Supergravity from 2 and 3-Algebra
Gauge Theory

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Work with: Zvi Bern, John Joseph Carrasco;
Yu-tin Huang, Sangmin Lee
de Donder gauge:

\[ \mathcal{L} = \frac{2}{\kappa^2} \sqrt{g} R, \quad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \]

\[ = \frac{1}{2} \left[ \eta_{\mu_1 \nu_1} \eta_{\mu_2 \nu_2} + \eta_{\mu_1 \nu_2} \eta_{\nu_1 \mu_2} - \frac{2}{D-2} \eta_{\mu_1 \mu_2} \eta_{\nu_1 \nu_2} \right] \frac{i}{p^2 + i\epsilon} \]

After symmetrization
\~ 100 terms!

higher order vertices...

\~10^3 terms

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On-shell simplifications

Graviton plane wave:
\[ \varepsilon^\mu(p) \varepsilon^\nu(p) e^{ip \cdot x} \]

On-shell 3-graviton vertex:
\[ \kappa \left( \eta_{\mu_1 \mu_2} (k_1 - k_2) \nu_3 + \text{cyclic} \right) \left( \eta_{\nu_1 \nu_2} (k_1 - k_2) \nu_3 + \text{cyclic} \right) \]

Gravity scattering amplitude:
\[ M_{\text{tree}}^{\text{GR}}(1, 2, 3, 4) = \frac{st}{u} A_{\text{tree}}^{\text{YM}}(1, 2, 3, 4) \otimes A_{\text{tree}}^{\text{YM}}(1, 2, 3, 4) \]
\[ \overset{d=3}{\longrightarrow} A_{\text{tree}}^{\text{CSm}}(1, 2, 3, 4) \otimes A_{\text{tree}}^{\text{CSm}}(1, 2, 3, 4) \]

Gravity processes = squares of gauge theory ones - entire S-matrix

H. Johansson  Bern, Carrasco, HJ  [BCJ]
Outline

- Motivation $D=3$ amplitudes
- Duality between Color and Kinematics
  - Kinematical Lie 2-Algebra (Yang-Mills theory)
  - Kinematical Lie 3-Algebra (Chern-Simons-matter theory)
  - Gravity as a Double Copy of YM and CSm theories
- Amplitudes in BLG, ABJM and $D=2$ SUGRA
  - Tree-Amplitude relations
  - Dimensional reduction: $D=2$ ABJM
  - Integrability of $D=2$ SUGRA?
- Conclusions

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Why Amplitudes in $D=3$ (or $D=2$)

- $N=8$ Bagger-Lambert-Gustavsson (BLG) theory
- $N=6$ Aharony-Bergman-Jafferis-Maldacena (ABJM) theory
- Chern-Simons-matter (CSM) theories – enticing gauge theories
- The celebrated AdS$_4$/CFT$_3$
- In $D=2$: supergravity integrability

Comparing CSM $\leftrightarrow$ SYM “Same but different”

Similar phenomena as in $D=4$ SYM

- Yangian/Dual conformal sym. (ABJM)
- Grassmannian formulation (ABJM)
- Color-kinematics duality (BLG, ABJM,...)

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2-algebra Color-Kinematics Duality

$D$-dim. Yang-Mills theories are controlled by a kinematic Lie algebra

- Amplitude represented by cubic graphs:

$$A_m^{(L)} = \sum_{i \in \text{cubic}} \int \frac{d^{LD} \ell}{(2\pi)^{LD}} \frac{1}{S_i} \frac{n_i c_i}{p_{i_1}^2 p_{i_2}^2 p_{i_3}^2 \cdots p_{i_l}^2}$$

Color & kinematic numerators satisfy same relations:

- Jacobi identity
- Antisymmetry

Duality: color ↔ kinematics

Bern, Carrasco, HJ

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Some details of color-kinematics duality

Bern, Carrasco, HJ

can be checked for 4pt on-shell ampl. using Feynman rules

Example with two quarks:

\[ \varepsilon_2 \cdot (\bar{u}_1 V u_3) \cdot \varepsilon_4 = \bar{u}_1 \gamma^\nu \gamma_5 u_2 \cdot u_3 - \bar{u}_1 \gamma^\nu \gamma_5 u_2 \cdot u_3 \]

\[ f^{cba \tau}_{ikc} = T_{ij}^{\alpha} T_{jk}^{\beta} - T_{ij}^{\alpha} T_{jk}^{\beta} \]

1. \((A^\mu)^4\) contact interactions absorbed into cubic graphs
   • by hand \(1=s/s\)
   • or by auxiliary field \(B \sim (A^\mu)^2\)

2. Beyond 4-pts duality not automatic \(\rightarrow\) Lagrangian reorganization

3. Known to work at tree level: all-\(n\) example Kiermaier; Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove

4. Enforces (BCJ) relations on partial amplitudes \(\rightarrow\) \((n-3)!\) Basis
   
   also in string theory: Bjerrum-Bohr, Damgaard, Vanhove; Stieberger

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Gravity is a double copy

- Gravity amplitudes obtained by replacing color with kinematics

\[
\mathcal{A}_m^{(L)} = \sum_{i \in \text{cubic}} \int \frac{d^{LD} \ell}{(2\pi)^{LD}} \frac{1}{S_i} \frac{n_i c_i}{p_{i_1}^2 p_{i_2}^2 p_{i_3}^2 \cdots p_{i_l}^2}
\]

\[
\mathcal{M}_m^{(L)} = \sum_{i \in \text{cubic}} \int \frac{d^{LD} \ell}{(2\pi)^{LD}} \frac{1}{S_i} \frac{n_i \tilde{n}_i}{p_{i_1}^2 p_{i_2}^2 p_{i_3}^2 \cdots p_{i_l}^2}
\]

- The two numerators can belong to different theories:

\( \mathcal{A}_m^{(L)} \) and \( \mathcal{M}_m^{(L)} \)

\( n_i \) \hspace{1cm} \( \tilde{n}_i \)

\( (\mathcal{N}=4) \times (\mathcal{N}=4) \rightarrow \mathcal{N}=8 \text{ sugra} \)

\( (\mathcal{N}=4) \times (\mathcal{N}=2) \rightarrow \mathcal{N}=6 \text{ sugra} \)

\( (\mathcal{N}=4) \times (\mathcal{N}=0) \rightarrow \mathcal{N}=4 \text{ sugra} \)

\( (\mathcal{N}=0) \times (\mathcal{N}=0) \rightarrow \text{Einstein gravity + axion+ dilaton} \)

BCJ

similar to Kawai-Lewellen-Tye but works at loop level

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3-algebra Color-Kinematics Duality

\[ D=3 \text{ Chern-Simons matter theories obey color-kinematics duality} \]

\[ 3\text{-algebra Fundamental identity (Jacobi identity):} \]

\[ f^{abc}[d f^{egh}]a = 0 \]

\[ C_S = C_t + C_u + C_v \iff n_S = n_t + n_u + n_v \]

4 and 6 point checks shows that the double copy of BLG
Is \( N = 16 \) \( E_{8(8)} \) SG of Marcus and Schwarz

\[ \text{BLG} = \text{‘square root’ of } N=16 \text{ SG} \quad A_{4}^{\text{BLG}} = \sqrt{M_{4}^{N=16}} = \sqrt[stu]{\delta^{16}(Q)} \]
**$D \leq 3$ supergravity is a double copy of CSM**

- Gravity amplitudes obtained by replacing color with kinematics

\[
\mathcal{A}_m^{(L)} = \sum_{i \in \text{quartic}} \int \frac{d^{LD} \ell}{(2\pi)^{LD}} \frac{1}{S_i} \frac{n_i c_i}{p_{i1}^2 p_{i2}^2 p_{i3}^2 \cdots p_{il}^2}
\]

\[
\mathcal{M}_m^{(L)} = \sum_{i \in \text{quartic}} \int \frac{d^{LD} \ell}{(2\pi)^{LD}} \frac{1}{S_i} \frac{n_i \tilde{n}_i}{p_{i1}^2 p_{i2}^2 p_{i3}^2 \cdots p_{il}^2}
\]

- No string understanding (cf. Kawai-Lewellen-Tye)

- Details more subtle than in SYM $\otimes$ SYM

  - BLG $\otimes$ BLG works in $D=3$ (verified: tree level $\leq 10$pts)
  - ABJM $\otimes$ ABJM works in $D=3$ at 4,6pts, but not $\geq 8$pts
  - ABJM $\otimes$ ABJM works in $D=2$ (verified: tree level $\leq 10$pts)

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Bargheer, He, McLoughlin; Huang, HJ.
Huang, HJ, Lee

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BLG, ABJM and $D=2$ SUGRA
ABJM and BLG theory

**ABJM:** $N=6$ CSm theory with $U(N) \times U(N)$ gauge group

Matter are the only propagating d.o.f.: bi-fundamental representation

**Chiral** $(N, \overline{N})$ multiplet:

$$\Phi = \phi^4 + \eta^A \psi_A + \frac{1}{2} \epsilon_{ABC} \eta^A \eta^B \phi^C + \frac{1}{3!} \epsilon_{ABCD} \eta^A \eta^B \eta^C \psi_4$$

In total 16 states (same spectrum as $N=4$ SYM, but chiral)

**BLG:** $N=8$ CSm theory with $SU(2) \times SU(2) = SO(4)$ gauge group

Matter is non-chiral $N = \overline{N}$

$$\Phi = \phi + \eta^A \psi_A + \frac{1}{2} \eta^A \eta^B \phi_{AB} + \frac{1}{3!} \eta^A \eta^B \eta^C \epsilon_{ABCD} \psi^D + \frac{1}{4!} \eta^A \eta^B \eta^C \eta^D \epsilon_{ABCD} \phi$$

In total 16 states
ABJM and BLG are three-algebras

Bi-fundamental matter theories are three-algebra theories

Bagger, Lambert; Bagger, Bruhn

Triple product of $N \times M$ matrices;

$[M^a, M^b; M^c] \equiv (M^a M^\bar{c} M^b - M^b M^\bar{c} M^a)_{\alpha' \beta} \equiv f^{abc} d (M^d)_{\alpha' \beta}$

Structure constants satisfy fundamental identity (Jacob identity)

$f^{ab\bar{f}} g fgcd\bar{e} - f^{ac\bar{d}} g fgb\bar{f} \bar{e} - f^{bc\bar{d}} g fag\bar{f} \bar{e} + f gc\bar{d} \bar{f} f^{ab} \bar{e} = 0$

Obtained from Feynman diag.

Interesting choices: $g = -g'$ or $g = g'$
Symmetries of structure constants

- **ABJM theory**
  \[ f_{abc\bar{d}} = - f_{ab\bar{d}c} \]
  complex, antisymmetric in pairs

- **BLG theory**
  \[ f_{abcd} \]
  real and totally antisymmetric

- **N=5 CSM theory**
  \[ f_{abc\bar{d}} = - f_{ab\bar{d}c} \] or \[ f_{abc\bar{d}} = f_{ab\bar{d}c} \]
  real, (anti)symmetric in pairs

Consider amplitudes:

\[ A_m = i \left( \frac{2\pi}{k} \right)^{\frac{m-2}{2}} \sum_{i \in \text{quartic}} \frac{n_i c_i}{\prod_{\alpha_i} s_{\alpha_i}} \]

\[ c_i = f_{abc\bar{d}} f_{de\bar{f}g} \ldots f_{wx\bar{y}z} \]

What are their properties?

1) Kleiss-Kuijf relations
2) Color-kinematics duality \( \rightarrow \) BCJ relations
3) double copy = supergravity

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Consider ABJM at 6pts

\[ A_m = \sum_{i \in \text{quartic}} \frac{n_i c_i}{\prod_{\alpha_i} s_{\alpha_i}} \]

Solve Jacobi

\[ A(i) = \sum_{j=1}^{p} \Theta_{ij} n_j \]

At 6pts (ABJM):

\[ \Theta_{ij} = \begin{pmatrix}
\frac{1}{s_1} & \frac{1}{s_2} + \frac{1}{s_9} & \frac{1}{s_9} & -\frac{1}{s_9} & 0 \\
\frac{1}{s_8} & -\frac{1}{s_8} & -\frac{1}{s_8} & -\frac{1}{s_7} & \frac{1}{s_8} + \frac{1}{s_7} & \frac{1}{s_5} + \frac{1}{s_6} + \frac{1}{s_7} \\
\frac{1}{s_7} & -\frac{1}{s_7} & -\frac{1}{s_6} & -\frac{1}{s_9} & \frac{1}{s_6} + \frac{1}{s_7} & \frac{1}{s_5} + \frac{1}{s_6} + \frac{1}{s_7} \\
0 & -\frac{1}{s_9} & -\frac{1}{s_3} & -\frac{1}{s_9} & \frac{1}{s_3} + \frac{1}{s_9} & \frac{1}{s_4} + \frac{1}{s_6} + \frac{1}{s_7} \\
0 & -\frac{1}{s_2} & -\frac{1}{s_6} & -\frac{1}{s_6} & \frac{1}{s_6} & \frac{1}{s_2} + \frac{1}{s_6} + \frac{1}{s_7} \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \]

5×5 matrix has rank 4, but only in \( D=3 \) and on-shell!

5-term amplitude relation:

\[ \text{Ker}(\Theta^T) \cdot A = \sum_{i=1}^{5} C_{ik} A(i) = 0 \]

\[ \text{Det}(\Theta_{i1}, \Theta_{i2}, \ldots, A(i), \ldots, \Theta_{ip}) = 0 \]

H. Johansson
Consider BLG at 6pts

\[ A_m = \sum_{i \in \text{quartic}} \frac{n_i c_i}{\prod_{\alpha_i} s_{\alpha_i}} \]

Solve Jacobi

\[ A(i) = \sum_{j=1}^{p} \Theta_{ij} n_j \]

At 6pts (BLG):

\[ \Theta_{ij} = \begin{pmatrix} \frac{1}{s_{123}} + \frac{1}{s_{126}} + \frac{1}{s_{156}} & \frac{1}{s_{124}} + \frac{1}{s_{135}} & \frac{1}{s_{134}} - \frac{1}{s_{145}} & \frac{1}{s_{135}} + \frac{1}{s_{145}} + \frac{1}{s_{146}} \\ \frac{1}{s_{124}} + \frac{1}{s_{135}} & \frac{1}{s_{125}} + \frac{1}{s_{136}} & \frac{1}{s_{136}} - \frac{1}{s_{145}} & \frac{1}{s_{135}} + \frac{1}{s_{145}} + \frac{1}{s_{146}} \\ \frac{1}{s_{134}} - \frac{1}{s_{145}} & \frac{1}{s_{136}} - \frac{1}{s_{145}} & \frac{1}{s_{135}} + \frac{1}{s_{145}} + \frac{1}{s_{146}} & \frac{1}{s_{135}} + \frac{1}{s_{145}} + \frac{1}{s_{146}} \\ \frac{1}{s_{135}} + \frac{1}{s_{145}} + \frac{1}{s_{146}} & \frac{1}{s_{135}} + \frac{1}{s_{145}} + \frac{1}{s_{146}} & \frac{1}{s_{135}} + \frac{1}{s_{145}} + \frac{1}{s_{146}} & \frac{1}{s_{135}} + \frac{1}{s_{145}} + \frac{1}{s_{146}} \end{pmatrix} \]

5x5 matrix has rank 3, but only in D=3 and on-shell!

4-term amplitude relation:

\[ \text{Ker}(\Theta^T) \cdot A = \sum_{i=1}^{4} C_{ik} A(i) = 0 \]

\[ \text{Det}(\Theta_{i1}, \Theta_{i2}, \ldots, A(i), \ldots, \Theta_{ip}) = 0 \]

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BLG and ABJM amplitude relations

**BLG: 4-term amplitude relation:**

\[ 0 = \sum_{i=2}^{5} S_i A_{(i)} \]

\[ S_2 = s_{124}(s_{145}s_{146} - s_{135}s_{136}) + s_{126}(s_{146}(s_{135} + s_{156}) - s_{136}(s_{145} + s_{156})) , \]

\[ S_3 = s_{145}(s_{156}(s_{124} + s_{126} + s_{135}) + s_{146}(s_{136} - s_{126}) - s_{126}s_{146}(s_{124} + s_{135} + s_{136})) , \]

\[ S_4 = s_{156}(s_{136}s_{145}(s_{124} + s_{126} + s_{135}) + s_{146}(s_{136}(s_{126} + s_{135}) + s_{145}(s_{135} + s_{136}) + s_{124}(s_{126} + s_{145})) , \]

\[ S_5 = -s_{126}(s_{145}s_{146}(s_{124} + s_{135} + s_{156}) + s_{136}(s_{135}(s_{145} + s_{146}) + s_{124}(s_{145} + s_{156}) + s_{146}(s_{145} + s_{156}))) , \]

(plus one additional relation)

**ABJM-type theory (in D=2): 4-term amplitude relations:**

\[ 0 = (A_{(1)}s_{123} - A_{(2)}s_{124})(s_{126}s_{146} - s_{136}s_{156}) + (A_{(4)}s_{146} - A_{(5)}s_{136})(s_{126}s_{156} + s_{123}s_{126} + s_{123}s_{156}) \]

\[ 0 = (A_{(2)}s_{124} - A_{(3)}s_{145})(s_{126}s_{146} - s_{136}s_{156}) + (A_{(4)}s_{156} - A_{(5)}s_{126})(s_{145}s_{146} + s_{136}s_{145} + s_{136}s_{146}) \]

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# ABJM and BLG data collection

## ABJM theory counts:

<table>
<thead>
<tr>
<th>external legs</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>$m = 2k + 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>quartic diagrams</td>
<td>1</td>
<td>9</td>
<td>216</td>
<td>9900</td>
<td>$(3k)!/(k+1)!$</td>
</tr>
<tr>
<td>planar amplitudes</td>
<td>1</td>
<td>6</td>
<td>72</td>
<td>1440</td>
<td>$\frac{1}{2}(k+1)!k!$</td>
</tr>
<tr>
<td>diagrams in $A^{\text{planar}}$</td>
<td>1</td>
<td>3</td>
<td>12</td>
<td>55</td>
<td>$\frac{(3k)!}{k!(2k+1)!}$</td>
</tr>
<tr>
<td>distinct fundamental id’s</td>
<td>0</td>
<td>9</td>
<td>432</td>
<td>29700</td>
<td>$\frac{(k-1)(3k)!((k+1)^2)}{(2k+1)!}$</td>
</tr>
<tr>
<td>KK-indep. ampls.</td>
<td>1</td>
<td>5</td>
<td>57</td>
<td>1144</td>
<td>*</td>
</tr>
<tr>
<td>BCJ-indep. ampls. $D = 2$</td>
<td>1</td>
<td>3</td>
<td>38</td>
<td>987</td>
<td>*</td>
</tr>
</tbody>
</table>

## BLG theory counts:

<table>
<thead>
<tr>
<th>external legs</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>$m = 2k + 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>quartic diagrams</td>
<td>1</td>
<td>10</td>
<td>280</td>
<td>15400</td>
<td>$\frac{(3k)!}{k!(3!)^k}$</td>
</tr>
<tr>
<td>distinct fundamental id’s</td>
<td>0</td>
<td>15</td>
<td>840</td>
<td>69300</td>
<td>$\frac{3}{2}(k-1)\frac{(3k)!}{k!(3!)^k}$</td>
</tr>
<tr>
<td>KK-indep. ampls.</td>
<td>1</td>
<td>5</td>
<td>56</td>
<td>1077</td>
<td>*</td>
</tr>
<tr>
<td>BCJ-indep. ampls. $D = 3$</td>
<td>1</td>
<td>3</td>
<td>38</td>
<td>1029</td>
<td>*</td>
</tr>
</tbody>
</table>

Note: no simple combinatorial patterns for KK and BCJ counts.

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Same $D=3$ Supergravity Either Way

In $D=3$, supergravity from two different double copies:

\[ CSM \otimes CSM = SYM \otimes SYM \quad \text{(kinematic parts)} \]

- The extra propagators in $SYM \otimes SYM$ compensates for dimension mismatch
- $SYM$ has even and odd matrix elements, $CSM$ only even!
- R-symmetry constrains ensure that double copy kills odd SYM contributions

For $N=16$ SUGRA: all states are $SO(16)$ spinors $\rightarrow$ no odd S-matrix elements

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D=2 Supergravity and Integrability

- Easy access to $D=2$ supergravity S-matrix
  
  Huong, HJ, Lee

- Problem: a $D=2$ massless S-matrix has severe IR divergences

- Possible to restrict to amplitudes without soft or collinear div’s

Note: vanishing soft channel

Can check integrability of $D=2$ supergravity S-matrix (in restricted momenta)

Nicolai, Warner

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\(D=2\) ABJM and supergravity ampls

**D=2 ABJM amplitudes:**

Huang, HJ, Lee

4pts:

\[
A_{D=2}^{\text{ABJM}}(\bar{1}, 2, \bar{3}, 4) = i \frac{\delta^{(3)}(\sum_{\text{even}} \lambda_i \eta_i) \delta^{(3)}(\sum_{\text{odd}} \bar{\lambda}_i \eta_i)}{\lambda_1 \lambda_2 \bar{\lambda}_3 \lambda_4}
\]

6pts:

\[
A_{D=2}^{\text{ABJM}}(\bar{1}2\bar{3}456) = i \frac{\delta^{(3)}(\sum_{\text{even}} \lambda_i \eta_i) \delta^{(3)}(\sum_{\text{odd}} \bar{\lambda}_i \eta_i)}{\lambda_1 \lambda_2 \bar{\lambda}_3 \lambda_4 \lambda_5 \lambda_6} \sum_{s=\pm} \delta^{(3)} \left( s \frac{\bar{\lambda}_3 \eta_1 - \bar{\lambda}_1 \eta_3}{\bar{\lambda}_5} + i \frac{\lambda_6 \eta_4 - \lambda_4 \eta_6}{\lambda_2} \right)
\]

**D=2 supergravity:**

4pts:

\[
M_{D=2}(\bar{1}, 2, \bar{3}, 4) = [A_{D=2}^{\text{ABJM}}(1, \bar{2}, 3, 4)]^2 \quad \text{(finite and non-zero)}
\]

6pts:

\[
\mathcal{M}_6(\bar{1}2\bar{3}456) = \frac{s_{12} s_{34} s_{56}}{(s_{23} - s_{14})(s_{36} - s_{12})(s_{34} - s_{16})} \left( (s_{34} - s_{16}) A_{(1)} \tilde{A}_{(1)} + (s_{36} - s_{12}) A_{(2)} \tilde{A}_{(2)} \right)
\]

\[
+ (s_{23} - s_{14}) A_{(3)} \tilde{A}_{(3)}
\]

6pt amplitude vanishes \(\rightarrow\) consistent with integrability

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We would like to check the Yang-Baxter Eqn:

\[ \quad = \quad \]

Problem: one line is massive $\rightarrow$ take massless limit

\[ \quad = \quad \]

Holds in $D=3$ ABJM and $D=3$ sugra

$\rightarrow$ more checks are needed, as well as better understanding of IR div.

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Conclusions

- **Yang-Mills theories** are controlled by a **kinematic Lie 2-algebra**
- **Chern-Simons-matter theories** controlled by a **kinematic Lie 3-algebra**
- The explicit kinematic algebra is still missing for all but the simplest case of self-dual Yang-Mills.
- With duality manifest: **Gravity becomes double copy.**
  double copy of CSM theory = double copy of $D \leq 3$ SYM
- **BCJ relations/double copy** present in $D=3$ for BLG theories
- **BCJ relations/double copy** present in $D=2$ for ABJM theories
- **Simple access to** $D=2$ supergravity S-matrix $\rightarrow$ checks of integrability
- C-K duality is a key tool for nonplanar gauge and gravity calculations.
  - Loop amplitudes in BLG (ABJM)...
  - $N=8$ supergravity UV behavior at 5 (and 7) loops...
  - $N=4$ supergravity UV behavior at 3,4 loops ...

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