mathsf{V}(\mathsf{q}') =$$
$$-\sqrt{k}\sqrt{k-1} \int_{\mathsf{R}^n} \int_{\mathsf{R}^n}\ _2(\mathsf{q})\ _1(\mathsf{q}')\ _k(\mathsf{q}'; \mathsf{q}; \mathsf{q}_1; \dots; \mathsf{q}_{k-2})\ d\mathsf{V}(\mathsf{q})\ d\mathsf{V}(\mathsf{q}') = -\Psi(\ _2)\Psi(\ _1)\ _k$$

In the second line we pick up minus sign because skew-symmetry of $\ _k$. In the third line we make a change of variables $q \to q'$ $q' \to q$.

Operators $\Psi(\ )$ and $\Psi^\dagger(\ )$ are adjoint. We have to verify that $(\Psi(\ )\ _k;\ '_{k-1}) = (\ _k; \Psi^\dagger(\ )\ '_{k-1})$:

$$(\Psi(\ )\ _k;\ '_{k-1}) =$$
$$= \sqrt{k} \int \left(\int\ (q)\ \ _k(q; q_1; \ldots; q_{k-1})\, dV(q) \right) \overline{\ }'_{k-1}(q_1; \ldots; q_{k-1})\, dV(q_1) \ldots dV(q_{k-1})$$

We change indices: variables $q; q_1; \ldots; q_{k-1}$ will be denoted $q_1; q_2; \ldots; q_k$.We continue:

$$= \sqrt{k} \int\ \ _k(q_1; q_2; \ldots; q_k) \overline{\ }(q_1)\ \ '_{k-1}(q_2; \ldots; q_k) dV(q_1) \ldots dV(q_k)$$

If we apply a a power $\ ^i$ of cyclic permutation $\ = (1; 2 \ldots; k)$ to variables of $\ _k(q_1; q_2; \ldots; q_k)$ the function up to a sign $(-1)^{(k-1)i}$ stays invariant. We continue:

$$= \frac{\sqrt{k}}{k} \sum_{i=1}^{k} (-1)^{(k-1)i} \int\ \ _k(q_{\ ^i(1)}; q_{\ ^i(2)}; \ldots; q_{\ ^i(k)}) \overline{\ }(q_1)\ \ '_{k-1}(q_2; \ldots; q_k)\, dV(q_1) \ldots dV(q_k)$$
$$= \frac{1}{\sqrt{k}} \sum_{i=1}^{k} (-1)^{(k-1)i} \int\ \ _k(q_1; q_2; \ldots; q_k) \overline{\ }(q_{\ ^{-i}(1)})\ \ '_{k-1}(q_{\ ^{-i}(2)}; \ldots; q_{\ ^{-i}(k)})\, dV(q_1) \ldots dV(q_k)$$
$$= \int\ \ _k(q_1; q_2; \ldots; q_k) \left(\overline{\frac{1}{\sqrt{k}} \sum_{i=0}^{k-1} (-1)^{(k-1)i} \overline{\ }(q_{\ ^i(1)})\ \ '_{k-1}(q_{\ ^i(2)}; \ldots; q_{\ ^i(k)})} \right) dV(q_1) \ldots dV(q_k)$$
$$= (\ _k; \Psi^\dagger(\ )\ '_{k-1})$$

From this we immediately obtain

$$\Psi^\dagger(\ _1)\Psi^\dagger(\ _2) + \Psi^\dagger(\ _2)\Psi^\dagger(\ _1) = (\Psi(\ _2)\Psi(\ _1) + \Psi(\ _1)\Psi(\ _2))^\dagger = 0 \tag{94}$$

Next we are going to verify that

$$\Psi(\ _1)\Psi^\dagger(\ _2) + \Psi^\dagger(\ _1)\Psi(\ _2) = \mathrm{id} \int_{R^n}\ \ _1(q) \overline{\ }_2(q)\, dV(q) \tag{95}$$

$$
\begin{aligned}
\Psi(\ _1)\Psi^\dagger(\ _2)\ _k = &\frac{\sqrt{k+1}}{\sqrt{k+1}}\int\ _1(\mathbf{q})(\bar{}_2(\mathbf{q})\ _k(\mathbf{q}_1;\mathbf{q}_2;\dots;\mathbf{q}_k)+\\
&+(-1)^k\bar{}_2(\mathbf{q}_1)\ _k(\mathbf{q}_2;\dots;\mathbf{q}_k;\mathbf{q})+\\
&+(-1)^{2k}\bar{}_2(\mathbf{q}_2)\ _k(\mathbf{q}_3;\dots;\mathbf{q}_k;\mathbf{q};\mathbf{q}_1)+\cdots)\,dV(\mathbf{q})\\
= &\ _k\int\ _1\bar{}_2\,dV(\mathbf{q})+\\
&+(-1)^k\bar{}_2(\mathbf{q}_1)\int\ _1(\mathbf{q})\ _k(\mathbf{q}_2;\dots;\mathbf{q}_k;\mathbf{q})\,dV(\mathbf{q})+\\
&+(-1)^{2k}\bar{}_2(\mathbf{q}_2)\int\ _1(\mathbf{q})\ _k(\mathbf{q}_3;\dots;\mathbf{q}_k;\mathbf{q};\mathbf{q}_1)\,dV(\mathbf{q})+\cdots\\
= &\ _k\int\ _1\bar{}_2\,dV(\mathbf{q})+\\
&+(-1)^{k+k-1}\bar{}_2(\mathbf{q}_1)\int\ _1(\mathbf{q})\ _k(\mathbf{q};\mathbf{q}_2;\dots;\mathbf{q}_k)\,dV(\mathbf{q})+\\
&+(-1)^{2k+k-2}\bar{}_2(\mathbf{q}_2)\int\ _1(\mathbf{q})\ _k(\mathbf{q};\mathbf{q}_3;\dots;\mathbf{q}_k;\mathbf{q}_1)\,dV(\mathbf{q})+\cdots\\
= &\ _k\int\ _1\bar{}_2\,dV(\mathbf{q})-\\
&-\frac{1}{\sqrt{k}}((-1)^{2(k-1)}\bar{}_2(\mathbf{q}_1)\int\sqrt{k}\ _1(\mathbf{q})\ _k(\mathbf{q};\mathbf{q}_2;\dots;\mathbf{q}_k)\,dV(\mathbf{q})+\\
&+(-1)^{3(k-1)}\bar{}_2(\mathbf{q}_2)\int\sqrt{k}\ _1(\mathbf{q})\ _k(\mathbf{q};\mathbf{q}_3;\dots;\mathbf{q}_k;\mathbf{q}_1)\,dV(\mathbf{q})+\cdots)\\
= &\ _k\int\ _1\bar{}_2\,dV(\mathbf{q})-\Psi^\dagger(\ _2)\Psi(\ _1)\ _k
\end{aligned}
$$

A very useful choice of $\ (\mathbf{q})$ is a $\ $-function $\ ^n(\mathbf{q}-\mathbf{r})$. Denote $\Psi(\ ^n(\mathbf{q}-\mathbf{r}))$ by $\Psi(\mathbf{r})$ and $\Psi^\dagger(\ ^n(\mathbf{q}-\mathbf{r}))$ by $\Psi^\dagger(\mathbf{r})$. Obviously $\Psi^\dagger(\ )$ and $\Psi(\ )$ are superpositions $\Psi(\mathbf{r})$ and $\Psi^\dagger(\mathbf{r})$:

$$
\Psi^\dagger(\ )=\int_{\mathbf{R}^n}\Psi^\dagger(\mathbf{q})\bar{}(\mathbf{q})\,dV(\mathbf{q});\quad \Psi(\ )=\int_{\mathbf{R}^n}\Psi(\mathbf{q})\ (\mathbf{q})\,dV(\mathbf{q})
$$

Operators $\Psi^\dagger(\mathbf{q})$ and $\Psi(\mathbf{q})$ satisfy

$$
\begin{aligned}
&\Psi(\mathbf{q}')\Psi^\dagger(\mathbf{q})+\Psi^\dagger(\mathbf{q})\Psi(\mathbf{q}')=\ ^n(\mathbf{q}-\mathbf{q}')\\
&\Psi^\dagger(\mathbf{q})\Psi^\dagger(\mathbf{q}')+\Psi^\dagger(\mathbf{q}')\Psi^\dagger(\mathbf{q})=0\\
&\Psi(\mathbf{q})\Psi(\mathbf{q}')+\Psi(\mathbf{q}')\Psi(\mathbf{q})=0
\end{aligned}
\tag{96}
$$

These formulas reduce to (93),(94),(95) after multiplication on $\ _1(\mathbf{q})$ and $\ _2(\mathbf{q}')$ and integration.

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