The Schwarzian and black hole physics

Thomas Mertens

Ghent University

Based on arXiv:1606.03438 with J. Engelsöy and H. Verlinde
arXiv:1705.08408 with G.J. Turiaci and H. Verlinde
arXiv:1801.09605
arXiv:1804.09834 with H. Lam, G.J. Turiaci and H. Verlinde
Outline

Motivation
SYK
JT dilaton gravity

The Schwarzian path integral

Embedding in 2d Liouville CFT
Partition function
2-point function
4-point function
OTO 4-point function

Generalizations

Conclusion
Motivation: SYK at low energies

**SYK-model**: $N$ Majorana fermions with all-to-all random interactions of $q$ fermions
Motivation: SYK at low energies

**SYK-model**: $N$ Majorana fermions with all-to-all random interactions of $q$ fermions

**IR symmetry** of solution to SYK Dyson-Schwinger equations for two-point function $\langle T \psi_i(\tau_1) \psi_i(\tau_2) \rangle$:

$$G(\tau_1, \tau_2) \rightarrow f'(\tau_1)\Delta f'(\tau_2)\Delta G(f(\tau_1), f(\tau_2)), \quad \Delta = 1/q$$
Motivation: SYK at low energies

**SYK-model:** $N$ Majorana fermions with all-to-all random interactions of $q$ fermions

**IR symmetry** of solution to SYK Dyson-Schwinger equations for two-point function $\langle T \psi_i(\tau_1) \psi_i(\tau_2) \rangle$:

$$ G(\tau_1, \tau_2) \rightarrow f'(\tau_1)^\Delta f'(\tau_2)^\Delta G(f(\tau_1), f(\tau_2)), \quad \Delta = 1/q $$

IR conformal symmetry is spontaneously broken (by specific solution $G_c$) and explicitly broken (by UV)
**Motivation: SYK at low energies**

**SYK-model:** \( N \) Majorana fermions with all-to-all random interactions of \( q \) fermions

**IR symmetry** of solution to SYK Dyson-Schwinger equations for two-point function \( \langle T \psi_i(\tau_1)\psi_i(\tau_2) \rangle \):

\[
G(\tau_1, \tau_2) \rightarrow f'(\tau_1)\Delta f'(\tau_2)\Delta G(f(\tau_1), f(\tau_2)), \quad \Delta = 1/q
\]

IR conformal symmetry is spontaneously broken (by specific solution \( G_c \)) and explicitly broken (by UV)

Effective Action (master fields \( \Sigma, G \)) \( \sim N \) \( \Leftrightarrow \) suppressed at \( N \rightarrow \infty \), unless also \( \beta J \rightarrow \infty \) \( \Rightarrow \) Schwarzian action

\( \Rightarrow \) Path-integrate over low-energy reparametrizations \( f \) of the classical solution with **Schwarzian action**
Motivation: SYK at low energies

**SYK-model**: $N$ Majorana fermions with all-to-all random interactions of $q$ fermions

**IR symmetry** of solution to SYK Dyson-Schwinger equations for two-point function $\langle T \psi_i(\tau_1) \psi_i(\tau_2) \rangle$:

$$G(\tau_1, \tau_2) \rightarrow f'(\tau_1)^\Delta f'(\tau_2)^\Delta G(f(\tau_1), f(\tau_2)), \quad \Delta = 1/q$$

IR conformal symmetry is spontaneously broken (by specific solution $G_c$) and explicitly broken (by UV)

Effective Action (master fields $\Sigma, G$) $\sim N \Leftrightarrow$ suppressed at $N \rightarrow \infty$, unless also $\beta J \rightarrow \infty \Rightarrow$ Schwarzian action

$\Rightarrow$ Path-integrate over low-energy reparametrizations $f$ of the classical solution with Schwarzian action

$G_c$ has $SL(2, \mathbb{R})$-symmetry $\Rightarrow$ not to be path integrated $\equiv$ gauge redundancy
Motivation: 2d dilaton gravity

Jackiw-Teitelboim (JT) gravity is model of 2d dilaton gravity
JT appears in s-wave dim. red. from near-extremal black holes
Motivation: 2d dilaton gravity

Jackiw-Teitelboim (JT) gravity is model of 2d dilaton gravity
JT appears in s-wave dim. red. from near-extremal black holes

JT is holographically dual to Schwarzian theory
Motivation: 2d dilaton gravity

Jackiw-Teitelboim (JT) gravity is model of 2d dilaton gravity
JT appears in s-wave dim. red. from near-extremal black holes

 JT is holographically dual to Schwarzian theory

Derivation:

\[
S = \frac{1}{16\pi G} \int d^2 x \sqrt{-g} \Phi(R + 2) + \frac{1}{8\pi G} \int d\tau \sqrt{-\gamma} \Phi \text{bdy}K
\]

Path integrate over \( \Phi \)
\( \Rightarrow R = -2 : \text{Geometry fixed as AdS}_2 \)

Consider boundary curve \((f(\tau), z = \epsilon \dot{f}(\tau))\) as UV cutoff, carving out a shape from AdS,
\( f = \text{time reparametrization} \)

\( \Rightarrow S = -C \int dt \{f, \tau\}, C \to \infty \equiv G \to 0 \)

Compare to CS / WZW, 3d gravity / Liouville topological dualities
Motivation: 2d dilaton gravity

Jackiw-Teitelboim (JT) gravity is a model of 2d dilaton gravity. JT appears in s-wave dimension reduction from near-extremal black holes.

JT is holographically dual to Schwarzian theory.

Derivation:

\[
S = \frac{1}{16\pi G} \int d^2x \sqrt{-g} \Phi (R + 2) + \frac{1}{8\pi G} \int d\tau \sqrt{-\gamma} \Phi_{bdy} K
\]

Path integrate over \( \Phi \) \( \Rightarrow \) \( R = -2 \): Geometry fixed as AdS\(_2\).
Motivation: 2d dilaton gravity

Jackiw-Teitelboim (JT) gravity is a model of 2d dilaton gravity. JT appears in s-wave dimension reduction from near-extremal black holes. JT is holographically dual to Schwarzian theory.

Derivation:

\[ S = \frac{1}{16\pi G} \int d^2x \sqrt{-g} \Phi(R + 2) + \frac{1}{8\pi G} \int d\tau \sqrt{-\gamma} \Phi_{\text{bdy}} K \]

Path integrate over \( \Phi \Rightarrow R = -2 \): Geometry fixed as AdS\(_2\)

- Consider boundary curve \((f(\tau), z = \epsilon \dot{f}(\tau))\) as UV cutoff, carving out a shape from AdS, \(f = \text{time reparametrization}\)
Motivation: 2d dilaton gravity

Jackiw-Teitelboim (JT) gravity is a model of 2d dilaton gravity. JT appears in s-wave dimension reduction from near-extremal black holes. JT is holographically dual to Schwarzian theory.

Derivation:

\[
S = \frac{1}{16\pi G} \int d^2x \sqrt{-g} \Phi (R + 2) + \frac{1}{8\pi G} \int d\tau \sqrt{-\gamma} \Phi_{bdy} K
\]

Path integrate over \( \Phi \Rightarrow R = -2 \): Geometry fixed as AdS\(_2\)

- Consider boundary curve \((f(\tau), z = \epsilon \dot{f}(\tau))\) as UV cutoff, carving out a shape from AdS, \(f = \) time reparametrization
- Fix asymptotics of dilaton \(\Phi_{bdy} \sim 1/\epsilon\)

Using \(K = 1 + \epsilon^2 \{f, \tau\}\)

\[
\Rightarrow S = -C \int dt \{f, \tau\}, \quad C \to \infty \equiv G \to 0
\]
Motivation: 2d dilaton gravity

Jackiw-Teitelboim (JT) gravity is model of 2d dilaton gravity
JT appears in s-wave dim. red. from near-extremal black holes
JT is holographically dual to Schwarzian theory

Derivation:

\[ S = \frac{1}{16\pi G} \int d^2x \sqrt{-g} \Phi(R + 2) + \frac{1}{8\pi G} \int d\tau \sqrt{-\gamma} \Phi_{bdy} K \]

Path integrate over \( \Phi \Rightarrow R = -2 \): Geometry fixed as AdS\(_2\)

- Consider boundary curve \((f(\tau), z = \epsilon \dot{f}(\tau))\) as UV cutoff, carving out a shape from AdS, \( f = \text{time reparametrization} \)
- Fix asymptotics of dilaton \( \Phi_{bdy} \sim 1/\epsilon \)

Using \( K = 1 + \epsilon^2 \{f, \tau\} \)

\[ \Rightarrow S = -C \int dt \{f, \tau\}, \quad C \to \infty \equiv G \to 0 \]

Compare to CS / WZW, 3d gravity / Liouville topological dualities
The Schwarzian theory: goal

Main goal:
Compute all correlation functions:

\[
\langle O_{\ell_1} O_{\ell_2} \ldots \rangle_\beta = \frac{1}{Z} \int_M [\mathcal{D} f] O_{\ell_1} O_{\ell_2} \ldots e^{C \int_0^\beta d\tau \left( \{ f, \tau \} + \frac{2\pi^2}{\beta^2} f'^2 \right)}
\]

with action

\[
S[f] = -C \int_0^\beta d\tau \left\{ F, \tau \right\}, \quad F \equiv \tan \left( \frac{\pi f(\tau)}{\beta} \right)
\]

\[
= -C \int_0^\beta d\tau \left( \left\{ f, \tau \right\} + \frac{2\pi^2}{\beta^2} f'^2 \right)
\]

where \( \left\{ f, \tau \right\} = \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2 \) the Schwarzian derivative
The Schwarzian theory: goal

Main goal:
Compute all correlation functions:

$$\langle O_{\ell_1} O_{\ell_2} \ldots \rangle_\beta = \frac{1}{Z} \int_\mathcal{M} [\mathcal{D} f] O_{\ell_1} O_{\ell_2} \ldots e^{C \int_0^\beta d\tau \left( \{ f, \tau \} + \frac{2\pi^2}{\beta^2} f'^2 \right)}$$

with action

$$S[f] = -C \int_0^\beta d\tau \left\{ F, \tau \right\}, \quad F \equiv \tan \left( \frac{\pi f(\tau)}{\beta} \right)$$

$$= -C \int_0^\beta d\tau \left( \left\{ f, \tau \right\} + \frac{2\pi^2}{\beta^2} f'^2 \right)$$

where \( \left\{ f, \tau \right\} = \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2 \) the Schwarzian derivative

with \( \mathcal{M} = \text{Diff}(S^1)/\text{SL}(2, \mathbb{R}), \quad f(\tau + \beta) = f(\tau) + \beta \)
The Schwarzian theory: goal

Main goal:
Compute all correlation functions:

$$\langle O_{\ell_1} O_{\ell_2} \ldots \rangle_\beta = \frac{1}{Z} \int_{\mathcal{M}} [\mathcal{D} f] O_{\ell_1} O_{\ell_2} \ldots e^{C \int_0^\beta d\tau \left( \left\{ f, \tau \right\} + \frac{2\pi^2}{\beta^2} f'^2 \right)}$$

with action

$$S[f] = -C \int_0^\beta d\tau \left\{ F, \tau \right\}, \quad F \equiv \tan \left( \frac{\pi f(\tau)}{\beta} \right)$$

$$= -C \int_0^\beta d\tau \left( \left\{ f, \tau \right\} + \frac{2\pi^2}{\beta^2} f'^2 \right)$$

where $\left\{ f, \tau \right\} = \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2$ the Schwarzian derivative

with $\mathcal{M} = \text{Diff}(S^1)/\text{SL}(2, \mathbb{R}), \quad f(\tau + \beta) = f(\tau) + \beta$

In particular: finding gravitational physics in exact correlators
Bilocal operators:

\[ O_\ell(\tau_1, \tau_2) \equiv \left( \frac{F'(\tau_1)F'(\tau_2)}{(F(\tau_1) - F(\tau_2))^2} \right)^\ell \equiv \left( \frac{f'(\tau_1)f'(\tau_2)}{\frac{\beta}{\pi} \sin^2 \frac{\pi}{\beta} [f(\tau_1) - f(\tau_2)]} \right)^\ell \]

Think of this expression as two-point function

\[ O_\ell(\tau_1, \tau_2) = \langle O(\tau_1)O(\tau_2) \rangle_{\text{CFT}} \] of some 1D ‘matter CFT’ at finite temperature coupled to the Schwarzian theory
Stanford and Witten demonstrated that Schwarzian partition function is one-loop exact Stanford-Witten '17.

\[ Z(\beta) = \left(\frac{\pi \beta}{2}\right)^{3/2} \exp\left(\frac{\pi^2 \beta}{2}\right) = \int_{-\infty}^{+\infty} dk \sinh(2\pi k) e^{-\beta k^2} \]

Density of states \( \rho(E) = \sinh(2\pi\sqrt{E}) \)

⇒ Cardy scaling at high energies

\[ \rho(E) \sim e^{2\pi\sqrt{E}} \]

⇒ 2d CFT origin?

What about correlators? Need other techniques
Stanford and Witten demonstrated that Schwarzian partition function is **one-loop exact** Stanford-Witten '17:

\[
Z(\beta) = \left(\frac{\pi}{\beta}\right)^{3/2} \exp\left(\frac{\pi^2}{\beta}\right) = \int_0^{+\infty} dk^2 \sinh(2\pi k)e^{-\beta k^2} \quad (C = 1/2)
\]

Density of states \(\rho(E) = \sinh(2\pi \sqrt{E})\)
Stanford and Witten demonstrated that Schwarzian partition function is one-loop exact:  

\[ Z(\beta) = \left( \frac{\pi}{\beta} \right)^{3/2} \exp \left( \frac{\pi^2}{\beta} \right) = \int_0^{+\infty} dk^2 \sinh(2\pi k) e^{-\beta k^2} \quad (C = 1/2) \]

Density of states  
\[ \rho(E) = \sinh(2\pi \sqrt{E}) \]

\[ \Rightarrow \quad \text{Cardy scaling at high energies} \quad \rho(E) \sim e^{2\pi \sqrt{E}} \]

\[ \Rightarrow \quad \text{2d CFT origin ?} \]

What about correlators? Need other techniques
Observation:

\[ \text{Tr}_0(q^{L_0}) \equiv \chi_0(q) = q^{\frac{1-c}{24}} \frac{(1-q)}{\eta(\tau)} = \int \frac{dP}{|\Psi_{ZZ}(P)|^2}, \quad q = e^{-2\pi t} \]
Partition function: Vacuum character

Observation:
\[ \text{Tr}_0(q^{L_0}) \equiv \chi_0(q) = \frac{q^{1-c/24}(1-q)}{\eta(\tau)} = \left( \frac{\sqrt{2}}{\sqrt{3}} \right), \quad q = e^{-2\pi t} \]

Take double-scaling limit \( c \to \infty, \quad t = \frac{12\pi \beta}{c} \to 0 \)

The Schwarzian and black hole physics
Observation:
\[ \text{Tr}_0(q^{L_0}) \equiv \chi_0(q) = \frac{1-c}{24} \frac{(1-q)}{\eta(\tau)} = \int_{0}^{1} d\tau, \quad q = e^{-2\pi t} \]

Take double-scaling limit \( c \to \infty, \quad t = \frac{12\pi \beta}{c} \to 0 \)

\[ Z(\beta) \sim \left( \frac{\pi}{\beta} \right)^{3/2} \exp \left( \frac{\pi^2}{\beta} \right) \]
Observation:

\[ \text{Tr}_0(q^{L_0}) \equiv \chi_0(q) = \frac{q^{\frac{1-c}{24}}(1-q)}{\eta(\tau)} = 0, \quad q = e^{-2\pi t} \]

Take double-scaling limit \( c \to \infty, \quad t = \frac{12\pi \beta}{c} \to 0 \)

\[ Z(\beta) \sim \left( \frac{\pi}{\beta} \right)^{3/2} \exp \left( \frac{\pi^2}{\beta} \right) \]

**Schwarzian limit** from 2d CFT: Cylinder radius \( \to 0 \) while \( c \to \infty \) keeping product fixed

From Liouville perspective:
Observation:
\[ \text{Tr}_0 (q^{L_0}) \equiv \chi_0 (q) = \frac{1-c}{24} \frac{(1-q)}{\eta(\tau)} = q^{1/12}, \quad q = e^{-2\pi t} \]

Take double-scaling limit \( c \to \infty, \quad t = \frac{12\pi \beta}{c} \to 0 \)

\[ Z(\beta) \sim \left( \frac{\pi}{\beta} \right)^{3/2} \exp \left( \frac{\pi^2}{\beta} \right) \]

Schwarzian limit from 2d CFT: Cylinder radius \( \to 0 \) while \( c \to \infty \) keeping product fixed

From Liouville perspective:
\[ \chi_0 (q) = \text{Cyl. amplitude of Liouville between } ZZ = \langle ZZ | \tilde{q}^{L_0} | ZZ \rangle \]
Observation:
\[ \text{Tr}_0(q^{L_0}) \equiv \chi_0(q) = \frac{1-c}{24} (1-q) = \chi(\frac{i}{\tau}), \quad q = e^{-2\pi t} \]

Take double-scaling limit \( c \to \infty, \quad t = \frac{12\pi \beta}{c} \to 0 \)

\[ Z(\beta) \sim \left( \frac{\pi}{\beta} \right)^{3/2} \exp \left( \frac{\pi^2}{\beta} \right) \]

Schwarzian limit from 2d CFT: Cylinder radius \( \to 0 \) while \( c \to \infty \) keeping product fixed

From Liouville perspective:

\[ \chi_0(q) = \text{Cyl. amplitude of Liouville between } ZZ = \langle ZZ|\tilde{q}^{L_0}|ZZ \rangle \]

\[ |ZZ\rangle = \int_0^\infty dP \Psi_{ZZ}(P) \| P \rangle \Rightarrow \chi_0 = \int_0^\infty dP |\Psi_{ZZ}(P)|^2 e^{-\beta \frac{P^2}{b^2}} \]
Observation:
\[
\text{Tr}_0 (q^{L_0}) \equiv \chi_0(q) = \frac{q^{1-c}_{24}(1-q)}{\eta(e)} = \chi_0(q) = q^{1-e_{-2\pi t}}
\]
Take double-scaling limit \( c \to \infty, t = 12\pi \beta/c \to 0 \)

\[
Z(\beta) \sim \left( \frac{\pi}{\beta} \right)^{3/2} \exp \left( \frac{\pi^2}{\beta} \right)
\]
Schwarzian limit from 2d CFT: Cylinder radius \( \to 0 \) while \( c \to \infty \) keeping product fixed

From Liouville perspective:
\[
\chi_0(q) = \text{Cyl. amplitude of Liouville between } ZZ = \langle ZZ|\tilde{q}^{L_0}|ZZ \rangle
\]

\[
|ZZ\rangle = \int_{0}^{\infty} dP \psi_{ZZ}(P) ||P\rangle \quad \Rightarrow \quad \chi_0 = \int_{0}^{\infty} dP |\psi_{ZZ}(P)|^2 e^{-\beta \frac{P^2}{b^2}}
\]
with density
\[
|\psi_{ZZ}(P)|^2 = \sinh(2\pi bP) \sinh\left( \frac{2\pi P}{b} \right) \quad c = 1 + 6(b + \frac{1}{b})^2
\]
In Schwarzian limit: \( P = bk, b \to 0 \)
One can prove via Liouville phase space path integral between ZZ-branes:

\[
\int_{\phi(0)=\phi(T)} \mathcal{D}\phi \mathcal{D}\pi_\phi \ e^{\int_0^T d\tau \int d\sigma \left( i \pi_\phi \dot{\phi} - \mathcal{H}(\phi, \pi_\phi) \right)}
\]

- **Field redefinition** \((\phi, \pi_\phi) \rightarrow (A, B)\) Gervais-Neveu '82:

\[
e^\phi = -8 \frac{A_\sigma B_\sigma}{(A - B)^2}, \quad \pi_\phi = \frac{A_{\sigma\sigma}}{A_\sigma} - \frac{B_{\sigma\sigma}}{B_\sigma} - 2 \frac{A_\sigma + B_\sigma}{(A - B)}
\]
One can prove via Liouville phase space path integral between ZZ-branes:
\[ \int_{\phi(0) = \phi(T)} \mathcal{D}\phi \mathcal{D}\pi_\phi \ e^{\int_0^T d\tau \int d\sigma \left( i\pi_\phi \dot{\phi} - \mathcal{H}(\phi, \pi_\phi) \right)} \]

- **Field redefinition** \((\phi, \pi_\phi) \rightarrow (A, B)\) Gervais-Neveu '82:
  \[
  e^\phi = -8 \frac{A_\sigma B_\sigma}{(A - B)^2}, \quad \pi_\phi = \frac{A_{\sigma\sigma}}{A_\sigma} - \frac{B_{\sigma\sigma}}{B_\sigma} - 2 \frac{A_\sigma + B_\sigma}{(A - B)}
  \]

  \[ \Rightarrow \mathcal{H} = -\frac{c}{24\pi} \{A(\sigma, \tau), \sigma\} - \frac{c}{24\pi} \{B(\sigma, \tau), \sigma\} \]

- **Schwarzian limit**: \(\pi_\phi \dot{\phi} \rightarrow 0\)
- **ZZ-brane boundary conditions**: \(A, B\) written in terms of 1 doubled field \(F\)
One can prove via Liouville phase space path integral between ZZ-branes:

\[
\int_{\phi(0)=\phi(T)} \mathcal{D}[\phi] [\mathcal{D}[\pi_\phi]] e^{\int_0^T d\tau \int d\sigma (i\pi_\phi \dot{\phi} - \mathcal{H}(\phi, \pi_\phi))}
\]

- **Field redefinition** \((\phi, \pi_\phi) \rightarrow (A, B)\) Gervais-Neveu '82:

  \[
e^\phi = -8 \frac{A_\sigma B_\sigma}{(A - B)^2}, \quad \pi_\phi = \frac{A_\sigma \sigma}{A_\sigma} - \frac{B_\sigma \sigma}{B_\sigma} - 2 \frac{A_\sigma + B_\sigma}{(A - B)}
  \]

  \[
  \Rightarrow \mathcal{H} = -\frac{c}{24\pi} \left\{ A(\sigma, \tau), \sigma \right\} - \frac{c}{24\pi} \left\{ B(\sigma, \tau), \sigma \right\}
  \]

- **Schwarzian limit:** \(\pi_\phi \dot{\phi} \rightarrow 0\)

- **ZZ-brane boundary conditions:** \(A, B\) written in terms of 1 doubled field \(F\)

**Liouville-Schwarzian Dictionary:**

- \(T(w) \rightarrow -\frac{c}{24\pi} \left\{ F(\sigma), \sigma \right\}\)

- \(e^\phi \rightarrow \frac{F'_1 F'_2}{(F_1 - F_2)^2}\)
**2-point correlator**

**Idea:** \( V_\ell(\tau_1, \tau_2) \equiv e^{2\ell \phi(\tau_1, \tau_2)} \leftrightarrow O_\ell(\tau_1, \tau_2) \)
2-point correlator

**Idea:** \( V_\ell(\tau_1, \tau_2) \equiv e^{2\ell \phi(\tau_1, \tau_2)} \leftrightarrow O_\ell(\tau_1, \tau_2) \)

\[
\langle ZZ | V_\ell | ZZ \rangle = \int dPdQ \, \psi^*_ZZ(P) \psi_ZZ(Q) \langle \langle P || V_\ell || Q \rangle \rangle
\]
2-point correlator

**Idea:** \( V_\ell(\tau_1, \tau_2) \equiv e^{2\ell \phi(\tau_1, \tau_2)} \leftrightarrow O_\ell(\tau_1, \tau_2) \)

\[
\langle ZZ| V_\ell | ZZ \rangle = \int dPdQ \psi^*_ZZ(P)\psi_ZZ(Q)\langle \langle P||V_\ell||Q \rangle \rangle
\]

Two ways of computing matrix element:

- **Schwarzian limit:** \( ||Q\rangle \rightarrow |Q\rangle \)
  \( \Rightarrow 3\)-point function on sphere \( \Rightarrow \) large \( c \) limit of DOZZ formula
2-point correlator

Idea: \( V_\ell(\tau_1, \tau_2) \equiv e^{2\ell \phi(\tau_1, \tau_2)} \longleftrightarrow O_\ell(\tau_1, \tau_2) \)

\[ \langle ZZ | V_\ell | ZZ \rangle = \int dPdQ \psi^*_{ZZ}(P)\psi_{ZZ}(Q)\langle\langle P||V_\ell||Q\rangle\rangle \]

Two ways of computing matrix element:

- **Schwarzian limit:** \( ||Q\rangle \rightarrow |Q\rangle \)
  \[ \Rightarrow \text{3-point function on sphere } \Rightarrow \text{large } c \text{ limit of DOZZ formula} \]

- **Computation in minisuperspace** regime of 2d Liouville CFT:
  \[ \langle P|e^{-(\beta-\tau)H}e^{2\ell \phi}e^{-\tau H}|Q\rangle \]
  Explicitly evaluate minisuperspace integrals
2-point correlator

Idea: \[ V_\ell(\tau_1, \tau_2) \equiv e^{2\ell \phi(\tau_1, \tau_2)} \leftrightarrow O_\ell(\tau_1, \tau_2) \]

\[ \langle ZZ \vert V_\ell \vert ZZ \rangle = \int dPdQ \, \psi^*_Z(P) \psi_Z(Q) \langle \langle P \vert \vert V_\ell \vert \vert Q \rangle \rangle \]

Two ways of computing matrix element:

- **Schwarzian limit:** \[ \langle \langle P \rangle \rangle \rightarrow \langle Q \rangle \]
  \[ \Rightarrow \text{3-point function on sphere} \Rightarrow \text{large } c \text{ limit of DOZZ formula} \]

- **Computation in minisuperspace regime of 2d Liouville CFT:**
  \[ \langle P \vert e^{-(\beta - \tau)H} e^{2\ell \phi} e^{-\tau H} \vert Q \rangle \]
  Explicitly evaluate minisuperspace integrals

Both agree:

\[
G^\beta_\ell(\tau_1, \tau_2) = \frac{1}{Z(\beta)} \int d\mu(k_1) d\mu(k_2) e^{-\tau k_1^2 - (\beta - \tau)k_2^2} \frac{\Gamma(\ell \pm i(k_1 \pm k_2))}{2\sqrt{\pi} \Gamma(2\ell)}
\]

\[ d\mu(k) \equiv dk^2 \sinh(2\pi k) \]
Semi-classical regime $C \to \infty$, $\ell \ll C$: $k_1 \sim k_2 \gg 1$

Redefine $k_1^2 = M + \omega$, $k_2^2 = M$, $M \gg \omega$

$$G_\ell^\pm \sim \int_0^\infty dMe^{2\pi \sqrt{M} - \frac{\beta}{2C} M} \int \frac{d\omega}{2\pi} e^{\pm i \frac{\tau}{2C} \omega + \pi \frac{\omega}{2\sqrt{M}}} \frac{\Gamma\left(\ell \pm i \frac{\omega}{2\sqrt{M}}\right)}{\Gamma(2\ell)} (2\sqrt{M})^{2\ell - 1}$$
Application: Semiclassics for light operators

Semi-classical regime $C \to \infty$, $\ell \ll C$: $k_1 \sim k_2 \gg 1$

Redefine $k^2_1 = M + \omega$, $k^2_2 = M$, $M \gg \omega$

\[ G^\pm_\ell \sim \int_0^\infty dM e^{2\pi \sqrt{M - \frac{\beta}{2C}M}} \int \frac{d\omega}{2\pi} e^{\pm i \frac{\tau}{2C} \omega + \pi \frac{\omega}{2\sqrt{M}}} \frac{\Gamma(\ell \pm i \frac{\omega}{2\sqrt{M}})}{\Gamma(2\ell)} (2\sqrt{M})^{2\ell - 1} \]

Interpretation:

- $M$-integral has saddle: $M_0 = 4\pi^2 C^2 / \beta^2$, the JT black hole
- $E(T)$-relation

Remaining $\omega$-integral is done explicitly to yield:

\[ G^\pm, cl_\ell(\tau_1, \tau_2) = \left( \frac{\pi}{\beta \sinh \frac{\pi}{\beta} \tau_{12}} \right)^{2\ell} \]
Semi-classical regime $C \to \infty$, $\ell \ll C$: $k_1 \sim k_2 \gg 1$

Redefine $k_1^2 = M + \omega$, $k_2^2 = M$, $M \gg \omega$

$$G_{\ell}^{\pm} \sim \int_0^{\infty} dMe^{2\pi \sqrt{M} \frac{\beta}{2C} M} \int \frac{d\omega}{2\pi} e^{\pm i \frac{\tau}{2C} \omega + \pi \frac{\omega}{2\sqrt{M}}} \frac{\Gamma(\ell \pm i \frac{\omega}{2\sqrt{M}})}{\Gamma(2\ell)} (2\sqrt{M})^{2\ell-1}$$

Interpretation:

- $M$-integral has saddle: $M_0 = 4\pi^2 C^2 / \beta^2$, the JT black hole $E(T)$-relation

Remaining $\omega$-integral is done explicitly to yield:

$$G_{\ell}^{\pm,cl}(\tau_1, \tau_2) = \left( \frac{\pi}{\beta \sinh \frac{\pi}{\beta} \tau_{12}} \right)^{2\ell}$$

- $\Gamma$-functions give poles: $\ell - i \frac{\omega}{2\sqrt{M}} = -n$

Matches with quasi-normal modes of AdS$_2$ BH metric Keeler-Ng '14: $\omega = -i \frac{2\pi}{\beta} (n + \ell)$
Semi-classical regime $C \to \infty$, $\ell \ll C$: $k_1 \sim k_2 \gg 1$

Redefine $k_1^2 = M + \omega$, $k_2^2 = M$, $M \gg \omega$

\[ G_\ell^\pm \sim \int_0^\infty dM e^{2\pi \sqrt{M} - \frac{\beta}{2C} M} \int \frac{d\omega}{2\pi} e^{\pm i \frac{\tau_1}{2C} \omega + \pi \frac{\omega}{2\sqrt{M}} \frac{\Gamma(\ell \pm i \frac{\omega}{2\sqrt{M}})}{\Gamma(2\ell)} (2\sqrt{M})^{2\ell} - 1} \]

**Interpretation:**

- $M$-integral has saddle: $M_0 = 4\pi^2 C^2 / \beta^2$, the JT black hole $E(T)$-relation

Remaining $\omega$-integral is done explicitly to yield:

\[ G_\ell^{\pm, cl}(\tau_1, \tau_2) = \left( \frac{\pi}{\beta \sinh \frac{\pi}{\beta} \tau_1 \tau_2} \right)^{2\ell} \]

- $\Gamma$-functions give poles: $\ell - i \frac{\omega}{2\sqrt{M}} = -n$

Matches with quasi-normal modes of AdS$_2$ BH metric Keeler-Ng '14: $\omega = -i \frac{2\pi}{\beta} (n + \ell)$

- Quantum black hole $M$ emits and reabsorbs excitation with mass $\sim \ell$ and energy $\omega$
Time-ordered 4-point function

\[ \langle ZZ| V_{\ell_1} V_{\ell_2} |ZZ\rangle \]

Evaluated using Conformal blocks
Time-ordered 4-point function

\[ \langle ZZ \mid V_{\ell_1} V_{\ell_2} \mid ZZ \rangle \]

Evaluated using Conformal blocks

\[ G_{\ell_1 \ell_2} = \int dP \, dQ \, dP_s \, \Psi_{ZZ}^*(P) \Psi_{ZZ}(Q) \times \]

\[ \begin{array}{c}
Q \\
\ell_1 \\
\ell_1 \\
Q
\end{array} \times \begin{array}{c}
P_s \\
\ell_2 \\
\ell_2 \\
P_s
\end{array} \times \begin{array}{c}
P \\
\ell_1 \\
\ell_1 \\
P
\end{array} \]

Conformal blocks dominated by primary in the intermediate channel ⇒ reduce to DOZZ OPE coefficients

\[ G_{\beta \ell_1 \ell_2} = \int dk_1 \, dk_2 \, 4 \, dk_3 \, \sinh^2 \pi k_1 \sinh^2 \pi k_2 \sinh^2 \pi k_3 \times \]

\[ e^{-k_1^2 (\tau_2 - \tau_1)} - k_2^2 (\tau_4 - \tau_3) - k_3^2 (\beta - \tau_2 + \tau_3 - \tau_4 + \tau_1) \Gamma(\ell_2 \pm ik_2 \pm ik_3) \Gamma(\ell_1 \pm ik_1 \pm ik_3) \Gamma(2 \ell_1) \Gamma(2 \ell_2) \]
Time-ordered 4-point function

\[ \langle ZZ | V_{\ell_1} V_{\ell_2} | ZZ \rangle \]

Evaluated using Conformal blocks

\[ G_{\ell_1 \ell_2} = \int dP dQ dP_s \ \psi_{ZZ}^*(P) \psi_{ZZ}(Q) \times \]

Conformal blocks dominated by primary in the intermediate channel \( \Rightarrow \) reduce to DOZZ OPE coefficients
Time-ordered 4-point function

\[ \langle ZZ \mid V_{\ell_1} V_{\ell_2} \mid ZZ \rangle \]

Evaluated using Conformal blocks

\[ G_{\ell_1 \ell_2} = \int \, dP \, dQ \, dP_s \, \psi^{*}_{ZZ}(P) \psi_{ZZ}(Q) \times \]

\[ \begin{array}{cccc}
Q & Q \\
\ell_1 & \ell_1 \\
P_s & \times & P_s \\
\ell_2 & \ell_2 \\
P & P \\
\end{array} \]

Conformal blocks dominated by primary in the intermediate channel \( \Rightarrow \) reduce to DOZZ OPE coefficients

\[ G^\beta_{\ell_1 \ell_2} = \int \, dk_1^2 \, dk_4^2 \, dk_s^2 \, \sinh 2\pi k_1 \sinh 2\pi k_4 \sinh 2\pi k_s \]

\[ \times \, e^{-k_1^2(\tau_2 - \tau_1) - k_4^2(\tau_4 - \tau_3) - k_s^2(\beta - \tau_2 + \tau_3 - \tau_4 + \tau_1)} \, \frac{\Gamma(\ell_2 \pm ik_4 \pm ik_s) \Gamma(\ell_1 \pm ik_1 \pm ik_s)}{\Gamma(2\ell_1) \Gamma(2\ell_2)} \]
Diagrammatic decomposition

Rules:

1. $\tau_2 = e^{-k^2(\tau_2 - \tau_1)}$, $\ell_{k_1/k_2} = \gamma_{\ell}(k_1, k_2) = \sqrt{\Gamma(\ell \pm ik_1 \pm ik_2)}/\Gamma(2\ell)$

2. Integrate over intermediate momenta $k_i$ with measure $d\mu(k) = dk^2 \sinh(2\pi k)$.
Diagrammatic decomposition

Rules:

- \[ \tau_2 \rightarrow e^{-k^2(\tau_2 - \tau_1)} , \quad \ell \rightarrow k_1 \]

- Integrate over intermediate momenta \( k_i \) with measure
  \[ d\mu(k) = dk^2 \sinh(2\pi k) \]

Examples:

- \[ Z(\beta) = \quad \langle O_\ell(\tau_1, \tau_2) \rangle = \tau_2 \ell \tau_1 \]

- \[ \langle O_{\ell_1}(\tau_1, \tau_2) O_{\ell_2}(\tau_3, \tau_4) \rangle = \tau_2 \ell_1 \ell_2 \tau_4 \]

Note: non-perturbative in Schwarzian coupling \( C \)
Swapping two operators in 4-point correlator, means the conformal block is dominated by its primary in a different channel:

$$\mathcal{F}_{P_s[2 \ 3 \ 4]}(z') = \int dP_t \ R_{P_sP_t} \mathcal{F}_{P_t[3 \ 2 \ 4]}(1/z')$$
Swapping two operators in 4-point correlator, means the conformal block is dominated by its primary in a different channel:

\[ \mathcal{F}_{P_s}[\begin{array}{c} 2 \\ 1 \\ 3 \\ 4 \end{array}](z') = \int dP_t \ R_{P_s P_t}[\begin{array}{c} 2 \\ 3 \\ 1 \\ 4 \end{array}] \ \mathcal{F}_t[\begin{array}{c} 3 \\ 1 \\ 2 \\ 4 \end{array}](1/z') \]

Blocks unknown in general, but \( R \)-matrix is known as quantum 6j-symbol of \( U_q(SL(2, \mathbb{R})) \) \cite{Teschner-Ponsot '99}
Swapping two operators in 4-point correlator, means the conformal block is dominated by its primary in a different channel:

$$\mathcal{F}_{P_s}[^2_1^3_4](z') = \int dP_t \; R_{P_sP_t}[^2_1^3_4] \; \mathcal{F}_{P_t}[^3_2_1_4](1/z')$$

Blocks unknown in general, but \(R\)-matrix is known as quantum 6j-symbol of \(U_q(SL(2, \mathbb{R}))\) \(\text{Teschner-Ponsot '99}\)

\(\Rightarrow\) **Schwarzian limit** of quantum 6j-symbol is 6j-symbol with 4 continuous and 2 discrete \(SL(2, \mathbb{R})\) labels
Swapping two operators in 4-point correlator, means the conformal block is dominated by its primary in a different channel:

$$\mathcal{F}_{P_s}[\begin{array}{c} 2 \\ 1 \\ 3 \\ 4 \end{array}](z') = \int dP_t \ R_{P_s P_t}[\begin{array}{c} 2 \\ 1 \\ 3 \\ 4 \end{array}] \mathcal{F}_{P_t}[\begin{array}{c} 3 \\ 2 \\ 1 \\ 4 \end{array}](1/z')$$

Blocks unknown in general, but $R$-matrix is known as quantum 6j-symbol of $U_q(SL(2, \mathbb{R}))$ Teschner-Ponsot ’99

$\Rightarrow$ Schwarzian limit of quantum 6j-symbol is 6j-symbol with 4 continuous and 2 discrete $SL(2, \mathbb{R})$ labels

Explicitly:

$$R_{P_s P_t}[\begin{array}{c} 2 \\ 1 \\ 3 \\ 4 \end{array}] \sim \left\{ \begin{array}{c} \ell_1 \\ \ell_3 \\ k_2 \\ k_4 \end{array} \right\} \frac{\Gamma(\ell_1 + ik_2 \pm ik_s) \Gamma(\ell_3 - ik_2 \pm ik_t) \Gamma(\ell_1 - ik_4 \pm ik_t) \Gamma(\ell_3 + ik_4 \pm ik_s)}{\sqrt{\Gamma(\ell_1 - ik_2 \pm ik_s) \Gamma(\ell_3 + ik_2 \pm ik_t) \Gamma(\ell_1 + ik_4 \pm ik_t) \Gamma(\ell_3 - ik_4 \pm ik_s)}}$$

$$\times \frac{i}{2\pi i} \int_{-i\infty}^{i\infty} du \ \frac{\Gamma(u) \Gamma(u-2ik_s) \Gamma(u+ik_2+4-s+t) \Gamma(u-ik_{s+t}-2-4) \Gamma(\ell_1+ik_{s-2}-u) \Gamma(\ell_3+ik_{s-4}-u)}{\Gamma(u+\ell_1-ik_{s-2}) \Gamma(u+\ell_3-ik_{s-4})}$$
At the Schwarzian level, this procedure is captured by the diagram:

\[
e^{-k_1^2(\tau_3-\tau_1)-k_t^2(\tau_3-\tau_2)-k_4^2(\tau_4-\tau_2)-k_s^2(\beta-\tau_4+\tau_1)}
\times \gamma_{\ell_1}(k_1, k_s) \gamma_{\ell_2}(k_s, k_4) \gamma_{\ell_1}(k_4, k_t) \gamma_{\ell_2}(k_t, k_1) \times \left\{ \begin{array}{ccc} \ell_1 & k_1 & k_s \\ \ell_2 & k_4 & k_t \end{array} \right\}
\]
Application: Shockwaves from semiclassics

Time delay:

\[ \tilde{t}_2 - t_2 \sim e^{\lambda_M(t_2 - t_1)}, \quad \lambda_M = \frac{2\pi}{\beta_M} \]
Application: Shockwaves from semiclassics

Time delay:
\[ \tilde{t}_2 - t_2 \sim e^{\lambda_M(t_2-t_1)}, \quad \lambda_M = \frac{2\pi}{\beta_M} \]

Shenker-Stanford: OTO-correlator \[ \langle V_1 W_3 V_2 W_4 \rangle \] in boundary theory captures this behavior

Shenker-Stanford '15
Time delay:
\[ \tilde{t}_2 - t_2 \sim e^{\lambda_M (t_2 - t_1)}, \quad \lambda_M = \frac{2\pi}{\beta_M} \]

Shenker-Stanford: OTO-correlator \[ \langle V_1 W_3 V_2 W_4 \rangle \] in boundary theory captures this behavior \cite{Shenker-Stanford '15}

\[ \langle V_1 W_3 V_2 W_4 \rangle, \text{ written using shockwaves in the AdS}_2 \text{ bulk as} \]
\[ \int_{0}^{+\infty} dq_+ \int_{0}^{+\infty} dp_- \Psi_{1}^*(q_+)\Phi_{3}^*(p_-)S(p_-, q_+)\Psi_{2}(q_+)\Phi_{4}(p_-) \]
**Application: Shockwaves from semiclassics**

Time delay:
\[ \tilde{t}_2 - t_2 \sim e^{\lambda_M(t_2-t_1)} \]
where
\[ \lambda_M = \frac{2\pi}{\beta M} \]

Shenker-Stanford: OTO-correlator
\[ \langle V_1 W_3 V_2 W_4 \rangle \] in boundary theory captures this behavior

Shenker-Stanford '15

\[ \langle V_1 W_3 V_2 W_4 \rangle, \text{written using shockwaves in the AdS}_2 \text{ bulk as} \]
\[ \int_0^{+\infty} dq_+ \int_0^{+\infty} dp_- \psi_1^*(q_+)\Phi_3^*(p_-)S(p_, q_+)\psi_2(q_+)\Phi_4(p_-) \]

- \( \psi, \Phi = \text{Kruskal wavefunctions} = \text{bulk-to-boundary propagators} \)
- \( S = \exp \left( \frac{i\beta}{4\pi C} p_- q_+ \right) \) the Dray-'t Hooft shockwave S-matrix
Large $C$ limit of complete OTO 4-point function, with light $\ell$, gives full eikonal shockwave expressions

**Exact match $\Rightarrow$ Derived full shockwave in semiclassical regime!**
Summary: Structural link theories

- 3d Gravity
- 2d Liouville
- 2d JT Gravity
- 1d Schwarzian

Holography

Dimensional Reduction

The Schwarzian and black hole physics

Thomas Mertens
Generalization: Group models

Dim. Red.
- 3d Chern-Simons
- 2d BF Theory

Holography
- 2d WZW
- 1d particle on group

Relevant for SYK-type models with internal symmetries

Correlators determined using 2d WZW CFT techniques

$\tau_1, \tau_2, j_m$:

$e^{-C j_m (\tau_2 - \tau_1)}$

$J M = (j_1 j_2 J m_1 m_2 M)$
Generalization: Group models

Holography

3d Chern-Simons ↔ 2d WZW
Dim. Red.

2d BF Theory ↔ 1d particle on group

Relevant for SYK-type models with internal symmetries
Generalization: Group models

Holography

3d Chern-Simons ↔ 2d WZW

Dim. Red.

2d BF Theory ↔ 1d particle on group

Relevant for SYK-type models with internal symmetries

Correlators determined using 2d WZW CFT techniques

\[ e^{-C_j(\tau_2 - \tau_1)} , \]

\[ JM_{j_1m_1}^{j_2m_2} = \left( \begin{array}{ccc} j_1 & j_2 & J \\ m_1 & m_2 & M \end{array} \right) \]

The Schwarzian and black hole physics

Thomas Mertens
JT is Hamiltonian reduction of \( SL(2, \mathbb{R}) \) BF
Wilson line perpective

JT is Hamiltonian reduction of SL(2, \(\mathbb{R}\)) BF

with action \(S = \int dt \text{Tr}\Phi F\)

Boundary-anchored Wilson lines are bilocal operators: in progress

\[\left\langle \mathcal{P} e^{\int A} \right\rangle = \left\langle g(\tau_f)g^{-1}(\tau_i) \right\rangle\]

Reduce to Schwarzian bilocals for the constrained SL(2, \(\mathbb{R}\)) case
Wilson line perspective

JT is Hamiltonian reduction of SL(2, \(\mathbb{R}\)) BF

with action \(S = \int dt \text{Tr} \Phi F\)

Boundary-anchored Wilson lines are bilocal operators: in progress

\[ \left\langle \mathcal{P} e^{\int A} \right\rangle = \left\langle g(\tau_f)g^{-1}(\tau_i) \right\rangle \]

Reduce to Schwarzian bilocals for the constrained SL(2, \(\mathbb{R}\)) case

Crossing of Wilson lines \(\Rightarrow 6j\)-symbol of group \(G\)
Schwarzian QM is relevant as

- Low-energy universal gravity sector of all SYK-type models
  Cfr. Liouville compared to holographic 2d CFT
Schwarzian QM is relevant as

- **Low-energy universal gravity** sector of all SYK-type models
  Cfr. Liouville compared to holographic 2d CFT
- holographically dual to JT dilaton gravity
Schwarzian QM is relevant as

- **Low-energy universal gravity** sector of all SYK-type models
  Cfr. Liouville compared to holographic 2d CFT
- holographically dual to JT dilaton gravity

Theory is **exactly solvable** and displays a wide variety of black hole quantum physics:

- Virtual black hole intermediate states
- Quasinormal modes and shockwaves in the semi-classical regime
Schwarzian QM is relevant as

- **Low-energy universal gravity** sector of all SYK-type models
  Cfr. Liouville compared to holographic 2d CFT
- holographically dual to JT dilaton gravity

Theory is **exactly solvable** and displays a wide variety of black hole quantum physics:

- Virtual black hole intermediate states
- Quasinormal modes and shockwaves in the semi-classical regime

*Naturally embedded within Liouville theory*
Thank you!