QCD, Wilson loop and the interquark potential

Marco Frasca

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Plan of the talk

- *Classical field theory*
  - Scalar field theory
  - Yang-Mills theory
  - Yang-Mills Green function
- *Quantum field theory*
  - Scalar field theory
  - Yang-Mills theory
  - QCD in the infrared limit
  - Wilson loop
  - Interquark potential
- *Conclusions*

Scalar field

- A classical field theory for a massless scalar field is given by

$$\Box\phi+\lambda\phi^3=j$$

- The homogeneous equation can be solved by

Exact solution

$$\phi=\mu\left(\frac{2}{\lambda}\right)^{\frac{1}{4}}\mathrm{sn}(p\cdot x+\theta,i) \qquad p^2=\mu^2\sqrt{\frac{\lambda}{2}}$$

being sn an elliptic Jacobi function and μ and θ two constants. This solution holds provided the given **dispersion relation** holds and represents a free massive solution notwithstanding we started from a massless theory.

- Mass arises from the nonlinearities when λ is taken to be finite.

Scalar field

- When there is a current we ask for a solution in the limit $\lambda\rightarrow\infty$ as our aim is to understand a strong coupling limit. So, we check a solution

$$\phi=\kappa\int d^4x' G(x-x')j(x')+\delta\phi$$

being $\delta\phi$ all higher order corrections.

- One can prove that this is indeed so provided

Next-to-leading Order (NLO)

$$\delta\phi=\kappa^2\lambda\int d^4x' d^4x'' G(x-x')[G(x'-x'')]^3 j(x')+O(j(x)^3)$$

with the identification $\kappa=\mu$, the same of the exact solution, and $\Box G(x-x')+\lambda[G(x-x')]^3=\mu^{-1}\delta^4(x-x')$.

- The correction $\delta\phi$ is known in literature and yields a **sunrise diagram** in momenta. **This needs a regularization**.
- Our aim is **to compute the propagator** $G(x-x')$ to NLO.

Scalar field

- In order to solve the equation

$$\Box G(x-x')+\lambda[G(x-x')]^3=\mu^{-1}\delta^4(x-x')$$

we can start from the $d=1+0$ case $\partial_t^2 G_0(t-t')+\lambda[G_0(t-t')]^3=\mu^2\delta(t-t')$.

- It is straightforwardly obtained the Fourier transformed solution

$$G_0(\omega)=\sum_{n=0}^{\infty}(2n+1)\frac{\pi^2}{K^2(i)}\frac{(-1)^n e^{-(n+\frac{1}{2})\pi}}{1+e^{-(2n+1)\pi}}\frac{1}{\omega^2-m_n^2+i\epsilon}$$

being $m_n=(2n+1)\frac{\pi}{2K(i)}\left(\frac{\lambda}{2}\right)^{\frac{1}{4}}\mu$ and $K(i)=1.311028777\ldots$ an elliptic integral.

- We are able to recover the fully covariant propagator by boosting from the rest reference frame obtaining finally

$$G(p)=\sum_{n=0}^{\infty}(2n+1)\frac{\pi^2}{K^2(i)}\frac{(-1)^n e^{-(n+\frac{1}{2})\pi}}{1+e^{-(2n+1)\pi}}\frac{1}{p^2-m_n^2+i\epsilon}.$$

Yang-Mills field

- A classical field theory for the Yang-Mills field is given by

$$\partial^\mu\partial_\mu A^a_\nu-\left(1-\frac{1}{\xi}\right)\partial_\nu(\partial^\mu A^a_\mu)+gf^{abc}A^{b\mu}(\partial_\mu A^c_\nu-\partial_\nu A^c_\mu)+gf^{abc}\partial^\mu(A^b_\mu A^c_\nu)+g^2f^{abc}f^{cde}A^{b\mu}A^d_\mu A^e_\nu=-j^a_\nu.$$

- For the homogeneous equations, we want to study it in the formal limit $g\to\infty$. We note that a class of exact solutions exists if we take the potential A^a_μ just depending on time, **after a proper selection of the components** [see Smilga (2001)]. **These solutions are the same of the scalar field** when spatial coordinates are set to zero (rest frame).
- Differently from the scalar field, **we cannot just boost away these solutions** to get a general solution to Yang-Mills equations **due to gauge symmetry**. But we can try to find a set of similar solutions with the proviso of a gauge choice.
- This kind of solutions will permit us to prove that **a set of them exists supporting a trivial infrared fixed point** to build up a quantum field theory.

Yang-Mills field

- Exactly as in the case of the scalar field we assume the following solution to our field equations

$$A_\mu^a = \kappa \int d^4x' D_{\mu\nu}^{ab}(x-x') j^{b\nu}(x') + \delta A_\mu^a$$

- Also in this case, apart from a possible correction, this boils down to an **expansion in powers of the currents** as already guessed in the '80 [R. T. Cahill and C. D. Roberts, Phys. Rev. D 32, 2419 (1985)].
- This implies that the corresponding quantum theory, in a very strong coupling limit, takes a Gaussian form and is trivial.
- The **crucial point**, as already pointed out in the eighties [T. Goldman and R. W. Haymaker, Phys. Rev. D 24, 724 (1981), T. Cahill and C. D. Roberts (1985)], is the **determination of the gluon propagator in the low-energy limit**.

Yang-Mills field

- The question to ask is: Does a set of classical solutions exist for Yang-Mills equations supporting a trivial infrared fixed point for the corresponding quantum theory?
- The answer is yes! These solutions are instantons in the form $A_\mu^a = \eta_\mu^a \phi$ with η_μ^a a set of constants and ϕ a scalar field.
- By direct substitution into Yang-Mills equations one recovers the equation for ϕ that is

$$\partial^\mu \partial_\mu \phi - \frac{1}{N^2-1}\left(1-\frac{1}{\xi}\right)(\eta^a \cdot \partial)^2 \phi + N g^2 \phi^3 = -j_\phi$$

  being $j_\phi = \eta_\mu^a j^{\mu a}$ and use has been made of the formula $\eta^{\nu a} \eta_\nu^a = N^2 - 1$.
- In the Landau gauge (Lorenz gauge classically) this equation is exactly that of the scalar field given before and we get again a current expansion.
- So, **a set of solutions of the Yang-Mills equations exists supporting a trivial infrared fixed point**. Our aim is to study the theory in this case.

Yang-Mills-Green function

- The instanton solutions given above permit us to write down immediately the propagator for the Yang-Mills equations in the Landau gauge for SU(N) being exactly the same given for the scalar field:

Gluon propagator in the infrared limit

$$\Delta_{\mu\nu}^{ab}(p)=\delta_{ab}\left(\eta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\sum_{n=0}^{\infty}\frac{B_n}{p^2-m_n^2+i\epsilon}+O\left(\frac{1}{\sqrt{N}g}\right)$$

being

$$B_n=(2n+1)\frac{\pi^2}{K^2(i)}\frac{(-1)^{n+1}e^{-(n+\frac{1}{2})\pi}}{1+e^{-(2n+1)\pi}}$$

and

$$m_n=(2n+1)\frac{\pi}{2K(i)}\left(\frac{Ng^2}{2}\right)^{\frac{1}{4}}\Lambda$$

- Λ is an integration constant as μ for the scalar field.
- **This is the propagator of a massive field theory**

Scalar field

- We can formulate a quantum field theory for the scalar field starting from the generating functional

$$Z[j]=\int[d\phi]\exp\left[i\int d^4x\left(\frac{1}{2}(\partial\phi)^2-\frac{\lambda}{4}\phi^4+j\phi\right)\right].$$

- We can rescale the space-time variable as $x\rightarrow\sqrt{\lambda}x$ and rewrite the functional as

$$Z[j]=\int[d\phi]\exp\left[i\frac{1}{\lambda}\int d^4x\left(\frac{1}{2}(\partial\phi)^2-\frac{1}{4}\phi^4+\frac{1}{\lambda}j\phi\right)\right].$$

  Then we can seek for a solution series as $\phi=\sum_{n=0}^{\infty}\lambda^{-n}\phi_n$ and rescale the current $j\rightarrow j/\lambda$ being this arbitrary.

- The leading order correction can be computed solving the classical equation

$$\Box\phi_0+\phi_0^3=j$$

  that we already know how to manage. This is completely consistent with our preceding formulation [M. Frasca, Phys. Rev. D 73, 027701 (2006)] but now all is fully covariant.

Scalar field

- Using the approximation holding at strong coupling

$$\phi_0 = \mu \int d^4x G(x - x')j(x') + \ldots$$

it is not difficult to write the generating functional at the leading order in a Gaussian form

$$\mathbf{Z_0[j] = \exp\left[\frac{i}{2}\int d^4x' d^4x'' j(x') G(x' - x'') j(x'')\right].}$$

- This conclusion is really important: It says that the scalar field theory in d=3+1 is trivial in the infrared limit!
- This functional describes a set of free particles with a mass spectrum

$$m_n = (2n+1)\frac{\pi}{2K(i)}\left(\frac{\lambda}{2}\right)^{\frac{1}{4}}\mu$$

that are the poles of the propagator, the same of the classical theory.

Yang-Mills field

- We can now take the form of the propagator given above, e.g. in the Landau gauge, as

$$D^{ab}_{\mu\nu}(p)=\delta_{ab}\left(\eta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\sum_{n=0}^{\infty}\frac{B_n}{p^2-m_n^2+i\epsilon}+O\left(\frac{1}{\sqrt{N}g}\right)$$

to do a formulation of Yang-Mills theory in the infrared limit.

- Then, the next step is to use the approximation that holds in a strong coupling limit

$$A^a_\mu=\Lambda\int d^4x' D^{ab}_{\mu\nu}(x-x')j^{b\nu}(x')+O\left(\frac{1}{\sqrt{N}g}\right)+O(j^3)$$

- and we note that, in this approximation, the ghost field just decouples and becomes free and one finally has at the leading order

$$Z_0[j]=N\exp\left[\frac{i}{2}\int d^4x' d^4x'' j^{a\mu}(x')D^{ab}_{\mu\nu}(x'-x'')j^{b\nu}(x'')\right].$$

Yang-Mills theory has an infrared trivial fixed point in the limit of the coupling going to infinity and we expect the running coupling to go to zero lowering energies. So, **the leading order propagator cannot confine**.

Yang-Mills field

- Now, we can take a look at the ghost part of the action. We just note that, for this particular form of the propagator, inserting our approximation into the action produces an action for a free ghost field.
- Indeed, we will have

$$S_g = -\int d^4x \left[\bar{c}^a \partial_\mu \partial^\mu c^a + O\left(\frac{1}{\sqrt{N}g}\right) + O(j^3)\right]$$

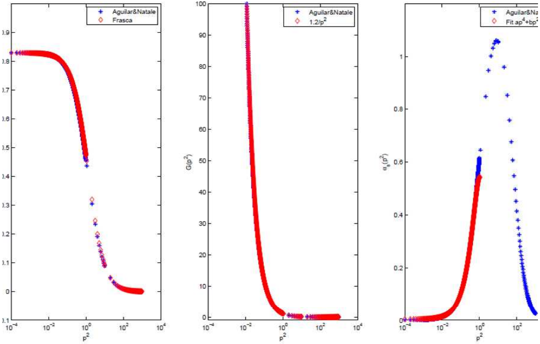
- A ghost propagator can be written down as

$$G_{ab}(p) = -\frac{\delta_{ab}}{p^2 + i\epsilon} + O\left(\frac{1}{\sqrt{N}g}\right).$$

- Our conclusion is that, in a strong coupling expansion $1/\sqrt{N}g$, we get the so called **decoupling solution**.

Yang-Mills field

A direct comparison of our results with numerical Dyson-Schwinger equations gives the following:



that is strikingly good (ref. A. Aguilar, A. Natale, JHEP 0408, 057 (2004)).

QCD at infrared limit

- When use is made of the trivial infrared fixed point, QCD action can be written down quite easily.
- Indeed, we will have for the gluon field

$$S_{gf}=\frac{1}{2}\int d^4x'd^4x''\left[j^{\mu a}(x')D^{ab}_{\mu\nu}(x'-x'')j^{\nu b}(x'')+O\left(\frac{1}{\sqrt{N}g}\right)+O\left(j^3\right)\right]$$

- and for the quark fields

$$S_q=\sum_q\int d^4x\bar{q}(x)\left[i\partial\!\!\!/-m_q-g\gamma^\mu\frac{\lambda^a}{2}\int d^4x'D^{ab}_{\mu\nu}(x-x')j^{\nu b}(x')\right.$$

$$\left.-g^2\gamma^\mu\frac{\lambda^a}{2}\int d^4x'D^{ab}_{\mu\nu}(x-x')\sum_{q'}\bar{q}'(x')\frac{\lambda^b}{2}\gamma^\nu q'(x')+O\left(\frac{1}{\sqrt{N}g}\right)+O\left(j^3\right)\right]q(x)$$

- We recognize here an explicit Yukawa interaction and a Nambu-Jona-Lasinio non-local term. Already at this stage we are able to recognize that **NJL is the proper low-energy limit for QCD**.

QCD at infrared limit

- Now we operate the Smilga's choice $\eta^a_\mu \eta^b_\nu = \delta_{ab}(\eta_{\mu\nu} - p_\mu p_\nu / p^2)$ for the Landau gauge.
- We are left with the infrared limit QCD using conservation of currents

$$S_{gf} = \frac{1}{2}\int d^4x' d^4x'' \left[j^a_\mu(x')\Delta(x'-x'')j^{\mu a}(x'') + O\left(\frac{1}{\sqrt{N}g}\right) + O(j^3)\right]$$

- and for the quark fields

$$S_q = \sum_q \int d^4x \bar{q}(x) \left[i\partial\!\!\!/ - m_q - g\gamma^\mu \frac{\lambda^a}{2} \int d^4x' \Delta(x-x') j^a_\mu(x') \right.$$

$$\left. - g^2\gamma^\mu \frac{\lambda^a}{2} \int d^4x' \Delta(x-x') \sum_{q'} \bar{q}'(x') \frac{\lambda^a}{2} \gamma_\mu q'(x') + O\left(\frac{1}{\sqrt{N}g}\right) + O(j^3)\right] q(x)$$

- We want to give explicitly the contributions from gluon resonances. In order to do this, we introduce the bosonic currents $j^a_\mu(x) = \eta^a_\mu j(x)$ with the current $j(x)$ that of the gluonic excitations.

QCD at infrared limit

- Using the relation $\eta_\mu^a \eta^{\mu a} = 3(N_c^2 - 1)$ we get in the end

$$S_{gf} = \frac{3}{2}(N_c^2 - 1) \int d^4x' d^4x'' \left[j(x')\Delta(x' - x'')j(x'') + O\left(\frac{1}{\sqrt{N}g}\right) + O\left(j^3\right) \right]$$

- and for the quark fields

$$S_q = \sum_q \int d^4x \bar{q}(x) \left[i\partial\!\!\!/ - m_q - g\eta_\mu^a \gamma^\mu \frac{\lambda^a}{2} \int d^4x' \Delta(x - x')j(x') \right.$$

$$\left. - g^2 \gamma^\mu \frac{\lambda^a}{2} \int d^4x' \Delta(x - x') \sum_{q'} \bar{q}'(x') \frac{\lambda^a}{2} \gamma_\mu q'(x') + O\left(\frac{1}{\sqrt{N}g}\right) + O\left(j^3\right) \right] q(x)$$

- Now, we recognize that the propagator is just a sum of Yukawa propagators weighted by exponential damping terms. So, we introduce the σ field and truncate at the first excitation. This is a somewhat rough approximation but will be helpful in the following analysis.
- This means the we can write the bosonic currents contribution as coming from a boson field and written down as $\sigma(x) = \sqrt{3(N_c^2 - 1)/B_0} \int d^4x' \Delta(x - x')j(x')$.

QCD at infrared limit

- So, low-energy QCD yields a NJL model as given in [M. Frasca, PRC 84, 055208 (2011)]

$$S_\sigma = \int d^4x \left[\frac{1}{2}(\partial\sigma)^2 - \frac{1}{2}m_0^2\sigma^2\right]$$

- and for the quark fields

$$S_q = \sum_q \int d^4x \bar{q}(x)\left[i\not\partial - m_q - g\sqrt{\frac{B_0}{3(N_c^2-1)}}\eta_\mu^a\gamma^\mu\frac{\lambda^a}{2}\sigma(x)\right.$$

$$\left. - g^2\gamma^\mu\frac{\lambda^a}{2}\int d^4x'\Delta(x-x')\sum_{q'}\bar{q}'(x')\frac{\lambda^a}{2}\gamma_\mu q'(x') + O\left(\frac{1}{\sqrt{N}g}\right) + O\left(j^3\right)\right]q(x)$$

- Now, we obtain directly from QCD ($2\mathcal{G}(0) = G$ **is the standard NJL coupling**)

$$\mathcal{G}(p) = -\frac{1}{2}g^2\sum_{n=0}^\infty \frac{B_n}{p^2 - (2n+1)^2(\pi/2K(i))^2\sigma + i\epsilon} = \frac{G}{2}\mathcal{C}(p)$$

with $\mathcal{C}(0) = 1$ fixing in this way the value of G using the gluon propagator. This yields an almost perfect agreement with the case of an instanton liquid (see Ref. in this page).

Wilson loop

- Low-energy QCD, being at infrared fixed point, is not confining (NJL model is not confining). This agrees with the analysis given in [P. González, V. Mathieu, and V. Vento, PRD 84, 114008 (2011)] for the decoupling solution of the propagators in the Landau gauge. Indeed, one has

$$W[\mathcal{C}]=\exp\left[-\frac{g^2}{2}C_2(R)\int\frac{d^4p}{(2\pi)^4}\Delta(p^2)\left(\eta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\oint_{\mathcal{C}}dx^\mu\oint_{\mathcal{C}}dy^\nu e^{-ip(x-y)}\right].$$

- For the **decoupling solution** (at infrared fixed point) one has

$$W[\mathcal{C}]\approx\exp\left[-T\frac{g^2}{2}C_2(R)\int\frac{d^3p}{(2\pi)^3}\Delta(\mathbf{p},0)e^{-i\mathbf{p}\cdot\mathbf{x}}\right]=\exp[-TV_{YM}(r)]$$

- The potential is (assuming a fixed point value for g in QCD)

$$V_{YM}(r)=-C_2(R)\frac{g^2}{2}\sum_{n=0}^{\infty}(2n+1)\frac{\pi^2}{K^2(i)}\frac{(-1)^n e^{-(n+\frac{1}{2})\pi}}{1+e^{-(2n+1)\pi}}\frac{e^{-m_n r}}{r}$$

  and due to massive excitations one gets a screened but not confining potential. **This agrees very well with González&al.**

Wilson loop

- The leading order of the gluon propagator, as also **emerging from lattice computations**, is insufficient to give reason for confinement. We need to compute the **sunrise diagram** going to NLO:

$$\Delta_R(p^2)-\Delta(p^2)=\lambda\frac{1}{\mu^2}\Delta(p^2)\int\frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\sum_{n_1,n_2,n_3}\frac{B_{n_1}}{p_1^2-m_{n_1}^2}\frac{B_{n_2}}{p_2^2-m_{n_2}^2}\frac{B_{n_3}}{(p-p_1-p_2)^2-m_{n_3}^2}.$$

- This integral is well-known [Caffo&al. Nuovo Cim. A 111, 365 (1998)] At small momenta will yield

Field renormalization factor

$$Z_\phi(p^2)=1-\frac{1}{\lambda^{\frac{1}{2}}}\frac{27}{\pi^8}+\frac{1}{\lambda}\frac{3.3\cdot 48}{\pi^8}\left(1+\frac{3}{16}\frac{p^2}{\mu^2}\right)+O\left(\lambda^{-\frac{3}{2}}\right).$$

- This implies for the gluon propagator $\left(\lambda=C_2(R)g^2,\ Z_0=Z_\phi(0)\right)$

$$D_{\mu\nu}^{ab}(p^2)=\delta_{ab}\left(\eta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\sum_{n=0}^\infty\frac{Z_0^{-1}B_n}{p^2+\frac{1}{\lambda}\frac{3.3\cdot 9}{\pi^8}\frac{p^4}{\mu^2}+m_n^2(p^2)}+O\left(\lambda^{-\frac{3}{2}}\right)$$

Wilson loop

- We note that

$$m_n^2(p^2)=m_n^2(0)\left[Z_0+\frac{1}{\lambda}\frac{3.3\cdot 9}{\pi^8}\frac{p^2}{\mu^2}+O\left(\lambda^{-\frac{3}{2}}\right)\right]$$

  that provides **very good agreement** with the scenario by **Dudal&al.** obtained by **postulating condensates**. Here we have an **existence proof**. Masses run with momenta.

- This correction provides the needed p^4 Gribov contribution to the propagator to get a linear term in the potential.
- Now, from Wilson loop, we have to evaluate

$$V_{YM}(r)=-\frac{g^2}{2}C_2(R)\int\frac{d^3p}{(2\pi)^3}\Delta_R(\mathbf{p},0)e^{-i\mathbf{p}\cdot\mathbf{x}}.$$

  being

$$D'^{ab}_{\mu\nu}(p^2)=\delta_{ab}\left(\eta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\Delta_R(p^2)$$

  the renormalized propagator.

Interquark potential

- So,

$$V_{YM}(r) = -\frac{g^2}{8\pi r} C_2(R) Z_0^{-1} \sum_{n=0}^{\infty} B_n \int_{-\infty}^{\infty} dp \frac{p\sin(pr)}{p^2 + \frac{1}{\lambda}\frac{3.3\cdot 9}{\pi^8}\frac{p^4}{\mu^2} + m_n^2(p^2)}.$$

- We rewrite it as

$$V_{YM}(r) \approx -\frac{g^2}{8\pi r} C_2(R) Z_0^{-1} \frac{\pi^8\lambda\mu^2}{3.3\cdot 9} \int_{-\infty}^{\infty} dp \frac{p\sin(pr)}{(p^2+\kappa^2)^2 - \kappa^4}$$

being $\kappa^2 = \frac{\pi^8\lambda\mu^2}{3.3\cdot 9}$, **neglecting running masses** that go like $\sqrt{\lambda}$.

- Finally, for $\kappa r \ll 1$, this yields the well-known linear contribution:

$$V_{YM}(r) \approx -\frac{g^2}{8r} C_2(R) e^{-\frac{\kappa}{\sqrt{2}}r} \sinh\left(\frac{\kappa}{\sqrt{2}} r\right) \approx -\frac{g^2}{8\pi} C_2(R)\left[\frac{\pi}{\sqrt{2}}\kappa - \frac{\pi}{2}\kappa^2 r + O\left((\kappa r)^2\right)\right].$$

Interquark potential

- From the given potential it is not difficult to evaluate the string tension, similarly to what is done in d=2+1 for pure Yang-Mills theory.
- The linear rising term gives

$$\sigma = \frac{\pi}{4}\frac{g^2}{4\pi}C_2(R)\kappa^2.$$

- Remembering that $\lambda = d(R)g^2$,

String tension for SU(N) in d=3+1:

$$\sqrt{\sigma} \approx \frac{\pi^{\frac{9}{2}}}{11} g^2 \sqrt{\frac{C_2(R)d(R)}{4\pi}}\mu$$

that compares really well with the case in d=2+1 [D. Karabali, V. P. Nair and A. Yelnikov, Nucl. Phys. B **824**, 387 (2010)] being $\sqrt{\sigma_{d=2+1}} \approx g^2\sqrt{\frac{C_2(R)d(R)}{4\pi}}$.

Conclusions

- We provided a **strong coupling expansion** both for classical and quantum field theory **for scalar field and QCD**.
- A **low-energy limit of QCD** is so obtained that reduces to a **non-local Nambu-Jona-Lasinio model** with all the parameters and the form factor properly fixed.
- We showed how the **leading order for the gluon propagator is not confining** and we need to compute **Next-to-Leading Order** approximation given by a **sunrise diagram**.
- **Next-to-Leading Order correction** provides the $\mathbf{p^4}$ **Gribov contribution** granting a **confining potential**.
- **String tension** can be computed and appears to be **consistent with** expectations from **d=2+1 case**.

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Thanks a lot for your attention!