Transverse-momentum
dependence of gluon
distributions at small-x

Cyrille Marquet

Centre de Physique Théorique
Ecole Polytechnique & CNRS

Contents of the talk

• The need for TMDs at colliders
  TMDs = transverse-momentum-dependent parton distributions

• The context for this talk: forward di-jets at the LHC
  their structure of may be modified in p+Pb vs p+p collisions

• Gluon TMDs in the small-x limit
  their (non-linear) QCD evolution can be obtained from the so-called JIMWLK equation

• Numerical results
  new insight regarding the low-momentum behavior (gluon saturation regime)
The need for TMDs at hadron (and other) colliders
Collinear factorization

In standard pQCD calculations, the incoming parton transverse momenta are set to zero in the matrix element and are integrated over in the parton densities.

\[ d\sigma_{AB \rightarrow X} = \sum_{ij} \int dx_1 dx_2 \, f_i/A(x_1, \mu^2) f_j/B(x_2, \mu'^2) \, d\hat{\sigma}_{ij \rightarrow X} + O\left(\Lambda_{QCD}^2/M^2\right) \]

- \(k_T\) integrated quantities
- The incoming partons are taken collinear to the projectile hadrons
- Some hard scale
Collinear factorization

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\[ d\sigma_{AB\to X} = \sum_{ij} \int dx_1 dx_2 \ f_{i/A}(x_1, \mu^2) f_{j/B}(x_2, \mu'^2) \ d\hat{\sigma}_{ij\to X} + \mathcal{O}\left(\frac{\Lambda_{QCD}^2}{M^2}\right) \]

\[ \text{k}_T \text{ integrated quantities} \]

the incoming partons are taken collinear to the projectile hadrons

in general for a hard process, this approximation is accurate in some cases however, this is not good enough (examples follow)

TMD factorization is a more advanced QCD factorization framework which can be useful and sometimes is even necessary
the transverse momentum of the lepton pair $q_T$ is the sum of the transverse momenta of the incoming partons

$$d\hat{\sigma} \propto \delta(k_{1t} + k_{2t} - q_T)$$

so in collinear factorization

$$d\sigma^{AB \rightarrow l^+ l^- X} \propto \delta(q_T) + \mathcal{O}(\alpha_s)$$

and TMDs could be useful here
Drell-Yan process

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and TMDs could be useful here

naively, TMD factorization is

$$d\sigma^{AB \rightarrow l^+ l^- X} = \sum_{i,j} \int dx_1 dx_2 d^2k_{1T} d^2k_{2T} f_{i/A}(x_1, k_{1T}) f_{j/B}(x_2, k_{2T}) d\hat{\sigma}^{ij \rightarrow l^+ l^- X}$$

(but unfortunately, there are complications)
In order to be able to trace the relative distance between the partons, one has to use the mixed longitudinal momentum–impact parameter representation which, in the momentum language, reduces to introduction of a mismatch between the transverse momentum of the parton in the amplitude and that of the same parton in the amplitude conjugated.

Multiple parton interactions keeping track of partonic transverse momenta is also crucial to describe multiple partonic interactions.

consider for instance: 4-jet production coming from a double hard scattering of two partons in each incoming hadron
In order to be able to trace the relative distance between the partons, one has to use the mixed longitudinal momentum–impact parameter representation which, in the momentum language, reduces to introduction of a mismatch between the transverse momentum of the parton in the amplitude and that of the same parton in the amplitude conjugated.

Multiple parton interactions keeping track of partonic transverse momenta is also crucial to describe multiple partonic interactions.

Consider for instance: 4-jet production coming from a double hard scattering of two partons in each incoming hadron. There is a kinematical domain in which this is as important as the leading-twist process of 4-jet production in one hard scattering with two partons coming from each hadron, the transverse momentum $\Delta$ can be non-zero.
Spin physics

TMDs are crucial to describe hard processes in polarized collisions (e.g. Drell-Yan and semi-inclusive DIS)

8 leading-twist TMDs

- Sivers function: correlation between transverse spin of the nucleon and transverse momentum of the quark
- Boer-Mulders function: correlation between transverse spin and transverse momentum of the quark in unpolarized nucleon
Our context: forward di-jets

- Large-x projectile (proton) on small-x target (proton or nucleus)

Incoming partons’ energy fractions:

\[
\begin{align*}
  x_1 &= \frac{1}{\sqrt{s}} \left( |p_{1t}|e^{y_1} + |p_{2t}|e^{y_2} \right) \\
  x_2 &= \frac{1}{\sqrt{s}} \left( |p_{1t}|e^{-y_1} + |p_{2t}|e^{-y_2} \right)
\end{align*}
\]

Gluon’s transverse momentum \((p_{1t}, p_{2t} \text{ imbalance})\):

\[
|k_t|^2 = |p_{1t} + p_{2t}|^2 = |p_{1t}|^2 + |p_{2t}|^2 + 2|p_{1t}||p_{2t}|\cos \Delta \phi
\]

\[|p_{1t}, p_{2t}| \gg |k_t|, Q_s\]

Prediction: modification of the \(k_t\) distribution in p+Pb vs p+p collisions
The gluon TMDs involved in the di-jet process
TMD gluon distributions

- the naive operator definition is not gauge-invariant

\[ \mathcal{F}_{g/A}(x_2, k_t) \overset{\text{naive}}{=} 2 \int \frac{d\xi^+ d^2\xi_t}{(2\pi)^3 p_A^-} e^{ix_2 p_A^- \cdot \xi^+ - ik_t \cdot \xi_t} \left\langle A | \text{Tr} \left[ F_{i-}^{a} (\xi^+, \xi_t) F_{i-}^{a} (0) \right] | A \right\rangle \]
TMD gluon distributions

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- a theoretically consistent definition requires to include more diagrams

They all contribute at leading power and need to be resummed.

this is done by including gauge links in the operator definition
Process-dependent TMDs

- the proper operator definition(s)

The path \([\alpha, \beta]\) depends on the hard process.

Gluon TMD, \(F\), is in general process-dependent.

Cross section for dijet production in hadron-hadron collisions cannot be written down with just a single gluon!

\[ F \]qg, \( F \)qg, \( F \)gg, \( F \)gg, \( F \)gg, \( F \)gg

\[ \mathcal{F}_{g/A}(x_2, k_t) = 2 \int \frac{d\xi_+ d^2\xi_t}{(2\pi)^3 p_A^-} e^{ix_2 p_A^- \xi_+ - i k_t \cdot \xi_t} \langle A| \text{Tr} \left[ F^{i-} (\xi_+, \xi_t) U_{[\xi, 0]} F^{i-} (0) \right] |A\rangle \]

- \( U_{[\alpha, \beta]} \) renders gluon distribution gauge invariant

\[ \mathcal{P} \exp \left[ -i g \int_{\alpha}^{\beta} d\eta^{\mu} A^a(\eta) T^a \right] \]
Process-dependent TMDs

- the proper operator definition(s)

\[ \mathcal{F}_{g/A}(x_2, k_t) = 2 \int \frac{d^2 \xi^+ \xi^-}{(2\pi)^3 p_A^-} e^{i x_2 p_A^- \xi^+ - i k_t \cdot \xi_t} \langle A | \text{Tr} \left[ F^{i-}(\xi^+, \xi_t) U_{[\xi,0]} F^{i-}(0) \right] | A \rangle \]

- some gauge link

\[ \mathcal{P} \exp \left[ -ig \int_{\alpha} d\eta A^a(\eta) T^a \right] \]

- \( U_{[\alpha,\beta]} \) renders gluon distribution gauge invariant

- however, the precise structure of the gauge link is process-dependent:
  - it is determined by the color structure of the hard process H

- in the large \( k_t \) limit, the process dependence of the gauge links disappears (like for the integrated gluon distribution), and a single gluon distribution is sufficient
TMDs for forward di-jets

- several gluon distributions are needed already for a single partonic sub-process

example for the $qg^* \rightarrow qg$ channel

each diagram generates a different gluon distribution
TMDs for forward di-jets

- several gluon distributions are needed already for a single partonic sub-process

example for the $qg^* \rightarrow qg$ channel

2 unintegrated gluon distributions per channel, 6 in total: $\Phi_{ag\rightarrow cd}^{(i)}(x_2, k_t^2)$

Kotko, Kutak, CM, Petreska, Sapeta and van Hameren (2015)
The six TMD gluon distributions

- correspond to a different gauge-link structure

\[ \mathcal{F}_{g/A}(x_2, k_t) = 2 \int d\xi^+ d^2\xi_t e^{i x_2 p_A^- \xi^+ - i k_t \cdot \xi_t} \langle A | \text{Tr} \left[ F^{i-} (\xi^+, \xi_t) U_{[\xi,0]} F^{i-} (0) \right] | A \rangle \]

several paths are possible for the gauge links

examples:

- when integrated, they all coincide

\[ \int \mu^2 d^2 k_t \ \Phi^{(i)}_{ag \rightarrow cd}(x_2, k_t^2) = x_2 f(x_2, \mu^2) \]
The six TMD gluon distributions

- correspond to a different gauge-link structure

\[
\mathcal{F}_{g/A}(x_2, k_t) = 2 \int \frac{d\xi^+ d^2 \xi_t}{(2\pi)^3 p_A^-} e^{i x_2 p_A^- \xi^+ - i k_t \cdot \xi_t} \langle A | \text{Tr} \left( F^{i-} (\xi^+, \xi_t) U_{[\xi,0]} F^{i-} (0) \right) | A \rangle
\]

several paths are possible for the gauge links

examples:

- when integrated, they all coincide

\[
\int d^2 k_t \quad \Phi_{ag \to cd}^{(i)}(x_2, k_t^2) = x_2 f(x_2, \mu^2)
\]

- they are independent and in general they all should be extracted from data

only one of them has the probabilistic interpretation of the number density of gluons at small \(x_2\)
Evaluating the gluon TMDs at small-\(x\)
Gluon TMDs at small-x

- the gluon TMDs involved in the di-jet process are:
  
  (showing here the $qg^* \rightarrow qg$ channel TMDs only)

\[
\mathcal{F}_{qg}^{(1)} = 2 \int \frac{d\xi^+ d^2 \xi}{(2\pi)^3 p_A^-} e^{ix_2 p_A^- \xi^+ - ik_t \cdot \xi} \left\langle \text{Tr} \left[ F_{i-}^{--} (\xi) U^{[-]\dagger} F_{i-}^{--} (0) U^{[+]} \right] \right\rangle
\]

\[
\mathcal{F}_{qg}^{(2)} = 2 \int \frac{d\xi^+ d^2 \xi}{(2\pi)^3 p_A^-} e^{ix_2 p_A^- \xi^+ - ik_t \cdot \xi} \left\langle \text{Tr} \left[ F_{i-}^{--} (\xi) \frac{\text{Tr} [U^{[\Box]}]}{N_c} U^{[+]\dagger} F_{i-}^{--} (0) U^{[+]} \right] \right\rangle
\]
Gluon TMDs at small-x

• the gluon TMDs involved in the di-jet process are:
  (showing here the $qg^* \rightarrow qg$ channel TMDs only)

\[
F^{(1)}_{qq} = 2 \int \frac{d\xi^+ d^2 \xi}{(2\pi)^3 p_A^-} e^{ix_2 p_A^- \xi^+ - i k_t \cdot \xi} \left\langle \text{Tr} \left[ F^\gamma_- (\xi) U_{[-]}^\dagger F^\gamma_- (0) U_{[+]} \right] \right\rangle
\]

\[
F^{(2)}_{qq} = 2 \int \frac{d\xi^+ d^2 \xi}{(2\pi)^3 p_A^-} e^{ix_2 p_A^- \xi^+ - i k_t \cdot \xi} \left\langle \text{Tr} \left[ F^\gamma_- (\xi) \frac{\text{Tr}[U^{[\square]]}}{N_c} U_{[+]}^\dagger F^\gamma_- (0) U_{[+]} \right] \right\rangle
\]

• at small $x$ they can be written as:

\[
U_x = \mathcal{P} \exp \left[ ig \int_{-\infty}^{\infty} dx^+ A_a^- (x^+, x) t^a \right]
\]

\[
F^{(1)}_{qq} (x_2, |k_t|) = \frac{4}{g^2} \int \frac{d^2 x d^2 y}{(2\pi)^3} e^{-ik_t \cdot (x-y)} \left\langle \text{Tr} \left[ (\partial_i U_y) (\partial_i U_x^\dagger) \right] \right\rangle_{x_2}
\]

\[
F^{(2)}_{qq} (x_2, |k_t|) = -\frac{4}{g^2} \int \frac{d^2 x d^2 y}{(2\pi)^3} e^{-ik_t \cdot (x-y)} \frac{1}{N_c} \left\langle \text{Tr} \left[ (\partial_i U_x) U_y^\dagger (\partial_i U_y) U_x^\dagger \right] \right\rangle_{x_2} \text{Tr} [U_y U_x^\dagger]
\]

these Wilson line correlators also emerge directly in CGC calculations when $|p_{1t}|, |p_{2t}| \gg |k_t|, Q_s$ (the regime of validity of TMD factorization)

Dominguez, CM, Xiao and Yuan (2011)
Outline of the derivation

- using $\langle p|p' \rangle = (2\pi)^3 \ 2p^- \delta(p^- - p'^-) \delta^{(2)}(p_t - p'_t)$ and translational invariance

$$\int \frac{d\xi^+ d^2\xi}{(2\pi)^3 p_A^-} e^{ix_2 p^- A \xi^- - i k_t \cdot \xi} \langle A|O(0, \xi)|A \rangle = \frac{2}{\langle A|A \rangle} \int \frac{d^3\xi d^3\xi'}{(2\pi)^3} e^{ix_2 p^- A (\xi^+ - \xi'^+) - i k_t \cdot (\xi - \xi')} \langle A|O(\xi', \xi)|A \rangle.$$
Outline of the derivation

- using \( \langle p|p' \rangle = (2\pi)^3 \ 2p^- \delta(p^- - p'^-)\delta^{(2)}(p_t - p'_t) \) and translational invariance

\[
\int \frac{d\xi^+ d^2\xi}{(2\pi)^3 p^-_A} e^{ix_2 p^-_A \xi^+ - ik_t \cdot \xi} \langle A|O(0, \xi)|A \rangle = \frac{2}{\langle A|A \rangle} \int \frac{d^3\xi d^3\xi'}{(2\pi)^3} e^{ix_2 p^-_A (\xi^+ - \xi'^+)} \langle A|O(\xi', \xi)|A \rangle .
\]

- setting \( \exp[ix_2 p^-_A (\xi^+ - \xi'^+)] = 1 \) and denoting \( \frac{\langle A|O(\xi', \xi)|A \rangle}{\langle A|A \rangle} = \langle O(\xi', \xi) \rangle_{x_2} \)

we obtain e.g.

\[
F_{qq}^{(1)}(x_2, k_t) = 4 \int \frac{d^3 x d^3 y}{(2\pi)^3} e^{-ik_t \cdot (x-y)} \left\langle \text{Tr} \left[ F^{i-}(x) U^{[-]} U^{[+]\dagger} F^{i-}(y) U^{[+]} \right] \right\rangle_{x_2}
\]
Outline of the derivation

• using $\langle p|p' \rangle = (2\pi)^3 \ 2p^- \delta(p^- - p'^-) \delta^{(2)}(p_t - p'_t)$ and translational invariance

$$\int \frac{d\xi^+ d^2 \xi}{(2\pi)^3 p_A^-} \ e^{ix_2p_A^-(\xi^+ - i k_t \cdot \xi)} \langle A|O(0, \xi)|A \rangle = \frac{2}{\langle A|A \rangle} \int \frac{d^3 \xi d^3 \xi'}{(2\pi)^3} e^{ix_2p_A^-(\xi^+ - \xi'^+ - i k_t \cdot (\xi - \xi')}} \langle A|O(\xi', \xi)|A \rangle.$$  

• setting $\exp[ix_2p_A^- (\xi^+ - \xi'^+)] = 1$ and denoting $\frac{\langle A|O(\xi', \xi)|A \rangle}{\langle A|A \rangle} = \langle O(\xi', \xi) \rangle_{x_2}$ we obtain e.g.

$$F_{qg}^{(1)}(x_2, k_t) = 4 \int \frac{d^3 x d^3 y}{(2\pi)^3} \ e^{-ik_t \cdot (x-y)} \left\langle \text{Tr} \left[ F^{i-}(x) U[-]^\dagger F^{i-}(y) U[+] \right] \right\rangle_{x_2}$$

• then performing the $x^+$ and $y^+$ integrations using

$$\partial_i U_y = ig \int_{-\infty}^{\infty} dy^+ U[-, y^+; y] F^{i-}(y) U[y^+, +\infty; y]$$

we finally get $F_{qg}^{(1)}(x_2, |k_t|) = \frac{4}{g^2} \int \frac{d^2 x d^2 y}{(2\pi)^3} \ e^{-ik_t \cdot (x-y)} \left\langle \text{Tr} \left[ (\partial_i U_y)(\partial_i U_y^\dagger) \right] \right\rangle_{x_2}$
The other TMDs at small-x

- involved in the $gg^* \to q\bar{q}$ and $gg^* \to gg$ channels

\[
\mathcal{F}_{gg}^{(1)}(x_2, k_t) = \frac{4}{g^2} \int \frac{d^2x d^2y}{(2\pi)^3} e^{-ik_t \cdot (x-y)} \frac{1}{N_c} \left\langle \text{Tr} \left[ (\partial_i U_y) (\partial_i U_y^\dagger) \right] \text{Tr} \left[ U_x U_y^\dagger \right] \right\rangle_{x_2},
\]
\[
\mathcal{F}_{gg}^{(2)}(x_2, k_t) = -\frac{4}{g^2} \int \frac{d^2x d^2y}{(2\pi)^3} e^{-ik_t \cdot (x-y)} \frac{1}{N_c} \left\langle \text{Tr} \left[ (\partial_i U_x) U_y^\dagger \right] \text{Tr} \left[ (\partial_i U_y) U_x^\dagger \right] \right\rangle_{x_2},
\]
\[
\mathcal{F}_{gg}^{(4)}(x_2, k_t) = -\frac{4}{g^2} \int \frac{d^2x d^2y}{(2\pi)^3} e^{-ik_t \cdot (x-y)} \left\langle \text{Tr} \left[ (\partial_i U_x) U_x^\dagger (\partial_i U_y) U_y^\dagger \right] \right\rangle_{x_2},
\]
\[
\mathcal{F}_{gg}^{(5)}(x_2, k_t) = -\frac{4}{g^2} \int \frac{d^2x d^2y}{(2\pi)^3} e^{-ik_t \cdot (x-y)} \left\langle \text{Tr} \left[ (\partial_i U_x) U_y^\dagger U_x^\dagger U_y^\dagger U_x^\dagger U_y^\dagger U_x^\dagger \right] \right\rangle_{x_2},
\]
\[
\mathcal{F}_{gg}^{(6)}(x_2, k_t) = -\frac{4}{g^2} \int \frac{d^2x d^2y}{(2\pi)^3} e^{-ik_t \cdot (x-y)} \frac{1}{N_c^2} \left\langle \text{Tr} \left[ (\partial_i U_x) U_y^\dagger U_x^\dagger (\partial_i U_y) U_x^\dagger \right] \text{Tr} \left[ U_x U_y^\dagger \right] \text{Tr} \left[ U_y U_x^\dagger \right] \right\rangle_{x_2}.
\]

with a special one singled out: the Weizsäcker-Williams TMD

\[
\mathcal{F}_{gg}^{(3)}(x_2, k_t) = -\frac{4}{g^2} \int \frac{d^2x d^2y}{(2\pi)^3} e^{-ik_t \cdot (x-y)} \left\langle \text{Tr} \left[ (\partial_i U_x) U_y^\dagger (\partial_i U_y) U_x^\dagger \right] \right\rangle_{x_2}.
\]
the evolution of Wilson line correlators with decreasing $x$ can be computed from the so-called JIMWLK equation

$$\frac{d}{d \ln(1/x_2)} \langle O \rangle_{x_2} = \langle H_{JIMWLK} O \rangle_{x_2}$$

a functional RG equation that resums the leading logarithms in $y = \ln(1/x_2)$
x evolution of the gluon TMDs

The evolution of Wilson line correlators with decreasing $x$ can be computed from the so-called JIMWLK equation:

$$\frac{d}{d \ln(1/x_2)} \langle O \rangle_{x_2} = \langle H_{\text{JIMWLK}} O \rangle_{x_2}$$

A functional RG equation that resums the leading logarithms in $y = \ln(1/x_2)$

- Qualitative solutions for the gluon TMDs:

The curve translates to the right with decreasing $x$.

The distribution of partons as a function of $x$ and $k_T$.

Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner
JIMWLK numerical results

using a code written by Claude Roiesnel

initial condition at $y=0$: MV model

evolution: JIMWLK at leading log

CM, Petreska, Roiesnel (2016)

saturation effects impact the various gluon TMDs in very different ways
Conclusions

- different processes involve different gluon TMDs, with different operator definitions
- given an initial condition, they can all be obtained at smaller values of x, from the JIMWLK equation
- as expected, the various gluon TMDs coincide at large transverse momentum, in the linear regime
- however, they differ significantly from one another at low transverse momentum, in the non-linear saturation regime
- we have quantified these differences and they are not negligible