Some insights into the magnetic “QCD” phase diagram from the Sakai-Sugimoto model

DAVID DUDAL

Department of Physics and Astronomy
Ghent University, Belgium

Twelfth Workshop on Non-Perturbative Quantum Chromodynamics, l’Institut d’Astrophysique de Paris, June 10-13, 2013

In collaboration with:
Nele Callebaut (Ghent/Brussels)
Overview

1. Motivation

2. (Magnetic) holographic setup

3. The $\rho$ meson mass in a magnetic field

4. Chiral transition in a magnetic field
Overview

1. Motivation

2. (Magnetic) holographic setup

3. The $\rho$ meson mass in a magnetic field

4. Chiral transition in a magnetic field
Why study strong magnetic fields?

- experimental relevance: appearance in QGP after a heavy ion collision (order $eB \sim 1 - 15m^2_\pi$) (work of Skokov, Tuchin, Kharzeev, McLerran, Deng, Huang)

  - lifetime$_{QGP} \sim 1 - 10$ fm $\rightarrow$ Incentive to take $eB$ constant (ignoring “spatial decay” as well!)
  - from a holographic viewpoint: interesting for comparison with recent lattice efforts
Why study strong magnetic fields?

Figure: McLerran, Skokov, arXiv:1305.0774

Figure: Tuchin, arXiv:1305.5806
Studied effects

• split between $T_c(eB)$ and $T_\chi(eB)$?

• $\rho$ meson condensation?

![Expected phase diagram](image)
ρ meson condensation

Studied effect: ρ meson condensation in vacuum ($T = 0$) (see: Chernodub, Van Doorsslaere, Verschelde; PRD85 (2012) 045002; Chernodub, PRL106 (2011) 142003, first suggestion made in Schramm, Muller, Schramm, MPLA7 (1992) 973, inspiration from Ambjorn, Olesen on $W$-condensation (80ies))

QCD vacuum unstable towards forming a superconducting state of condensed charged $\rho$ mesons at critical magnetic field $eB_c$

Small note: academic exercise (“hubris”) since by the time $\rho$ enters, $B$ might have already (long) left...
ρ meson condensation: Landau levels

The energy levels $\varepsilon$ of a free relativistic spin-$s$ particle moving in a background of the external magnetic field $\vec{B} = B\hat{e}_z$ are the Landau levels

\[ \varepsilon_{n,s_z}(p_z) = p_z^2 + m^2 + (2n - 2s_z + 1)|eB|. \]

Appropriate polarization combinations (spin $s_z = 1$ parallel to $\vec{B}$) can condense, since in the lowest energy state ($n = 0$, $p_z = 0$):

\[ M_\rho^2(eB) = m_\rho^2 - eB, \]

→ tachyonic if the magnetic field is strong enough. Important: gyromagnetic ratio = 2!
ρ meson condensation: Landau levels

\[ M_\rho^2(eB) = m_\rho^2 - eB, \]

\[ \Rightarrow \] The fields ρ and ρ† should condense at the critical magnetic field

\[ eB_c = m_\rho^2. \]
ρ meson condensation

- phenomenological models: \( eB_c = m^2_\rho = 0.6 \text{ GeV}^2 \) (effective DSGS \( \rho \)-model, Chernodub, PRD82 (2010) 085011), \( eB_c \approx 1 \text{ GeV}^2 \) (NJL) Chernodub, PRL106 (2011) 142003
- lattice simulation: \( eB_c \approx 0.9 \text{ GeV}^2 \) Braguta et al, PLB718 (2012) 667
- \( \rightarrow \) holographic approach:
  - can the \( \rho \) meson condensation be modeled?
  - can this approach deliver new insights? e.g. taking into account strong magnetic effects on constituents (\( \rightarrow \rho \)-substructure), effect on \( eB_c \)

Callebaut, Dudal, Verschelde, JHEP 1303 (2013) 033; work in progress
Overview

1. Motivation

2. (Magnetic) holographic setup

3. The ρ meson mass in a magnetic field

4. Chiral transition in a magnetic field
Holographic QCD

- **What is holographic QCD?**
  "QCD" $^{\text{dual}} \equiv$ (super)gravitation in a higher-dimensional background:
  $4D$ "QCD" "lives" on the boundary of a $5D$ space where the
  (super)gravitation theory is defined

- **Origin of the QCD/gravitation duality idea?**
  AdS/CFT-correspondence (Maldacena 1997):
  supergravitation in AdS$_5$ space $^{\text{dual}} \equiv$ conformal $\mathcal{N}=4$ SYM theory
The D4-brane background

\[
\begin{align*}
    ds^2 &= \left( \frac{u}{R} \right)^{3/2} (\eta_{\mu\nu} dx^\mu dx^\nu + f(u) d\tau^2) + \left( \frac{R}{u} \right)^{3/2} \left( \frac{du^2}{f(u)} + u^2 d\Omega_4^2 \right), \\
    e^\phi &= g_s \left( \frac{u}{R} \right)^{3/4}, \quad F_4 = \frac{N_c}{V_4} \epsilon_4, \quad f(u) = 1 - \frac{u_K^3}{u^3},
\end{align*}
\]
The Sakai-Sugimoto model

\[ ds^2 = \left( \frac{u}{R} \right)^{3/2} \left( \eta_{\mu\nu} dx^\mu dx^\nu + f(u) d\tau^2 \right) + \left( \frac{R}{u} \right)^{3/2} \left( \frac{du^2}{f(u)} + u^2 d\Omega_4^2 \right), \]

\[ e^\phi = g_s \left( \frac{u}{R} \right)^{3/4}, \quad F_4 = \frac{N_c}{V_4} \epsilon_4, \quad f(u) = 1 - \frac{u_K^3}{u^3}, \]
### The Sakai-Sugimoto model

- **Flavour** → $N_f$ pairs of D8-$\overline{D8}$ flavour branes are added to the D4-brane background.  
  

- **Probe approximation** $N_f \ll N_c$: backreaction of flavour branes on background is ignored \sim quenched “QCD”.

- **Stack of** $N_f$ coinciding pairs of D8-$\overline{D8}$ flavour branes → $U(N_f)_L \times U(N_f)_R$ theory, to be interpreted as the chiral symmetry in QCD

- **background geometry** (∪-shape) enforces “$L = R$” (joining of flavour branes): $U(N_f)_L \times U(N_f)_R \rightarrow U(N_f)_V$.

Sakai, Sugimoto, PTP113 (2005) 843; PTP114 (2005) 1083
The flavour gauge field

The $U(N_f)$ gauge field $A_\mu(x^\mu, u)$ that lives on the flavour branes describes a tower of vector mesons $v_{\mu,n}(x^\mu)$ in the dual QCD-like theory:

$$A_\mu(x^\mu, u) = \sum_{n \geq 1} v_{\mu,n}(x^\mu)\psi_n(u)$$

with $v_{\mu,n}(x^\mu)$ a tower of vector mesons with masses $m_n$, and $\{\psi_n(u)\}_{n\geq 1}$ a complete set of functions of $u$, satisfying the eigenvalue equation

$$u^{1/2}gamma_B^{-1/2}(u)partial_u [u^{5/2}gamma_B^{-1/2}(u)partial_u \psi_n(u)] = -R^3 m_n^2 \psi_n(u),$$
Why use the Sakai-Sugimoto model

• the way it works:

dynamics of the flavour D8/D8-branes: 5D YM theory $S_{DBI}[A_{\mu}] = \cdots$, 
$A_{\mu}(x^\mu, u) = \sum_{n \geq 1} v_{\mu,n}(x^\mu) \psi_n(u)$

\[\downarrow\] integrate out the extra radial dimension $u$

effective 4D meson theory for $v_{\mu}^n(x^\mu)$

• ideal holographic QCD model to study low-energy QCD
  • confinement and chiral symmetry breaking
  • effective low-energy QCD models drop out: Skyrme ($\pi$, also: baryons as skyrmions), HLS ($\pi, \rho$ coupling), VMD

Magnetized (holographic) “QCD”
Approximations of the model

Duality is valid in the limit $N_c \to \infty$ and large 't Hooft coupling \( \lambda = g_{YM}^2 N_c \gg 1 \), and at low energies (where redundant massive d.o.f. decouple).

Approximations (inherent to the model):

- quenched approximation ($N_f \ll N_c$)
- chiral limit ($m_\pi = 0$, bare quark masses zero)

Choices of parameters:

- $N_c = 3$
- $N_f = 2$ to model charged mesons
How to turn on the magnetic field

A non-zero value of the flavour gauge field $A_m(x^\mu, z)$ on the boundary,

$$A_m(x^\mu, u \to \infty) = \bar{A}_\mu,$$

corresponds to an external gauge field in the boundary field theory that couples to the quarks

$$\bar{\psi} i \gamma_\mu D_\mu \psi \quad \text{with} \quad D_\mu = \partial_\mu + \bar{A}_\mu.$$

To apply an external electromagnetic field $A_{\mu}^{em}$, put

$$A_\mu(u \to +\infty) = -iQ_{em}A_{\mu}^{em} = \bar{A}_\mu.$$
How to turn on the magnetic field

To apply a magnetic field along the $x_3$-axis,

$$A_{2}^{em} = x_1 B,$$

in the $N_f = 2$ case,

$$Q_{em} = \begin{pmatrix} 2/3 & 0 \\ 0 & -1/3 \end{pmatrix} = \frac{1}{6} \mathbf{1}_2 + \frac{1}{2} \sigma_3,$$

we set

$$\bar{A}_\mu = -ieQ_{em}A_{\mu}^{em} = -ieQ_{em}x_1 B\delta_{\mu 2}.$$
Overview

1 Motivation

2 (Magnetic) holographic setup

3 The $\rho$ meson mass in a magnetic field

4 Chiral transition in a magnetic field
Plan

DBI action:

\[ S_{DBI} = -T_8 \int d^4 x \ 2 \int_{u_0}^\infty du \int \epsilon_4 \ e^{-\phi} \ \text{STr} \sqrt{-\det [g_{mn} + (2\pi\alpha')iF_{mn}]} , \]

with

- \text{STr}(F_1 \cdots F_n) = \frac{1}{n!} \text{Tr}(F_1 \cdots F_n + \text{all permutations}) the symmetrized trace,
- \( g^{D8}_{mn} = g_{mn} + g_{\tau\tau} (D_m \tau)^2 \) the induced metric on the D8-branes (with covariant derivative \( D_m \tau = \partial_m \tau + [A_m, \tau] \)),
- \( \tau = \) the brane embedding (+ fluctuations)
Simplest embedding to start: $u_0 = u_K$

1. Embedding trivial: $\partial_u \tau = 0$ for all values of the magnetic field

2. Determine EOM for $\rho_\mu$:
   - STr reduces to regular Tr (because of coincident branes)
   - $\tilde{A}_m$ and $\tilde{\tau}$ automatically decouple
   - $\tilde{A}_\mu = \rho_\mu(x)\psi(u)$ (retain only lowest meson of the tower, most likely to condense)

\[\mathcal{L}_{5D} = \int d^4x \int du \left\{ -\frac{1}{4} f_1 (F_{\mu\nu}^a)^2 - \frac{1}{2} f_2 (F_{\mu\nu}^a)^2 - \frac{1}{2} f_3 \sum_{\mu, \nu = 1}^2 \bar{F}_{\mu\nu}^3 \varepsilon_{3ab} \tilde{A}_\mu^a \tilde{A}_\nu^b \right\}\]
Motivation

(Magnetic) holographic setup

The \( \rho \) meson mass in a magnetic field

Chiral transition in a magnetic field

First approximation: Landau levels

The \( \rho \) meson mass in a magnetic field

Chiral transition in a magnetic field

EOM for \( \rho \) for \( u_0 = u_K \)

\[
\mathcal{L}_{5D} = \int d^4x \int du \left\{ -\frac{1}{4} f_1 (F_{\mu\nu}^a)^2 - \frac{1}{2} f_2 (F_{\mu\mu}^a)^2 - \frac{1}{2} f_3 \sum_{\mu,\nu=1}^2 \bar{F}_{\mu\nu}^3 \varepsilon_{3ab} \hat{A}_\mu^a \hat{A}_\nu^b \left( \rho^a_{\mu} \rho^b_{\nu} \right)^2 \right\}
\]

demand \( \int du f_1 \psi^2 = 1 \) and \( \int du f_2 (\partial_u \psi)^2 = m_\rho^2 \), then \( \int du f_3 \psi^2 = k \)

\[
\Rightarrow \mathcal{L}_{4D} = \int d^4x \left\{ -\frac{1}{4} (\mathcal{F}_{\mu\nu}^a)^2 - \frac{1}{2} m_\rho^2 (\rho^a_{\mu})^2 - \frac{1}{2} k \sum_{\mu,\nu=1}^2 \bar{F}_{\mu\nu}^3 \varepsilon_{3ab} \rho^a_{\mu} \rho^b_{\nu} \right\}
\]

(with \( \mathcal{F}_{\mu\nu}^a = D_\mu \rho^a_{\nu} - D_\nu \rho^a_{\mu} \))

standard 4D Lagrangian for a vector field in an external EM field
Landau levels for Sakai-Sugimoto $u_0 = u_K$

Standard 4D Lagrangian for a vector field in an external EM field with $k = 1$ ($f_3 = f_1$)

$\sim$ Landau levels and

$$M_\rho^2(eB) = m_\rho^2 - eB$$
General embedding $u_0 > u_K$

$u_0 > u_K$ to model non-zero constituent quark mass which is related to the distance between $u_0$ and $u_K$.

Numerical fixing of holographic parameters

There are three unknown free parameters \((u_K, u_0 \text{ and } \kappa(\sim \lambda N_c))\). In order to get results in physical units, we fix the free parameters by matching to

- the constituent quark mass \(m_q = 0.310 \text{ GeV}\),
- the pion decay constant \(f_\pi = 0.093 \text{ GeV}\) and
- the \(\rho\) meson mass in absence of magnetic field \(m_\rho = 0.776 \text{ GeV}\).

Results:

\[ u_K = 1.39 \text{ GeV}^{-1}, \quad u_0 = 1.92 \text{ GeV}^{-1} \text{ and } \kappa = 0.00678 \]

- cross-check: “QCD” string tension \(\sigma \approx 0.18 \text{ GeV}^2 (= \text{standard lattice estimate})\)
$eB$-dependent embedding for $u_0 > u_K$

Keep $L$ fixed: $u_0(eB)$ rises with $eB$. This models magnetic catalysis of chiral symmetry breaking Johnson, Kundu JHEP0812 (2008) 053; in general: Miransky et al.

Non-Abelian: $u_{0,u}(eB) > u_{0,d}(eB)! \ U(2) \to U(1)_u \times U(1)_d$
eB-dependent embedding for $u_0 > u_K$

Change in embedding models:

- chiral magnetic catalysis $\Rightarrow m_u(eB)$ and $m_d(eB)$ \(\uparrow\)
- $eB$ explicitly breaks global $U(2) \rightarrow U(1)_u \times U(1)_d$

Effect on $\rho$ mass?

- expect $m_\rho(eB) \uparrow$ as constituents get heavier
- split between branes generates other mass mechanism: 5D gauge field gains mass through holographic Higgs mechanism
**eB-induced Higgs mechanism**

The string associated with a charged ρ meson (\(\bar{u}d\), \(\bar{d}u\)) stretches between the now separated up- and down brane \(\Rightarrow\) because a string has tension it contributes to the mass.
EOM for $\rho$ for $u_0 > u_K$?

Non-trivial embedding

$$\tau(u) = \begin{pmatrix} \tau_u(u)\theta(u - u_0,u) & 0 \\ 0 & \tau_d(u)\theta(u - u_0,d) \end{pmatrix} \not\sim 1,$$

describing the splitting of the branes, utterly complicates the analysis (but necessary for a realistic modeling!).

$$L_{5D} = \text{STr} \left\{ \ldots (\tilde{A}_m, \tau) + D_m \tilde{\tau} \right\}^2 + \ldots (F_{\mu\nu})^2 + \ldots (F_{\mu u})^2 + \ldots \tilde{F}_{\mu\nu} [\tilde{A}_\mu, \tilde{A}_\nu] + \ldots (\partial_u \tau) \tilde{F} ( [\tilde{A}, \tau] + D\tilde{\tau} ) \tilde{F} \right\}$$

with all the .. different functions $\mathcal{H}(\partial_u \tau, \tilde{F}; u)$ of the background fields $\partial_u \tau, \tilde{F}$. 
STr-prescription


- \( \text{STr} = \text{symmetric average over all orderings of } F_{ab}, D_a \phi^i, [\phi^i, \phi^j] \) and the individual non-Abelian scalars \( \phi^k \) appearing in the non-Abelian Taylor expansions of the background fields.

\[
\text{STr} \left( \mathcal{H}(\mathcal{F}) \tilde{F}^2 \right) = - \frac{1}{2} \sum_{a=1}^{2} \tilde{F}_a^2 \quad I(\mathcal{H}) \quad + \sum_{a=0,3} \cdots
\]

with

\[
I(\mathcal{H}) = \frac{\int_0^1 d\alpha \mathcal{H}(F_0 + \alpha F_3) + \int_0^1 d\alpha \mathcal{H}(F_0 - \alpha F_3)}{2}
\]

- The integral functions \( I(\mathcal{H}) \) are complicated functions of \( eB \) and \( u \), even discontinuous in \( u \) (at \( u = u_{0,u} \)).
STr-prescription

After many pages of computations (analytical + numerical)
EOM for $\rho$ for $u_0 > u_K$

$$\mathcal{L}_{5D} = \int d^4x \int du \left\{ -\frac{1}{4} f_1(eB) (F_{\mu\nu}^a)^2 - \frac{1}{2} f_2(eB) (F_{\mu\nu}^a)^2 (J_{\mu\nu}^a)^2 \psi^2 - \frac{1}{2} m_\rho^2 (\rho_a^a)^2 (\partial_u \psi)^2 \right\}$$

$$- \frac{1}{2} f_3(eB) \sum_{\mu, \nu=1}^2 \mathcal{F}_{\mu\nu}^3 \varepsilon_{3ab} \bar{A}_\mu^a \bar{A}_\nu^b \frac{1}{2} f_4(eB) (\bar{A}_\mu^a)^2 (\bar{\tau}^3)^2 \psi^2$$

Demand $\int du f_1(eB) \psi^2 = 1$ and $\int du f_2(eB)(\partial_u \psi)^2 + f_4(eB)(\bar{\tau}^3)^2 \psi^2 = m_\rho^2(eB)$, then

$\int du f_3(eB) \psi^2 = k(eB) \neq 1(\Leftarrow f_3(eB) \neq f_1(eB))$

$\Rightarrow \mathcal{L}_{4D} = \int d^4x \left\{ -\frac{1}{4} (J_{\mu\nu}^a)^2 - \frac{1}{2} m_\rho^2(eB)(\rho_a^a)^2 - \frac{1}{2} k(eB) \sum_{\mu, \nu=1}^2 \mathcal{F}_{\mu\nu}^3 \varepsilon_{3ab} \rho_a^a \rho_b^b \right\}$

Modified 4D Lagrangian for a vector field in an external EM field, with $eB$-dependent gyromagnetic coupling!
Solve the eigenvalue problem

The normalization condition and mass condition on the $\psi$ combine to the eigenvalue equation

$$f_1^{-1} \partial_u (f_2 \partial_u \psi) - f_1^{-1} f_4 (\tau_3)^2 \psi = -m_\rho^2 \psi$$

with b.c. $\psi(x = \pm \pi/2) = 0, \psi'(x = 0) = 0$

which we solve with a numerical shooting method to obtain $m_\rho^2 (eB)$. 

![Graph](image)
Landau vs Sakai-Sugimoto $u_0 > u_K$

Modified 4D Lagrangian for a vector field in an external EM field with $k(eB) \neq 1 (\iff f_3(eB) \neq f_1(eB))$

$\sim$ modified Landau levels and, with $\xi = \frac{eB}{m_\rho^2(eB)}$, Tsai, Yildiz PRD4 (1971) 3643; Obukhov et al, Theor.Math.Phys. 55 (1983) 536

$$M_\rho^2(eB) = m_\rho^2(eB) - eB + (1 - k(eB))m_\rho^2(eB) \left( \frac{\xi^2}{2} + \xi \sqrt{1 - \xi + \frac{\xi^2}{4}} \right)$$
Summary $\rho$ meson

Studied effect: possibility for $\rho$ meson condensation

- phenomenological models: $eB_c = m_\rho^2 = 0.6 \text{ GeV}^2$
- lattice simulation: slightly higher value of $eB_c \approx 0.9 \text{ GeV}^2$
- $\Rightarrow$ holographic approach:
  - can the $\rho$ meson condensation be modeled? yes
  - can this approach deliver new insights? e.g. taking into account constituents, effect on $eB_c$

Up and down quark constituents of the $\rho$ meson can be modeled as separate branes, each responding to the magnetic field by changing their embedding. This is a modeling of the chiral magnetic catalysis effect. We take this into account and find also a string effect on the mass, leading to a $eB_c \approx 0.78 \text{ GeV}^2$
Overview

1. Motivation
2. (Magnetic) holographic setup
3. The $\rho$ meson mass in a magnetic field
4. Chiral transition in a magnetic field
**Chiral temperature**

\[ T_\chi = \text{temperature at which chiral symmetry is restored} \quad (\text{chiral limit is understood}) \]

Studied effect: possible split between \( T_c(eB) \) and \( T_\chi(eB) \)
Expected behaviour (Fig from ’08):

- \( T_\chi(eB) \uparrow \): “chiral magnetic catalysis” seen in chirally driven models (e.g. NJL)  
  Miransky, Shovkovy, PRD66 (2002) 045006

- \( T_c(eB) \downarrow \): quark gas thermodynamically favoured over pion gas (e.g. MIT bag)  
Some results in different models

PLSM$_q$ model Mizher, Chernodub, Fraga, PRD82 (2010) 105016

Lattice D’Elia, Mukherjee, Sanfilippo, PRD82 (2010) 051501

Different PNJL models Gatto, Ruggieri, PRD82 (2010) 054027; PRD83 (2011) 034016
Sakai-Sugimoto at finite temperature

“Black D4-brane background”

\[
\begin{align*}
\frac{ds^2}{R} &= \left(\frac{u}{R}\right)^{3/2} (\hat{f}(u) dt^2 + \delta_{ij} dx^i dx^j + d\tau^2) + \left(\frac{R}{u}\right)^{3/2} \left(\frac{du^2}{\hat{f}(u)} + u^2 d\Omega_4^2\right) \\
\hat{f}(u) &= 1 - \frac{u_T^3}{u^3}, \quad u_T \sim T^2
\end{align*}
\]
Numerical fixing of holographic parameters

Input parameters at $eB = 0$ $f_\pi = 0.093$ GeV and $m_\rho = 0.776$ GeV fix all holographic parameters except “brane separation” $L$.

Choice of $L$ a priori free, determines a kind of choice of holographic theory:

- $L$ small $\sim$ NJL-type boundary field theory
  
  Antonyan, Harvey, Jensen, Kutasov, hep-th/0604017

- $L = \delta \tau / 2$ maximal $\sim$ maximal probing of the gluon background
  
  (original antipodal Sakai-Sugimoto)
Sakai-Sugimoto at finite $T$ and $eB$

- no backreaction $\Rightarrow T_c$ independent of $eB$
- $eB$-dependent embedding of flavour branes $\Rightarrow T_\chi(eB)$:

\[
S_{\text{merged}} - S_{\text{separated}} = 0 \quad \Rightarrow \quad T_\chi
\]
Conclusion on $T_\chi(eB)$

The appearance of a split between $T_\chi$ (GeV) (blue) and $T_c$ (GeV) (purple) depends on the choice of $L$!

Plots for fixed $L$ (from small to large) respectively corresponding to $m_q(eB = 0) = 0.357, 0.310$ and 0.272 GeV and $T_c = 0.103, 0.115$ and 0.123 GeV Callebaut, Dudal, PRD87 (2013)106002

- Left: split for $L$ small enough $\sim$ NJL results
- Middle and right: split only at large $eB$ or no split at all for parameter values that match best to QCD $\sim SU(2), N_f = 2$ (chirally extrapolated) lattice data of Ilgenfritz et al, PRD85 (2012) 114504 (no split)
(Locally) Inverse magnetic catalysis

BIG BUT

Latest lattice data disagree with most previous results: $T_\chi(eB) \downarrow$

→ quenched vs. true QCD


Important task for future holographs: construct a “magnetized bulk geometry” that is sufficiently QCD-like
Fin

Merci!