Energy flow in $\mathcal{N} = 4$ SYM

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**\( e^+e^- \) annihilation in \( \mathcal{N} = 4 \) SYM**

- Define IR finite observables in \( \mathcal{N} = 4 \) SYM and evaluate them both at weak/strong coupling
- Are closely related to the QCD weighted cross-sections for the final states in \( e^+e^- \) – annihilation

From QCD to \( \mathcal{N} = 4 \) SYM: introduce an analog of the electromagnetic current:

(protected) half-BPS operator built from the six real scalars

\[
O_{20}^{IJ}(x) = \text{tr} \left[ \Phi^I \Phi^J - \frac{1}{6} \delta^{IJ} \Phi^K \Phi^K \right], \quad (I, J = 1, \ldots, 6)
\]

\[
O(x, Y) = Y^I Y^J O_{20}^{IJ}(x) = Y^I Y^J \text{tr}[\Phi^I(x)\Phi^J(x)]
\]

The null vector \( Y^I \) defines the orientation of the projected operator in the isotopic \( SO(6) \) space

**What are the properties of the final states created from the vacuum by the operator** \( O(x, Y) \)?
Final states in $\mathcal{N} = 4$ SYM

✔ To lowest order in the coupling, $O(x, Y)$ produces a pair of scalars out of the vacuum

✔ For arbitrary coupling, the state $O(x, Y)|0\rangle$ can be decomposed into an infinite sum over on-shell states with an arbitrary number of scalars ($s$), gauginos ($\lambda$) and gauge fields ($g$)

$$
\int d^4x \ e^{iqx} O(x, Y)|0\rangle = |ss\rangle + |ssg\rangle + |s\lambda\lambda\rangle + \ldots
$$

✔ The amplitude of creation of a particular final state $|X\rangle$ out of the vacuum

$$
\langle X| \int d^4x \ e^{iqx} O(x, Y)|0\rangle = (2\pi)^4 \delta^{(4)}(q - p_X) M_{O \rightarrow X}
$$

$p_X$ is the total momentum of the state $|X\rangle$

✔ The amplitude $M_{O \rightarrow X}$ has the meaning of a (IR divergent) form-factor

$$
M_{O \rightarrow X} = \langle X|O(0, Y)|0\rangle
$$
**Total cross-section of $O_{20} \rightarrow \text{everything}$**

- Analog of the QCD process $e^+ e^- \rightarrow \text{everything}$

\[
\sigma_{\text{tot}}(q) = \sum_X (2\pi)^4 \delta^{(4)}(q - p_X) |M_{O_{20} \rightarrow X}|^2
\]

- To lowest order in the coupling, the production of a pair of scalars

\[
\sigma_{\text{tot}}(q; Y) = \frac{1}{2} (N^2 - 1) (Y \bar{Y})^2 \int \frac{d^4 k}{(2\pi)^4} (2\pi)^2 \delta_+(k^2) \delta_+((q - k)^2) + \ldots
\]

- To higher order in the coupling, each term in the sum $\sum_X$ has IR / collinear divergences

- How to avoid divergences? Use the completeness condition $\sum_X |X\rangle\langle X| = 1$

\[
\sigma_{\text{tot}}(q) = \int d^4 x \ e^{iqx} \sum_X \langle 0|O(0, \bar{Y})|X\rangle \ e^{-ixp_X} \langle X|O(0, Y)|0\rangle
\]

\[
= \int d^4 x \ e^{iqx} \langle 0|O(x, \bar{Y})O(0, Y)|0\rangle \quad \text{The operators are not time ordered!}
\]

Wightman correlation function (protected for half-BPS operators)

- All-loop result in $\mathcal{N} = 4$ SYM [van Neerven]

\[
\sigma_{\text{tot}}(q) = \frac{1}{16\pi} (N^2 - 1) (Y \bar{Y})^2 \theta(q^0) \theta(q^2)
\]

Perturbative corrections cancel order by order IR finite to any order in the coupling
Weighted cross-section

✔ More refined information about the final states in $O_{20'} \rightarrow \text{everything}$

✔ Assign a weight factor $w(X)$ to the contribution of each state $|X\rangle$

$$\sigma_W(q) = \sigma_{\text{tot}}^{-1} \sum_X (2\pi)^4 \delta^4(q - p_X) w(X) |M_{O_{20'} \rightarrow X}|^2$$

$$= \sigma_{\text{tot}}^{-1} \int d^4x \ e^{i q x} \sum_X \langle 0 | O(x, \bar{Y}) | X \rangle w(X) \langle X | O(0, Y) | 0 \rangle$$

✔ Less inclusive quantity as compared with the total cross section, no optical theorem

✔ Choose of the weight factors $w(X)$ gives an access to the flow of various quantum numbers of particles (energy, charge, etc) in the final state

✔ Popular choice – energy-energy correlations

[Basham,Brown,Ellis,Love]

$$w(X) = \sum_{i,j} E_i E_j \delta(\cos \theta_{ij} - \cos \chi)$$

Are known in QCD up to 2 loops numerically
Energy flow

✔ The total energy in the final state $|X\rangle = |k_1, \ldots, k_\ell\rangle$ that flows into the detector located at spatial infinity in the direction of the vector $\vec{n}$.

$$w_\mathcal{E}(k_1, \ldots, k_\ell) = \sum_{i=1}^{\ell} k_i^0 \delta^{(2)}(\Omega_{\vec{k}_i} - \Omega_{\vec{n}}),$$

✔ Energy flow operator

$$\mathcal{E}(\vec{n})|X\rangle = w_\mathcal{E}(X)|X\rangle.$$ 

✔ Is expressed in terms of the energy-momentum tensor in $\mathcal{N} = 4$ SYM

[Sveshnikov,Tkachov],[GK,Oderda,Sterman]

$$\mathcal{E}(\vec{n}) = \int_0^\infty dt \lim_{r\to\infty} r^2 \vec{n}^i T_{0i}(t, r\vec{n})$$

✔ Representation for $\mathcal{E}(\vec{n})$ in terms of creation and annihilation operators of on-shell states

$$\mathcal{E}(\vec{n}) = \int \frac{d^4 k}{(2\pi)^4} 2\pi \delta_+(k^2) k^0 \delta^{(2)}(\Omega_{\vec{n}} - \Omega_{\vec{k}}) \sum_{i=s,\lambda,\bar{\lambda},g} a^\dagger_i(k) a_i(k),$$
Energy correlations

✔ Single correlator

$$\sum_X \langle 0 | O(x, \bar{Y}) | X \rangle w_E(X) \langle X | O(0, Y) | 0 \rangle = \sum_X \langle 0 | O(x, \bar{Y}) E(\vec{n}) | X \rangle \langle X | O(0, Y) | 0 \rangle$$

$$= \langle 0 | O(x, \bar{Y}) E(\vec{n}) O(0, Y) | 0 \rangle$$

Wightman correlation function (no time ordering!) due to real time evolution

✔ Single energy flow

$$\langle E(\vec{n}_1) \rangle = \sigma_{\text{tot}}^{-1} \int d^4 x \, e^{i q x} \langle 0 | O(x, \bar{Y}) E(\vec{n}_1) O(0, Y) | 0 \rangle$$

✔ Multi-energy correlations [GK, Sterman], [Belitsky, GK, Sterman], [Hofman, Maldacena]

$$\langle E(\vec{n}_1) \ldots E(\vec{n}_\ell) \rangle = \sigma_{\text{tot}}^{-1} \int d^4 x \, e^{i q x} \langle 0 | O(x, \bar{Y}) E(\vec{n}_1) \ldots E(\vec{n}_\ell) O(0, Y) | 0 \rangle$$

Energy flow in the direction of $\vec{n}_1, \ldots, \vec{n}_\ell$

Depends on the relative angles $\cos \theta_{ij} = (\vec{n}_i \cdot \vec{n}_j)$

✔ The goal is to find $\langle E(\vec{n}_1) \ldots E(\vec{n}_\ell) \rangle$ for arbitrary coupling in $\mathcal{N} = 4$ SYM
Weighted cross-sections from amplitudes I

Transition amplitude

\[ M_{O_{20}' \rightarrow X} = \begin{matrix} \text{1} & \text{s} \\ \text{s} & \text{s} \end{matrix} + \begin{matrix} \text{0} & \text{g} \\ \text{s} & \text{s} \end{matrix} + \begin{matrix} \text{0} & \text{\lambda} \\ \text{s} & \text{\lambda} \end{matrix} + \ldots \]

One-loop matrix elements (Mandelstam invariants \( s_{ij} = (p_i + p_j)^2 \) with \( p_i^2 = 0 \))

\[
|M_{O_{20}' \rightarrow ss}|^2 = |\langle s(p_1)s(p_2)|O(0,Y)|0\rangle|^2 = \frac{2}{s_{12}} \left[ 1 + aF_{\text{virt}}(q^2) \right]
\]

\[
|M_{O_{20}' \rightarrow ssg}|^2 = |\langle s(p_1)s(p_2)g(p_3)|O(0,Y)|0\rangle|^2 = a \frac{s_{12}}{s_{13}s_{23}}
\]

\[
|M_{O_{20}' \rightarrow s\lambda\lambda}|^2 = |\langle \lambda(p_1)\lambda(p_2)s(p_3)|O(0,Y)|0\rangle|^2 = a \frac{2}{s_{12}}
\]

The total transition amplitude to order \( O(a) \)

\[
\sigma_{\text{tot}}(q) = \int dP_S^2 |M_{O_{20}' \rightarrow ss}|^2 + \int dP_S^3 \left( |M_{O_{20}' \rightarrow ssg}|^2 + |M_{O_{20}' \rightarrow s\lambda\lambda}|^2 \right) + O(a^2)
\]

\[
= \frac{1}{8\pi} \left[ 1 + aF_{\text{virt}}(q^2) \right] + a \int dP_S^3 \frac{s_{12}^2 + 2s_{13}s_{23}}{s_{12}s_{13}s_{23}} + O(a^2) = \frac{1}{8\pi} + 0 \cdot a + O(a^2)
\]

Protected from perturbative corrections
Weighted cross-sections from amplitudes II

- Energy correlations

\[ \sigma_E(q) = \sigma_{\text{tot}}^{-1} \left[ \int \text{dPS}_2 \ w_E(p_1, p_2) \ |\mathcal{M}_{O_{20'}}\rightarrow ss|^2 \right. \\
\left. + \int \text{dPS}_3 \ w_E(p_1, p_2, p_3) \left( |\mathcal{M}_{O_{20'}}\rightarrow ssg|^2 + |\mathcal{M}_{O_{20'}}\rightarrow s\lambda\lambda|^2 \right) + O(a^2) \] 

- Single detector (space-time orientation is specified by the light-like vector \( n^\mu = (1, \vec{n}) \))

\[ \langle E(\vec{n}) \rangle = \frac{1}{4\pi} \frac{(q^2)^2}{(qn)^3} \]

Protected from loop corrections

- Two detectors (unprotected quantity) [Zhiboedov],[Engelund,Roiban]

\[ \langle E(\vec{n}_1)E(\vec{n}_2) \rangle = -\frac{a}{4(2\pi)^4} \frac{q^2}{(n_1n_2)^3} \frac{z \ln(1-z)}{(1-z)} + O(a^2) \]

The scaling variable in the rest frame of the source \( q^\mu = (q^0, \vec{0}) \)

\[ z = \frac{q^2(n_1n_2)}{2(qn_1)(qn_2)} = \frac{(1 - \cos \theta_{12})}{2}, \quad 0 < z < 1 \]

Two-loop corrections to \( \langle E(\vec{n}_1)E(\vec{n}_2) \rangle \) are hard to compute (\( \sim 10^2 \) diagrams)
Weighted cross-sections from correlation functions I

✔ Energy flow operator

\[
\langle \mathcal{E}(\vec{n}_1) \rangle \sim \int d^4 x \, e^{i q x} \langle 0 | O(x, \vec{Y}) \, \mathcal{E}(\vec{n}_1) \, O(0, Y) | 0 \rangle
\]

\[
= \int d^4 x \, e^{i q x} \int_0^\infty dt \, \lim_{r \to \infty} r^2 \langle 0 | O(x, \vec{Y}) \, T_0 \vec{n}_1(x_1) \, O(0, Y) | 0 \rangle \bigg|_{x_1 = (t, r \vec{n}_1)}
\]

✔ Generalization for \( \ell \) detectors

\[
\langle \mathcal{E}(\vec{n}_1) \ldots \mathcal{E}(\vec{n}_\ell) \rangle = \text{Fourier} \times \text{Limit} \left[ \langle 0 | O(x, \vec{Y}) \, T_0 \vec{n}_1(x_1) \ldots T_0 \vec{n}_\ell(x_\ell) \, O(0, Y) | 0 \rangle \bigg|_{x_i = (t_i, r_i \vec{n}_i)} \right]
\]

✔ How to compute energy flow correlators:

✗ Compute corr.function \( \langle O(x) T(x_1) \ldots T(x_\ell) O(0) \rangle \) in Euclid

✗ Continue to Minkowski with Wightman prescription

✗ Take detector limit + perform Fourier

✔ Correlation functions in \( \mathcal{N} = 4 \) SYM have a lot of symmetry :

✗ \( \langle O(x) T(x_1) O(0) \rangle \) is fixed by conformal symmetry → exact result for \( \langle \mathcal{E}(\vec{n}_1) \rangle \) [Hofman,Maldacena]

✗ \( \langle O(x) T(x_1) T(x_2) O(0) \rangle \) is not fixed by conformal symmetry
Hidden beauty of $\mathcal{N}=4$ SYM:

- Quantum corrections to various correlation functions are determined by the same scalar function

$$\langle O(x_1)O(x_2)O(x_3)O(x_4) \rangle_E = \frac{1}{x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2} \Phi(u, v; a)$$

$$\langle O(x_1)T(x_2)T(x_3)O(x_4) \rangle_E = \frac{1}{(x_{12}^2 x_{23}^2 x_{34}^2)^2} P(\partial_u, \partial_v) \Phi(u, v; a)$$

Conformal ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{23}^2 x_{41}^2}{x_{13}^2 x_{24}^2}$$

- Universal function in $\mathcal{N}=4$ SYM at weak coupling

$$\Phi(u, v) = a \Phi^{(1)}(u, v) + a^2 \left( \frac{1}{2} (1 + u + v) \left[ \Phi^{(1)}(u, v) \right]^2 + 2 \left[ \Phi^{(2)}(u, v) + \frac{1}{u} \Phi^{(2)}(v/u, 1/u) + \frac{1}{v} \Phi^{(2)}(1/v, u/v) \right] \right) + O(a^3)$$

$\Phi^{(1)}(u, v)$ ‘box’ integral, $\Phi^{(2)}(u, v)$ ‘double’ box integral

- ‘Permutation symmetry’ allows us to determine $\Phi_{\text{weak}}(u, v)$ to six loops

- AdS/CFT correspondences predicts $\Phi(u, v)$ at strong coupling
From Euclid to Minkowski

✔ Brute force method: compute anew using Schwinger-Keldysh technique (too hard)

✔ Better method: analytically continue correlation functions from Euclid to Minkowski+Wightman

✔ Warm-up example: free scalar propagator $D_{\text{Euclid}}(x) = \langle \phi(x)\phi(0) \rangle \sim 1/x^2$

$$
\langle 0|\phi(x)\phi(0)|0 \rangle = \sum_{n} \langle 0|\phi(x)|n \rangle \langle n|\phi(0)|0 \rangle \\
= \sum_{E_n > 0} e^{-iE_n(x^0-i0)+i\vec{p}\cdot\vec{x}} \langle 0|\phi(0)|n \rangle \langle n|\phi(0)|0 \rangle \sim \frac{1}{(x^0-i0)^2-x^2}
$$

✔ How to get Wightman correlation functions (‘magic’ recipe)

✗ Go to Mellin space:

$$
\Phi_{\text{Euclid}} = \int_{-\delta-i\infty}^{-\delta+i\infty} \frac{dj_1 dj_2}{(2\pi i)^2} M(j_1, j_2; a) u^{j_1} v^{j_2} , \quad u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} , \quad v = \frac{x_{23}^2 x_{41}^2}{x_{13}^2 x_{24}^2}
$$

✗ Substitute $x_{ij}^2 \rightarrow x_{ij}^2 = x_{ij}^2 - i0 \cdot x_{ij}^0$

$$
\Phi_{\text{Wightman}} = \int_{-\delta-i\infty}^{-\delta+i\infty} \frac{dj_1 dj_2}{(2\pi i)^2} M(j_1, j_2; a) \left( \frac{x_{12}^2 + x_{34}^2}{x_{13}^2 + x_{24}^2} \right)^{j_1} \left( \frac{x_{23}^2 + x_{41}^2}{x_{13}^2 + x_{24}^2} \right)^{j_2}
$$

✔ $M(j_1, j_2; a)$ is known both at weak and strong coupling in planar $\mathcal{N} = 4$ SYM
Energy correlations

✔ Energy correlations for arbitrary coupling

\[ \langle \mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2) \rangle = \frac{1}{(4\pi^2)^2} \frac{q^2}{(n_1 n_2)^3} \mathcal{F}_\mathcal{E}(z; a), \quad z = (1 - \cos \theta_{12})/2 \]

✗ Weak coupling:

\[ \mathcal{F}_\mathcal{E}(z; a < 1) = -\frac{a}{4} \frac{z \ln(1 - z)}{(1 - z)} + O(a^2) \]

[Zhboedov],[Engelund,Roiban]

✗ Strong coupling:

\[ \mathcal{F}_\mathcal{E}(z; a \to \infty) = 8z^3 + O(1/\sqrt{a}) \]

[Hofman,Maldacena]

✔ All-loop representation

\[ \mathcal{F}_\mathcal{E}(z; a) = \int_{-\delta-i\infty}^{-\delta+i\infty} \frac{dj_1 dj_2}{(2\pi i)^2} \left[ \frac{M(j_1, j_2; a) K_\mathcal{E}(j_1, j_2)}{\Gamma(1-j_1-j_2)} \right] K_\mathcal{E}(j_1, j_2) \left( \frac{1-z}{z} \right)^{j_1+j_2} \]

Detector function is independent on the coupling

\[ K_\mathcal{E}(j_1, j_2) \sim \frac{\Gamma(1-j_1-j_2)}{\Gamma(j_1+j_2)[\Gamma(1-j_1)\Gamma(1-j_2)]^2} \]

\[ M(j_1, j_2; a) = aM^{(1)}(j_1, j_2) + a^2 M^{(2)}(j_1, j_2) + \ldots \]

are known

✔ Analytical expression for \( \mathcal{F}_\mathcal{E}(z; a) \) at two loops, extension to higher loops is feasible
Conclusions and open questions

✔ Energy correlations are good/nontrivial physical observables in $\mathcal{N} = 4$ SYM

✔ Relation to energy flow correlations in QCD (most complicated part)?

✔ All symmetries of $\mathcal{N} = 4$ SYM are preserved, what is the manifestation of integrability?

✔ Interpolation between weak and strong coupling?

✔ Other proposals for ‘good’ observables?