Discontinuities of Feynman Integrals

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Outline

- Landau and Cutkosky

- Classic unitarity cuts
  - Dispersion relations
  - Modern unitarity method, with master integrals
  - Dimensional regularization and masses [recent work with Mirabella, Ochirov]

- Generalized or iterated cuts
  - Double dispersion relations
  - Cut integrals and discontinuities [work in progress with Abreu, Duhr, Gardi]
Singularities of Feynman integrals: Landau conditions

Denominators: \( A_i \equiv M_i^2 - q_i^2 \)

Feynman parameters \( \alpha_i \).

1st Landau condition:

\[ \alpha_i A_i = 0 \quad \forall i, \]

2nd Landau condition:

\[ \sum \alpha_i q_i = 0, \quad \text{for each closed loop.} \]
Cutkosky cuts

Discontinuities = Landau singularities = replace propagators by delta functions in integral

Any number of delta functions!

At one loop: geometric interpretation of 2nd Landau condition.

Polytope volume → 0. Point $Q$ falls into hyperplane of external momenta.
Scattering and interaction matrices:

\[ S = 1 + iT \]

The unitarity condition: \( S^\dagger S = 1 \).

\[ -i(T - T^\dagger) = T^\dagger T \]

\[ 2\text{Im} \, T = T^\dagger T \]

\[ 2 \text{Im} = \sum_t d\Pi_t \langle f | \, \overset{\text{\tiny t}}{\bullet} \, \rangle \langle \overset{\text{\tiny f}}{\bullet} | \, f \rangle \]

Cut across one channel, with any number of loops.
Dispersion relations

From the imaginary part, reconstruct the integral:

\[ A(K^2) = \frac{1}{\pi} \int_0^\infty ds \frac{\text{Im} \ A(s)}{s - K^2} \]

Classic example: On-shell vertex function, 2 loops. [Van Neerven, 1986]

Integration is still hard work. At least at one loop, one can do much better.
Master integrals

\[ A^{1-\text{loop}} = \sum_i c_i l_i + r, \quad c_i, r \text{ are rational functions.} \]

Analytically known at 1-loop, some special cases beyond.
Master integrals

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Analytically known at 1-loop, some special cases beyond.

- e.g. box: \( \int d^{4-2\epsilon} k \frac{1}{(\ell^2)(\ell-K_1)^2(\ell-K_1-K_2)^2(\ell-K_1-K_2-K_3)^2} \)
- scalar numerators
- max. 4 propagators in 4d
- can include masses
Master integrals

\[ A^{1-\text{loop}} = \sum_i c_i I_i + r, \quad c_i, r \text{ are rational functions.} \]

Analytically known at 1-loop, some special cases beyond.

e.g.: If \( K_3^2 = K_4^2 = 0, \)

\[ l_4^{2m h} = \frac{2r}{st} \frac{1}{\epsilon^2} \left[ \frac{1}{2} (-s)^{-\epsilon} + (-t)^{-\epsilon} - \frac{1}{2} (-K_1^2)^{-\epsilon} - \frac{1}{2} (-K_2^2)^{-\epsilon} \right] \]

\[ - \frac{2r}{st} \left[ -\frac{1}{2} \ln \left( \frac{s}{K_1^2} \right) \ln \left( \frac{s}{K_2^2} \right) + \frac{1}{2} \ln^2 \left( \frac{s}{t} \right) \right] \]

\[ + Li_2 \left( 1 - \frac{K_1^2}{t} \right) + Li_2 \left( 1 - \frac{K_2^2}{t} \right) \] + \( O(\epsilon). \)
Amplitudes from unitarity cuts

\[ A^{1\text{-loop}} = \sum c_i l_i \]

\[ A = c_1 + c_2 + c_3 + \cdots \]
Amplitudes from unitarity cuts

\[ \Delta A^{1-\text{loop}} = \sum c_i \Delta l_i \]

LHS: work at the level of tree amplitudes. RHS: known masters.

Matching 4-dimensional cuts can suffice to determine reduction coefficients! Logarithms with unique arguments. "cut-constructibility" [Bern, Dixon, Dunbar, Kosower]

But: we still get several coefficients together in the same equation. How do we evaluate a unitarity cut?
Amplitudes from unitarity cuts

\[ \Delta A^{1\text{-loop}} = \sum c_i \Delta I_i \]

Matching 4-dimensional cuts can suffice to determine reduction coefficients! Logarithms with unique arguments.

“cut-constructibility”

[Bern, Dixon, Dunbar, Kosower]

But: we still get several coefficients together in the same equation.

How do we evaluate a unitarity cut?
\[ \Delta A^{1-\text{loop}} = \int d\mu \ A^{\text{tree}}(-\ell, i, \ldots, j, \ell - K) \ A^{\text{tree}}(K - \ell, j + 1, \ldots, i - 1, \ell) \]

\[ d\mu = d^4\ell \ \delta(\ell^2) \ \delta((\ell - K)^2) \]

Change to homogeneous \((CP^1)\) spinor variables with

\[ \ell_{a\dot{a}} = t \ \lambda_a \bar{\lambda}_{\dot{a}}. \]

Integration measure:

\[ \int d^4\ell \ \delta(\ell^2) \ (\bullet) = \int_0^\infty dt \ t \int_{\bar{\lambda} = \lambda} \langle \lambda \ d\lambda \rangle \ [\bar{\lambda} \ d\bar{\lambda}] \ (\bullet) \]

[Cachazo, Svrček, Witten]
Systematic procedure: spinor integration

[Anastasiou, RB, Buchbinder, Cachazo, Feng, Kunszt, Mastrolia]

- Change variables, $\ell = t \lambda \tilde{\lambda}$, and use the spinor measure,

$$\int d^4 \ell \delta(\ell^2)\delta((\ell - K)^2) = \int dt \ t \int \langle \lambda \ d\lambda \rangle [\tilde{\lambda} \ d\tilde{\lambda}] \delta((t\lambda\tilde{\lambda} - K)^2)$$

- Use 2nd delta function to perform $t$-integral.

- $\lambda, \tilde{\lambda} \rightarrow z, \bar{z}$ familiar complex variables.

- Evaluate with residue theorem.

- Identify cuts of basis integrals and read off coefficients. $D$-dimensional cuts also treated, for complete amplitudes.

- We have given formulas for the resulting coefficients.
Dimensional regularization at one loop

In $D = 4 - 2\epsilon$ dimensions, the result of reduction is

$$A = \sum_i e_i \text{ (pentagon)} + \sum_i d_i \text{ (box)}$$

$$+ \sum_i c_i \text{ (triangle)} + \sum_i b_i \text{ (bubble)}$$

No extra rational term.
Unitarity in $D = 4 - 2\epsilon$ dimensions

Orthogonal decomposition, keeping external momenta in 4 dimensions. [Bern, Chalmers, Mahlon, Morgan]

$$\int d^{4-2\epsilon} \ell_{4-2\epsilon} = \frac{(4\pi)^\epsilon}{\Gamma(-\epsilon)} \int_0^1 du \ u^{-1-\epsilon} \int d^4 \ell_4.$$ 

where $\ell_{-2\epsilon}^2 = K^2/4 \ u$.

The integral over $u$ will remain. The $u$-dependence is controlled:

$$\Delta A = \int_0^1 du \ u^{-1-\epsilon} \int d^4 \ell \ \delta(\ell^2) \ \delta(\sqrt{1-u \ K^2} - 2K \cdot \ell)$$

Recognize and perform the 4-d integral as before.

(Cf. methods by Ossola, Papadopoulos, Pittau; Forde; Ellis, Giele, Kunszt; Kilgore; Giele, Kunszt, Melnikov; Badger)
Massive particles

Cut amplitude:

\[
\int \langle \lambda \ d\lambda \rangle [\tilde{\lambda} \ d\tilde{\lambda}] \left( \frac{\sqrt{\Delta[K^2, M_1^2, M_2^2]}}{K^2} \right) \frac{(K^2)^{n+1}}{\langle \lambda | K | \tilde{\lambda} \rangle^{n+2}} \prod_{j=1}^{n+k} \langle \lambda | R_j | \tilde{\lambda} \rangle \times \prod_{i=1}^{k} \langle \lambda | Q_i | \tilde{\lambda} \rangle
\]

- The integral coefficients have the same form.

  [RB, Feng, Mastrolia, Yang]

- New master integrals.
The special “massive” master integrals

These integrals do not have kinematic cuts.

\[ I_1 = m^{2-2\epsilon} \frac{\Gamma(1 + \epsilon)}{\epsilon(\epsilon - 1)} \]

\[ I_2(0; m^2, m^2) = m^{-2\epsilon} \frac{\Gamma(1 + \epsilon)}{\epsilon} \]

\[ I_2(m^2; 0, m^2) = m^{-2\epsilon} \frac{\Gamma(1 + \epsilon)}{\epsilon(1 - 2\epsilon)} \]
Divergent cuts for on-shell bubbles

- Try to apply unitary cuts to the special massive master integrals
- Cut of massless on-shell bubble diverges, due to internal on-shell propagator
- Must include the counterterms.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{sample.png}
\caption{Graphical representation of the cuts discussed in the text.}
\end{figure}
Isolate and remove the divergent diagrams

Implicit use of counterterm

Feynman-diagram decomposition is gauge dependent

Embedded in a numerical algorithm
Our method \cite{RB, Mirabella}

Use an off-shell continuation of the fermion mass. The cut is finite until we take the on-shell limit.

- Power series expansion in the off-shell parameter
- In the on-shell limit, divergences are guaranteed to cancel: keep only finite terms
- Explicit use of counterterms. Gauge dependence enters only in tree level currents.
- Clean analytic results
Momentum-conserving shift: \( k \rightarrow \hat{k} = k + \xi r, \quad r \rightarrow \hat{r} = r - \xi r. \)

Can choose \( \bar{k} = r \) for some null external momentum \( r \), so it stays on shell: \( \hat{r}^2 = r^2 = 0. \)

Propagator of interest: \( k^2 - m^2 \rightarrow \xi (2k \cdot r) \)

Cut diverges as \( 1/\xi \).
Reduction of the shifted divergent diagram

\[ A_L = \frac{1}{\hat{k}^2 - m^2} \left( \bar{u}_{\hat{k} - \ell} \frac{\hat{t}^*}{\ell} (m + \hat{k}) \hat{J} \right), \]

\[ A_3 = \bar{u}_{\hat{k}_\ell} \frac{\hat{t}^*}{\ell} u_{\hat{k} - \ell} \]

For internal helicity sum, use Feynman gauge as in EGKM:

\[ \sum_{\lambda=\pm} \varepsilon_{\mu}^\lambda \left( \varepsilon_{\nu}^\lambda \right)^* = -g_{\mu\nu} \]
Reduction of the shifted divergent diagram

Simple reduction of linear bubble gives

$$\int A_3 A_L \rightarrow \frac{1}{\xi(2k \cdot \bar{k})}(4m^2 \bar{u}_k \hat{J} + 2\xi m \bar{u}_k \bar{k} \hat{J}) B_0(\hat{k}^2).$$

Also expand off-shell current and scalar bubble to 1st order:

$$\hat{J} = J + \xi J',

B_0(\hat{k}^2) = B_0(m^2) + \xi(2k \cdot \bar{k})B'_0(m^2)$$

Result:

$$\frac{1}{\xi} \frac{4m^2 \bar{u}_k J B_0}{(2k \cdot \bar{k})} + \frac{4m^2(2k \cdot \bar{k}) \bar{u}_k J B'_0 + 4m^2 \bar{u}_k J' B_0 + 2m \bar{u}_k \bar{k} J B_0}{(2k \cdot \bar{k})}.$$
Reduction: tadpole part

Similar, except gluon polarization sum → propagator, $-ig_{\mu\nu}/\ell^2$.

Result:

$$\left[ \left( \frac{2}{\xi(2k \cdot k)} - \frac{1}{m^2} \right) \bar{u}_k \mathcal{J} + \frac{1}{(2k \cdot k)m} \bar{u}_k \ell \mathcal{J} + \frac{2}{(2k \cdot k)} \bar{u}_k \mathcal{J}' \right] A_0.$$
Counterterm

\[ -\frac{1}{\xi (2k \cdot \bar{k})} \bar{u}_k \left( k \delta Z_{\psi} + \xi \bar{k} \delta Z_{\psi} - m \delta Z_{\psi} - m \delta Z_m \right) (k + \xi \bar{k} + m) \hat{J} \]

Renormalization constants in on-shell scheme:

\[ \delta Z_m = \frac{A_0}{m^2} + 2B_0, \]
\[ \delta Z_{\psi} = \frac{A_0}{m^2} - 4m^2 B'_0. \]

Verify total cancellation of divergent diagram.
Small examples with Feynman Diagrams

- $H \rightarrow b\bar{b}$
  3 loop diagrams + 2 counterterm diagrams

- $q\bar{q} \rightarrow t\bar{t}$
  12 loop diagrams + 2 counterterm diagrams

1. Implemented momentum shift

2. Computed bubble and tadpole coefficients from unitarity cut

3. Checked cancellation of divergences against counterterm and agreement of finite result with Passarino-Veltman reduction.
The fermion-channel cut in the spinor-helicity formalism

The spinor-helicity convention for the polarization vectors requires axial gauge:

\[
\varepsilon^-(p) = -\sqrt{2} \frac{|p\rangle [q| + |q\rangle \langle p|}{[qp]} , \\
\varepsilon^+(p) = -\sqrt{2} \frac{|p| \langle q| + |q\rangle [p]}{\langle qp\rangle} .
\]

The completeness relation is

\[
\sum_{\lambda=\pm} \varepsilon^\lambda_\mu(p) \left( \varepsilon^\lambda_\nu(p) \right)^* = -g_{\mu\nu} + \frac{p_\mu q_\nu + q_\mu p_\nu}{p \cdot q}.
\]

Specific gauge choice = choice of \( q \) for each \( p \).
Additional counterterm in axial gauge, for spinors

The double cut gets an extra $O(\xi^0)$ contribution:

\[
\frac{1}{\xi(2k \cdot \bar{k})} \int d\mu_{2,k} \left[ \frac{\left( \bar{u}_k \not{l} u_{\hat{k}-\ell} \right) \left( \bar{u}_{\hat{k}-\ell} \not{q} \left( m + \hat{k} \right) \hat{J} \right)}{q \cdot \ell} + \frac{\left( \bar{u}_k \not{q} u_{\hat{k}-\ell} \right) \left( \bar{u}_{\hat{k}-\ell} \not{\ell} \left( m + \hat{k} \right) \hat{J} \right)}{q \cdot \ell} \right].
\]

Second term vanishes by Ward identity with cut gluon. First term is cancelled by a new (non-divergent) counterterm:

\[
\mathcal{M}^k = -\frac{1}{2k \cdot \bar{k}} \bar{u}_k \left[ (\bar{k} - m) \hat{k} \not{q} \delta Z'_k \right] (\hat{k} + m)\hat{J},
\]

\[
\delta Z'_k = \frac{B_0}{q \cdot k}
\]
Example from $t\bar{t} \rightarrow gg$ amplitude

Full analytic result computed previously by other methods. [Körner, Merabashvili; Badger, Sattler, Yundin]

- Color decomposition
- 3-point tree $\times$ 5-point tree for $m^2$ on-shell bubbles
- Checked cancellation of divergence and evaluated on-shell bubble coefficient, for equal-helicity gluons
Finite parts of the counterterm need off-shell $\hat{J}$ explicitly, not the gauge-invariant $A_L A_R$.

The current $\hat{J}$ depends on gauge choices of external gluons—these cancel among counterterms.

Generate $\hat{J}$ by BCFW-type relations, starting purely from 3-point vertices with the polarizations

$$\varepsilon^{-}(p) = -\sqrt{2} \frac{|p\rangle [q| + |q\rangle \langle p|}{[qp]} , \quad \varepsilon^{+}(p) = -\sqrt{2} \frac{|p\rangle \langle q| + |q\rangle [p|}{\langle qp\rangle} .$$

BCFW shifts available for any pair of massless quarks/gluons.
Example: Take $q_3 = q_4 = q$.

\[
i\mathcal{J} = -i \frac{|q\rangle \langle 3| + |3\rangle [q]}{[q^3]} \frac{p_2 - \hat{p}_3 + m}{(p_2 - p_3)^2 - m^2} \frac{|q\rangle \langle \hat{4}| + |\hat{4}\rangle [q]}{[q^4]} |2\rangle
\]

\[
= \frac{i}{[q^3][q^4]} \left\{ \frac{1}{\langle 3|2|3\rangle} \left( |4\rangle [q|2|3\rangle [q] - |q\rangle \langle 4|1|q\rangle \langle 3| \right) \\
+ m|q\rangle \langle 43| [q] \right) - \frac{1}{[34]} \left( [q^3] \left( |q\rangle \langle 3| + |3\rangle [q] \right) \\
+ [q^4] \left( |q\rangle \langle 4| + |4\rangle [q] \right) \right\} |2\rangle
\]
For a nice recursion, we need:

- Residue at infinity = 0
- Poles from propagators only

Residue at infinity can be made zero, no worse than on-shell case. Argument from groups of Feynman diagrams.

Reference spinors generically introduce “unphysical poles” which can be avoided for some gauge choices.

\[ \varepsilon^- (\hat{p}) = -\sqrt{2} \frac{|p\rangle [q| + |q\rangle \langle p|}{[qp] - z[qp']} \]

Result: recursion established for certain helicities, with preferred gauge choice.
More loops?

Conceptual challenges at two loops and beyond:

- Master integrals more numerous, not canonical, and not all known analytically
- Nonplanar topologies
- Need $D$-dimensional ingredients for cuts
Generalized cuts

- One-loop box coefficients  "quadruple cuts"
  [RB, Cachazo, Feng]

- Typically require complex momenta.

- One-loop: sequence of quadruple, triple, double, single cuts. "OPP method" underlies all state-of-the-art numerical codes. Samples complex momenta.  [Ossola, Papadopoulos, Pittau; Mastrolia; Forde; Kilgore; Ellis, Giele, Kunszt; Giele, Kunszt, Melnikov; RB, Mirabella]
Generalized cuts beyond one loop

- Extension of OPP method at 2 loops and beyond. Algebraic-geometric analysis of integrands and their relations. [Mastrolia, Mirabella, Ossola, Peraro; Badger, Frellesvig, Zhang]

- “Maximal cuts.” Multi-dimensional complex residues = leading singularities. [Buchbinder, Cachazo; Bern, Carrasco, Johansson, Kosower; Larsen, Kosower; Caron-Huot, Larsen; Johansson, Kosower, Larsen]

- Can we make use of non-maximal cuts? Work without master integrals?
Double dispersion relations

Previously computed at one loop, with strictly real momenta. [Mandelstam; Ball, Braun, Dosch]. From iterated cuts.

Spectral function: 3-cut of box with real momenta = 4-cut with complex momenta = volume of tetrahedron

Check with more dimensions or more loops.
Two or more loops [with Abreu, Duhr, Gardi]

- Use symbol of multiple polylogarithms. [Goncharov, Spradlin, Vergu, Volovich]
- Encodes discontinuities in various channels. [Gaiotto, Maldacena, Sever, Viera]
- Deeper entries in the symbol appear in terms of more natural variables
- Can match discontinuities to iterated cuts
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\[
y \otimes w \otimes (w \otimes \bar{w} + \bar{w} \otimes w - \bar{w} \otimes \bar{w}) + x \otimes (1 - w) \otimes (w \otimes w - w \otimes \bar{w} - \bar{w} \otimes w)
+ y \otimes \bar{w} \otimes (w \otimes w - w \otimes \bar{w} - \bar{w} \otimes w) + x \otimes (1 - \bar{w}) \otimes (w \otimes \bar{w} + \bar{w} \otimes w - \bar{w} \otimes \bar{w})
+ x \otimes x \otimes ((1 - \bar{w}) \otimes \bar{w} - (1 - w) \otimes w))
\]

\[
y = \frac{p_3^2}{p_1^2}, \quad x = \frac{p_2^2}{p_1^2}, \quad w = \frac{(p_1^2 + p_2^2 - p_3^2 + \sqrt{\lambda(p_1^2, p_2^2, p_3^2)})}{p_1^2}
\]
Two or more loops

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The correspondence seems inexact: do cuts still have more information?

One of many remaining challenges: control of complexified momentum.
Summary and Outlook

- Discontinuities of Feynman integrals indicated by Landau and Cutkosky

- Not always easy to compute, but powerful:
  - Dispersion relations
  - Unitarity method with master integrals
  - Generalized cuts with master integrals

- Hard parts at one loop: rational parts, massive contributions. Under control, though some parts need improvement analytically. Clean method for divergent bubbles from off-shell momentum continuation.

- Beyond one loop: master integrals largely unknown. Seeking systematic approach via on-shell methods (generalized cuts). Using mathematics of multiple polylogarithms where applicable.