Multi-Dimensional Screening: A Solution to a Class of Problems

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Abstract

We develop a general method for solving multi-dimensional screening problems in which the 'physical' allocation space is one-dimensional, and provide necessary and sufficient conditions for the existence of 'exclusion' in the optimal mechanism. We illustrate the application of our method to an example with quadratic utility and uniformly distributed types. Interestingly, the optimal solution exhibits discontinuity along the boundary of the region between exclusion and non-exclusion for a large set of parameter values.
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1 Introduction

This paper studies a screening problem in which the type space is multi-dimensional and the allocation space is one-dimensional. Such problems are common in economics, for two distinct reasons.

First, in many important economic environments agents typically differ along several dimensions on which there is private information. In the area of price discrimination, consumers differ both in demand intensity (intercept of demand) and price sensitivity (slope of demand). For example, high demand consumers can be price insensitive (because they are rich) or price sensitive (because they are poor and have large families). Similarly, an industrial customer's valuation for an input may depend both on the technology this firm uses to process the input, and the demand for the final product. Additionally, firms often have available multiple socioeconomic data that are imperfectly correlated with customers' purchase patterns. In other areas, multi-dimensionality of types is also prevalent. In insurance, customers differ both in risk aversion and the probability of having an accident. In labour taxation, the government may wish to differentially treat individuals who have low ability and those who have a high preference for leisure. And in the regulation of monopolies, the regulatory agency may wish to allow a different regulatory price and access charge for firms that have a high cost than for firms that have a low demand.

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Secondly, in many of these screening environments, the principal cannot discriminate between agents along more than one dimension. In price discrimination, firms can often differentially treat customers only by purchase quantity. For non-durable consumption goods there may be no opportunity to differentiate by quality, so quantity becomes the only instrument. Examples include soft drinks (which come in various sizes), residential electricity, and public transportation. On the other hand, for many consumer durables customers only purchase one unit, so then the only available dimension for discrimination becomes quality. Frequently, there is only one (or at least one dominant) dimension of quality, such as the speed of a microprocessor or internet connection, or the number of megapixels in a camera. In auctions, there is often only one unit offered for sale, and the single dimension then becomes the probability of obtaining the object. In areas other than price discrimination, the allocation space if often also one-dimensional. In insurance markets, the allocation consists of the amount of coverage, in labor taxation the instrument is the tax rate, and in regulation it is the regulatory price.

Our paper contains several methodological contributions. First, by correctly characterizing the isoquants, the set of agent types that consume the same quantity, we are able to reduce the multi-dimensional screening problem to a one-dimensional optimal control problem, whose solution is governed by an ordinary differential equation. Our solution method is therefore accessible to most economists, and generates analytical solutions. Second, we formulate the multi-dimensional screening problem as one of assigning agent types and tariff to the one-dimensional allocation. This approach is not only natural here, underscoring the one-dimensional nature of the principal's optimization problem, but also avoids some of the difficulties associated with discontinuities in the quantity allocation as a function of types that typically arise in our problem (see the discussion in the next paragraph). Our method also handles bunching in a straightforward and transparent way, without any need to resort to "ironing" or "sweeping". Finally, we present a novel condition, termed Single Crossing of Demand (SCD), which ensures that the solution to the principal's relaxed problem is globally incentive compatible.

The solution to our multi-dimensional screening problem exhibits several interesting properties. First, it may or may not be optimal to exclude some consumer types from consumption. The result that it can be optimal to have full consumer participation contrasts with established wisdom, which holds that when the type space is multi-dimensional, exclusion is generic (Armstrong (1996), Basov (2005)). Second the optimal quantity allocation is discontinuous at the boundary between the region of exclusion (where the optimal quantity is zero) and the region of non-exclusion (where the optimal quantity is generally bounded away from zero). Finally, and perhaps most surprisingly, we find that there can be a bunching of quantities allocated to a type located on the boundary between exclusion and non-exclusion, i.e. there can be a discontinuity of quantity as a function of type. The consumer type on which the quantities are bunched is then indifferent between all quantities in the bunch.

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 introduces the SCD condition, and characterizes the associated implementable allocations. Section 4 uses this characterization to reformulate the principal's problem to a one-dimensional screening problem in which the density of types is endogenous. Section 5 solves the associated optimal control problem, and presents necessary and sufficient conditions for exclusion to occur. Section 6 studies an example with linear quadratic utility and uniformly distributed types. Section 7 contains the conclusion.

2 Literature Review

Despite the existence of a rather voluminous literature on screening, relatively little is known about the type of problem we study. There are several reasons for this. First, as we will demonstrate, one of the dominant current approaches, the method of demand profiles (pioneered by Goldman, Leland and Sibley (1984), and further popularized by Brown and Sibley (1986) and, most forcefully, by Wilson (1993)) fails to adequately solve the problem. The difficulty with the demand profile method is that it requires that the derived marginal price schedule intersect a customer's demand schedule from below. In the one-dimensional type case, this is assured by the condition that marginal valuation is increasing in type ($u_{q\theta} > 0$), and that the assignment of quantities to types is nondecreasing (ensured by a monotonic inverse hazard rate, or by ironing). In the multi-dimensional case, no such sufficient condition is known. Furthermore, crossing from below is hard to ensure, because demand curves vary both in slope and intercept - sufficient variation in the intercept will thus necessarily lead to a violation of the required condition. As a consequence, the allocation will fail to be incentive compatible: the quantity assigned to customers whose demand curve intersects the tariff from above will correspond to a local minimum rather than a global maximum of their surplus maximization problem. Many of the worked out examples in the literature, such as the linear quadratic one studied in Wilson (1993, p. 196), therefore involve tariffs that are not incentive compatible.

To illustrate this, consider the following example.

Example 1 Suppose that a monopolistic seller of a good faces a consumer with utility function $u(q,\alpha,\theta) = \theta q - \frac{b-\alpha}{2}q^2$, where q is the quantity of the good, (α,θ) is privately known consumer type distributed uniformly over the unit square, and b is a constant satisfying $b < \frac{3}{2}$. The seller has zero cost of production. We are interested in the optimal pricing strategy or, equivalently, the optimal screening mechanism.

Following Wilson (1993) define the demand profile $N(p,q)$ as the fraction of consumers in the population whose demand price u_q exceeds p. A simple calculation yields:

$$N(p,q) = \begin{cases} \frac{1}{2q}\{(1-p-(b-1)q)^2 - (1-p-bq)^2\}, & \text{if } p+bq \le 1 \\ \frac{1}{2q}(1-p-(b-1)q)^2, & \text{if } p+bq \ge 1. \end{cases}$$

According to the demand profile approach, $N(p,q)$ represents the demand schedule for quantity increment q. Thus for the quantity increment q, monopolist should charge the price $p(q) = P'(q)$ to solve

$$\max_p\{(p-c)N(p,q)\}$$

Performing this maximization gives

$$p(q) = \begin{cases} \frac{1}{2} - \frac{1}{4}(2b-1)q, & \text{if } q \le \frac{2}{2b+1} \\ \frac{1}{3}(1-(b-1)q), & \text{if } q \ge \frac{2}{2b+1}. \end{cases} .$$

resulting in the tariff $P(q) = \int_0^q p(z)dz$

$$P(q) = \begin{cases} \frac{1}{2}q + \left(\frac{1}{8} - \frac{b}{4}\right)q^2, & \text{if } q \leq \frac{2}{2b+1} \\ \frac{1}{6(2b+1)} + \frac{q}{3} - \frac{b-1}{6}q^2, & \text{if } q \geq \frac{2}{2b+1}. \end{cases} .$$

For this approach to be correct, every consumer type whose demand price equals $p(q)$ should also be willing to purchase all increments $q' < q$ and not purchase any increments $q' > q$. This will be the case if the iso-price curves in type space, defined by the equation $u_q(q, \alpha, \theta) = p(q)$, do not intersect, for then every consumer type α, θ will have only one solution to the first order condition associated with her surplus maximization problem $\max_q\{u(q,t) - P(q)\}$.[1]

Let us therefore examine the iso-price curves associated with the schedule P. Solving the equation $\theta - (b - \alpha)q = p(q)$ yields

$$\theta(q, \alpha) = \begin{cases} \frac{1}{2} + \frac{1}{4}(2b + 1 - 4\alpha)q, & \text{if } q \leq \frac{2}{2b+1} \\ \frac{1}{3} + \frac{1}{3}(2b + 1 - 3\alpha)q, & \text{if } q \geq \frac{2}{2b+1}. \end{cases}$$

Figure 1 illustrates these iso-price curves. All iso-price curves are straight lines. For $q \in [0, \frac{2}{2b+1}]$, iso-price lines go through the point $(\alpha, \theta) = (\frac{2b+1}{4}, \frac{1}{2})$, rotating up form a flat line at the level $q = 0$ to the the quantity $q = \frac{2}{2b+1}$, where the northwest corner point $(\alpha, \theta) = (0, 1)$ is reached. For $q \geq \frac{2}{2b+1}$, all iso-price lines rotate up through the point $(\alpha, \theta) = (\frac{2b+1}{3}, \frac{1}{3})$, until the quantity $q = \frac{1}{b-1}$ is reached, when the north-east corner point $(\alpha, \theta) = (1, 1)$ is hit. This means that any point (α, θ) in the interior of triangle Δ defined by the inequalities $\frac{1+2b-2\alpha}{1+2b} \leq \theta \leq 1/2$ and $\alpha \geq \frac{2b+1}{4}$ is the intersection point of an iso-price line from the region $q < \frac{2}{2b+1}$ and an iso-price line from the region $q > \frac{2}{2b+1}$. The objective function of such a type therefore has two stationary points, one at a quantity $q_-(\alpha, \theta) < \frac{2}{2b+1}$ and one at a quantity $q_-(\alpha, \theta) > \frac{2}{2b+1}$. It is easy to see that q_- corresponds to a local minimum, and q_+ to a local maximum.

The presence of a local minimum to the consumer's objective function has two immediate consequences. First, the demand profile approach, in which consumers are presented with marginal price schedules $p(q)$, is no longer equivalent to the original approach, where consumers are presented with a nonlinear tariff $P(q)$. Indeed, any consumer in the above mentioned triangle would be unwilling to purchase any quantity increment in the interval $[0, q_-]$, whereas they might purchase this increment when presented with the nonlinear pricing schedule P. Secondly, and more damagingly, the quantity q_+ may no longer be a global maximum to the consumer's optimization problem.

Since the only other candidate for an optimum occurs at $q = 0$, this raises the important issue of whether all consumer types who are purchasing increment q_+ under the marginal schedule $p(q)$ would be willing to participate in the mechanism. As indicated above, this is

[1]More formally, consider any type (α, θ) on the iso-price curve at the quantity q, i.e. $u_q(q, \alpha, \theta) - p(q) = 0$. Since iso-price lines do not cross, iso-price curves at quantities $q' > q$ will lie to the northeast of the iso-price curve at quantity q, and iso-price curves at quantities $q' < q$ will lie to the southwest of the iso-price curve at quantity q. It then follows from assumption 1(iii) that $u_q(q', \alpha, \theta) - p(q') > 0$ for $q' < q$, and $u_q(q', \alpha, \theta) - p(q') > 0$ for $q' > q$. Consequently, type (α, θ)'s objective function is strictly quasiconcave, implying that q is a global maximum.

not an issue for consumer types with $\theta \geq \frac{1}{2}$, since iso-price lines do not cross for such types. For consumers types in the triangle Δ, only $q > \frac{2}{2b+1}$ can be a maximum, and for such q we have

$$u(q, \alpha, \theta(q, \alpha)) - P(q) = \frac{1}{6}\left((1+2b)q^2 - \frac{1}{1+2b}\right) - \frac{\alpha}{2}q^2$$

Setting this expression equal to zero yields

$$\begin{aligned} \underline{\alpha}(q) &= \frac{1+2b}{3} - \frac{1}{3(1+2b)q^2} \\ \underline{\theta}(q) &= \frac{1+(1+2b)q}{3(1+2b)q} \end{aligned}$$

These equations trace out a strictly decreasing a curve in type space, which may equivalently be expressed as

$$\underline{\theta}(\alpha) = \frac{1}{3} + \frac{\sqrt{(1+2b)(1+2b-3\alpha)}}{3(2b+1)}$$

Note that the participation constraint is violated for all types in Δ that lie below the curve $\underline{\theta}$. As a consequence, the demand profile approach necessarily fails whenever when $b < \frac{3}{2}$.

McAfee and McMillan (1988) propose a closely related approach. These authors introduce a condition termed "Generalized Single Crossing" which ensures that any solution satisfying the first and second order conditions of the agent's surplus maximization problem is globally incentive compatible. Generalized Single Crossing implies that iso-price curves are linear in the type space, thereby permitting a reduction to a one-dimensional screening problem. McAfee and McMillan's contribution is considerable, but suffers from a number of drawbacks. First, the limitation to *linear* iso-price curves is significant in our context. Second, their approach implicitly assumes that in equilibrium all agent types along an iso-price line will participate. Unfortunately, as our analysis will reveal, this assumption is often violated.[2]

Lewis and Sappington (1988) adopt the Generalized Single Crossing assumption, but instead of formulating the problem in terms of demand profiles use the direct method pioneered by Mussa and Rosen (1978), leading to an objective based on virtual utility functions. Because it is based upon McAfee and McMillan's method for reducing the problem to a one-dimensional screening problem, this approach suffers from the same drawbacks. In addition, Lewis and Sappington's analysis assumes that in equilibrium there is no exclusion. They do not provide conditions for exclusion to be absent, and unfortunately, as we will show, exclusion is rather prevalent. In particular, in the context of nonlinear pricing, absence of exclusion requires the aggregate demand curve to be perfectly inelastic at the seller's marginal cost of production.[3] Finally, Lewis and Sappington implicitly assume that the slope of iso-price curves in type

[2]Properly taking into account the agent's participation constraint changes the integrand of principal's objective function in an essential way: rather than depending only on the allocation $q(t_1)$ and its derivative $q'(t_1)$ it now also depends on $q(t_1')$ for all $t_1' > t_1$. As a consequence, McAfee and McMillan's formulation of the problem can no longer be solved by the method of calculus of variation.

[3]Armstrong (1999) already pointed out this deficiency, but he did not provide neccessary and sufficient conditions for exclusion to occur, nor did he solve the associated multi-dimensional screening problem.

space is constant, which (in the nonlinear pricing interpretation) can happen only if the slope of the agent's demand function is independent of type.[4]

The crossing of iso-price lines demonstrated in Example 1 also implies that the methods of McAfee and McMillan and Lewis and Sappington are flawed. Indeed, since a consumer type can lie on two distinct iso-price lines $u_q(q, \alpha, \theta) - p(q) = 0$, merely being located on an iso-price line generally cannot identify the quantity purchased by a consumer.

There are several approaches to resolving these difficulties. One could try to identify conditions under which iso-price curves never cross. This is a useful approach, and we pursue it elsewhere (Deneckere and Severinov, 2009b). The main drawback of this approach is that it fails to solve some of the most rudimentary examples, such as the one presented above. For this reason, the present paper concentrates on the more difficult question of solving the multi-dimensional screening problem when iso-price curves are allowed to intersect.

Rochet and Stole (2001) develop the direct method for arbitrary multi-dimensional screening problems. Their approach has two drawbacks. First, because the problem is not reduced to a one-dimensional screening problem, the associated first-order conditions require the solution of a partial differential equation, which cannot be solved analytically, except in very special cases. Secondly, because the direct approach only imposes the local incentive compatibility constraints, the solution typically violates the conditions for global incentive compatibility. A general method for solving our type of problem therefore remains lacking.

Lastly, a solution method for our problem has recently become available for the special case where the agent's utility function is linear in type. This was made possible by two breakthroughs in the analysis of multi-dimensional screening problems. First, Rochet and Choné (1998) developed a "sweeping" procedure (analogous to ironing for the one-dimensional case), which adjusts the solution derived by the direct method so as to ensure global incentive compatibility. Rochet and Choné's method requires that the dimension of the type space and allocation space coincide. However, by interpreting the coefficients on consumer types as artificial goods in the utility function, Basov (2001) was able to transform the problem from one where the number of consumer characteristics exceeds the dimension of the physical allocation space to one where the two dimensions coincide. While ingenious, this approach also has several drawbacks. It requires agents' utility functions to be linear in type, which is great for applications such as auctions, but quite limiting in the current context. The method also necessitates the solution of a partial differential equation, which generally can be solved only numerically. Finally, sweeping is a complicated procedure which does lend not itself to analytical solutions.

It is fair to conclude that because of all these issues, our type of screening problem has hitherto remained inaccessible to most economists, and therefore failed to generate interesting practical applications.

[4] As a consequence, Lewis and Sappington's characterization of an optimal mechanism (Proposition 1, p. 447) is generally incorrect. However, it does hold for the special example studied in Section 5 of their paper.

3 The Model and Characterization of Isoquants

A monopolist supplier of a single good faces a population of consumers. Consumers are distinguished by a two dimensional preference parameter $t = (\alpha, \theta)$, which is private information. The limitation to two dimensions of uncertainty is made here for ease of exposition and compactness in notation. With minor modifications, our results generalize to higher dimensions (for details, see Deneckere and Severinov (2009a). When consuming a quantity $q \in \mathbf{R}_+$ of the good, acquired at cost p, a consumer of type t receives net utility $u(q,t) - p$. Consumers' reservation utilities are equal to zero.

The distribution function $F(\alpha, \theta)$ of consumer types in the population is common knowledge. We assume that $F(.)$ is twice continuously differentiable function, with density function $f(\alpha, \theta) > 0$, and a rectangular support $[a, b] \times [c, d]$. Renormalizing, we can without loss of generality take the support to be $[0, 1] \times [0, 1]$.

We assume that the firm's marginal and average cost of production is constant at the level $c > 0$. To handle the case in which the monopolist's aggregate cost $C(Q)$ is an increasing function of aggregate output $Q = \int q(t)f(t)dt$ we would need one extra step. Precisely, for any given constant marginal cost level c, our model would predict the corresponding aggregate output level Q selected by the firm. In the optimal mechanism, $C'(Q) = c$.

We maintain the following assumptions on preferences throughout the paper:

Assumption 1 *The function $u(q, \alpha, \theta)$: $\mathbf{R}_+ \times [0, 1]^2$ is of class C^3. Furthermore,*
(i) $u(0, \alpha, \theta) = 0$ for all $(\theta, \alpha) \in [0, 1]^2$;
(ii) $u_q(q, \alpha, \theta) > 0$, $u_\theta(q, \alpha, \theta) > 0$ and $u_\alpha(q, \alpha, \theta) > 0$, for all $q > 0$ and $(\alpha, \theta) \in [0, 1]^2$;
(iii) $u_{\theta q}(q, \alpha, \theta) > 0$, $u_{\alpha q}(q, \alpha, \theta) > 0$, for all $(\alpha, \theta) \in (0, 1]^2$ and $q > 0$;
(iv) $u_{qq}(q, \alpha, \theta) < 0$ for all θ, α and q.

Assumption 1 is fairly standard. Part (iii) requires consumer's utility functions to be supermodular. Part (iv) ensures that consumers' demand functions are downward sloping.

We also make extensive use of a novel assumption, specific to the higher-dimensional type space, which we term "Single-Crossing of Demand":

Assumption 2 *(SCD) $\frac{d}{dq}\frac{u_{q\alpha}}{u_{q\theta}} > 0$ for all $q > 0$.*

The economic interpretation of Assumption 2 is that the inverse demand functions can intersect at most once, as the next Lemma demonstrates.

Lemma 1 *Suppose Assumption 2 holds and $\alpha' > \alpha$. Then $u_q(q, \alpha', \theta') = u_q(q, \alpha, \theta)$ implies $u_{qq}(q, \alpha', \theta') > u_{qq}(q, \alpha, \theta)$.*

Assumption 2 should not be confused with the single-crossing condition in one-dimensional screening problems, which guarantees that consumers' indifference curves in (q, t) space intersect at most once. In fact, the latter condition is extremely restrictive, as it implies that consumers' demand curves do not intersect at all, i.e. can be ranked. In the next section, we will show that Assumption 2 has many important consequences. In particular, it implies that isoquants in (θ, α) space cannot intersect, and must "fan out".

By the Revelation Principle, the monopolist's problem is to choose a direct mechanism $(q(\alpha,\theta),t(\alpha,\theta))$, where $q(\alpha,\theta)$ is the quantity assigned to type (α,θ) and $t(\alpha,\theta)$ is the transfer that this type pays to the firm, so as to maximize her expected profits subject to the consumer's incentive constraints and individual rationality. [5]. Formally, the firm's problem can be stated as follows:

$$\max \int_{[0,1]^2} t(\alpha,\theta)dF(\alpha,\theta) \quad (1)$$

$$u(q(\alpha,\theta),\alpha,\theta)-t(\alpha,\theta)\geq u(q(\alpha',\theta'),\alpha,\theta)-t(\alpha',\theta') \quad \text{for all } (\alpha,\theta),\ (\alpha',\theta'^2 \quad (2)$$

$$u(q(\alpha,\theta),\alpha,\theta)-t(\alpha,\theta)\geq 0 \quad \text{for all } (\alpha,\theta)\ \in [0,1] \quad (3)$$

The solution to this problem exists by standard arguments, since $u(.)$ is continuous and bounded in (α,θ).

Let $T^*(\tilde{q}|q(.),t(.))$ denote the set of all types (α,θ) who are assigned the same quantity $\tilde{q}$ in the mechanism, i.e. $T^*(\tilde{q}|q(.),t(.))=\{(\alpha,\theta)|q(\theta,\alpha)=\tilde{q}\}$. We will refer to $T^*(\tilde{q}|(q(.),t(.)))$ as an *isoquant* at quantity $\tilde{q}$ in the mechanism $(q(.),t(.))$. For brevity, we will drop the argument $(q(.),t(.))$ and simply write $T(\tilde{q})$.

The first goal of this section is to characterize the set $T(q)$ in an incentive compatible and individually rational mechanism. Note that several quantity pairs may provide the same net utility for a type (α,θ) who is assigned quantity $q(\alpha,\theta)$ in the mechanism. That is, for such type incentive constraints (2) may hold as equality for a set of (α',θ'). This point will turn out to be important for the characterization of isoquants $T(q)$, as we will show below. Accordingly, let us define

$$Q^*(\alpha,\theta) \quad = \quad \{q(\theta',\alpha')|(\theta',\alpha')\in \arg\max_{\theta',\alpha'} u(q(\theta',\alpha'),\alpha,\theta)-t(\theta',\alpha')\}$$

That is, $Q^*(\alpha,\theta)$ denotes the set of optimal quantities for type (α,θ) in the mechanism $(q(.),t(.))$. Clearly, $Q^*(\alpha,\theta)$ is non-empty for all (α,θ), since $q(\alpha,\theta)\in Q^*(\alpha,\theta)$. Furthermore, since the utility function $u(.)$ is strictly supermodular, every selection of $Q^*(\alpha,\theta)$ is an increasing function.[6]

The following Lemma provides two important results which constitute the basis for our characterization of the isoquants $T(q)$. The two main ingredients behind this characterization are the supermodularity of the agent's payoff function and the single-crossing of demand (SCD) property.

Lemma 2 *Suppose Assumption 2 holds.*

(i) Let $q_1\in Q^*(\theta_1,\alpha_1)$ *and*

$$u_q(q_1,\theta_2,\alpha_2)=u_q(q_1,\theta_1,\alpha_1) \ \textit{for some} \quad (\theta_2,\alpha_2) \quad \textit{s.t.}\ \alpha_1>\alpha_2. \quad (4)$$

Then $Q^*(\theta_2,\alpha_2)=q_1$.

(ii) Suppose in addition that Assumption 1(iii) holds. Let $(\theta,\alpha)\in[0,1]\times[0,1)$ *be such that* $\{q_1,q_2\}\in Q^*(\theta,\alpha)$ *for some* $q_1,q_2\in\mathbf{R}_{++}$, $q_1\neq q_2$. *Consider any* $(\alpha',\theta'^2$ *s.t.* $\alpha'>\alpha$ *and* $u_q(q_1,\alpha,\theta)=u_q(q_1,\alpha',\theta')$. *Then* $q_1\notin Q^*(\alpha',\theta')$.

[5] The Taxation principle implies that we can view this problem equivalently as the firm's choice of the optimal tariff $P(q)$

[6] Precisely, if q' is a selection from Q^* then $q'',\alpha')\geq q'(\theta,\alpha)$ whenever $\theta'\geq\theta$ and $\alpha'\geq\alpha$.

Lemma 2 provides a partial characterization of the set $T(q)$. Namely, starting from any type (α_1, θ_1) for which q is optimal, the Lemma identifies an interval of types to the "left" of α_1 (types with lower values of this parameter) for which q is the unique optimum and which, therefore, belong to $T(q)$. These types satisfy condition (4). Lemma 2 also implies that the types with higher values of the parameter α that satisfy condition (4) are not in $T(q)$ if more than one quantity is optimal at (α_1, θ_1). Accordingly, let us define:

$$I(q, \alpha, \theta) = \{(\alpha', \theta') : \ u_q(q, \alpha', \theta') = u_q(q, \alpha, \theta), \ \alpha' > \alpha \ \} \tag{5}$$

We will refer to the set $I(q, \alpha, \theta)$ as left iso-price interval for q from (α, θ). Lemma 2 implies that $I(q, \alpha, \theta) \subset T(q)$ if $q \in Q(\alpha, \theta)$. It follows that, if $Q^*(\alpha, \theta)$ is multi-valued for some (α, θ), then there are many left iso-price intervals emanating from (α, θ) at all quantities $q \in Q^*(\alpha, \theta)$, all of which correspond to optimal quantities on these intervals.

Our next goal is to characterize the boundary between the two regions in the mechanism $(q(.), t(.))$: the participation region which includes all types who consume a positive quantity and the exclusion region which includes all types who consume zero quantity. The boundary between these two regions is defined as follows:[7]

$$\underline{\theta}(\alpha) \equiv \inf\{\theta | \theta \geq 0, \ q(\alpha, \theta) > 0\}$$

$$\text{Also, let} \quad s(\alpha, \theta) = \max_{\theta', \alpha'} \{u(q(\theta', \alpha'), \alpha, \theta) - t(\theta', \alpha')\}$$

Note that without loss of generality $s(\alpha, \theta) = 0$ if $q(\alpha, \theta) = 0$, for otherwise the mechanism is suboptimal and can be improved by setting to zero the transfers paid by the types who get zero quantity. A characterization of the lower boundary is provided in the following Lemma:

Lemma 3 *Suppose that Assumptions 1 and 2 holds. The function $s(.)$ is continuous, with $s(\alpha, \theta) > 0$ if $\theta > \underline{\theta}(\alpha)$ and $s(\alpha, \theta) = 0$ if $\theta < \underline{\theta}(\alpha)$. The lower boundary $\underline{\theta}(\alpha)$ is continuous, non-increasing in α, and strictly decreasing in α whenever $q(\alpha, \underline{\theta}(\alpha)) > 0$ and $\underline{\theta}(\alpha) > 0$.*

The optimal quantity correspondence along the lower boundary, $Q^*(\underline{\theta}(\alpha), \alpha)$, is increasing in α and, for almost all α, $Q^*(\underline{\theta}(\alpha), \alpha)$ is a singleton i.e. $Q^*(\alpha, \underline{\theta}(\alpha)) = q(\alpha, \underline{\theta}(\alpha))$.

If, in addition, $\sup \frac{u_\alpha}{u_\theta}(q, \alpha, \theta) < \infty$, then $\underline{\theta}(\alpha)$ is absolutely continuous and, for almost all α s.t. $0 < \underline{\theta}(\alpha) < 1$, we have

$$\frac{d\underline{\theta}}{d\alpha} = -\frac{u_\alpha}{u_\theta}(q(\alpha, \underline{\theta}(\alpha)), \alpha, \underline{\theta}(\alpha)). \tag{6}$$

The lower boundary $\underline{\theta}(\alpha)$ can have flat segments. First, flat segments can be present because $\underline{\theta}(.)$ hits the lower ($\underline{\theta} = 0$) or upper ($\underline{\theta} = 1$) boundaries of the type space. Also, Lemma 3 implies that the lower boundary is flat i.e., $\underline{\theta}(.)$ is constant on an interval $[\alpha', \alpha'']$, if $-\frac{u_\alpha}{u_\theta}(q(\alpha, \underline{\theta}(\alpha)), \alpha, \underline{\theta}(\alpha))$ is constant on this interval, as illustrated in Example 1. Accordingly, let $\hat{\alpha} = \sup\{\alpha | \alpha \in [0, 1], \underline{\theta}(\alpha) > 0\}$, and $\hat{\theta} = \hat{\alpha}$.

Next, let

$$L \equiv \{(\alpha, \underline{\theta}(\alpha)) : 0 \leq \alpha \leq 1\} \cup \{(1, \theta) : \theta \geq \underline{\theta}(1)\},$$

[7] We adopt the convention that the infimum of an empty set equals 1.

In words, L traces out the lower boundary and the right boundary of the participation region.

The following Lemma establishes important properties of an optimal mechanism and of the optimal quantity correspondence $Q^*(.)$.

Lemma 4 *In an optimal mechanism,*
(i) the associated allocation $q(\alpha,\theta)$ is continuous at all (α,θ) s.t. $\theta > \underline{\theta}(\alpha)$.
(ii) $Q^*(.)$ is closed and upper-hemicontinuous (u.h.c) correspondence. $Q^*(\alpha,\theta)$ is convex-valued for all (α,θ) s.t. $\alpha > 0$ and $\theta < 1$.
(iii) For every, (α,θ) s.t. $q(\alpha,\theta) > 0$ there exists a unique $(\theta',\alpha') \in L$ s.t. $(\alpha,\theta) \in I(q',\alpha',\theta')$ where $q'^*(\alpha',\theta')$. So, $q(\alpha,\theta) = q'$.

Furthermore, $I(q',\alpha',\theta') \cap L = (\alpha',\theta')$ for all $(\alpha',\theta') \in L$ and $q'^*(\alpha',\theta')$.

According to part (iii) of Lemma 4, every isoquant emanates from the boundary L and never intersects L again, and each point in the participation region lies on some isoquant. As a consequence, we make an important conclusion that both the isoquants and the quantity allocation $q(.,.)$ are entirely determined by the behavior of the allocation $q(\alpha,\theta)$ along the curve L.

The results of the previous Lemmas can be summarized in the following Theorem:

Theorem 1 *Suppose that Assumptions 1 and 2 hold and $\sup \frac{u_\alpha}{u_\theta}(q,\alpha,\theta) < \infty$. Consider a direct incentive compatible and individually rational mechanism $(q(\alpha,\theta), t(\alpha,\theta))$ s.t. $q(\alpha,\theta)$ is continuous in the participation region.*

Then there exists an absolutely continuous function $\underline{\theta}(.) : [0,1] \to [0,1]$, a type $(\widehat{\alpha},\widehat{\theta})$, with either $\widehat{\alpha} = 1$ or $\widehat{\theta} = 0$, and two non-decreasing, u.h.c, convex-valued correspondences $\underline{Q}^*(\cdot) : [0,1] \to \mathbf{R}_+$ and $\overline{Q}^*(\cdot) : [\widehat{\theta},1] \to \mathbf{R}_+$ with $\underline{Q}^*(1) = \overline{Q}^*(\widehat{\theta})$ such that:

(i) $\underline{\theta}(\alpha) = \inf\{\theta | \theta \geq 0,\ q(\alpha,\theta) > 0\}$ for all $\alpha \in [0,1]$;

(ii) $\hat{\alpha} = \sup\{\alpha | \underline{\theta}(\alpha) > 0\}$, $\hat{\theta} = \underline{\theta}(\hat{\alpha})$;

(iii) $q(\alpha,\underline{\theta}(\alpha)) \in \underline{Q}^*(\alpha)$ for all $\alpha \in [0,1]$; $q(1,\theta) \in \overline{Q}^*(\theta)$ for all $\theta \in [\widehat{\theta},1]$

(iv) for all $(\alpha,\theta) \in [0,1]^2$ s.t. $\theta > \underline{\theta}(\alpha)$, there exists either:
(a) a unique $\tilde{\alpha} \in [0,1]$ s.t. $q(\alpha,\theta) \in \underline{Q}^*(\tilde{\alpha})$. In this case, $u_q(q(\alpha,\theta),\alpha,\theta) = u_q(q(\alpha,\theta),\tilde{\alpha},\underline{\theta}(\tilde{\alpha}))$,

$$t(\alpha,\theta) = \begin{cases} u(q(\alpha,\theta),\tilde{\alpha},\underline{\theta}(\tilde{\alpha})) & \text{if } \tilde{\alpha} \leq \hat{\alpha} \\ u(q(\hat{\alpha},\hat{\theta}),\hat{\alpha},\hat{\theta}) + \int_{\hat{\alpha}}^{\alpha} u_q(q(s,0),s,0)ds & \text{if } \tilde{\alpha} \in (\hat{\alpha},1]. \end{cases} \tag{7}$$

or (b) a unique $\tilde{\theta} \in [\widehat{\theta},1]$ s.t. $q(\alpha,\theta) \in \overline{Q}^*(\tilde{\theta})$. In this case, $u_q(q(\alpha,\theta),\alpha,\theta) = u_q(q(\alpha,\theta),1,\tilde{\theta})$, and

$$t(\alpha,\theta) = u(q(\hat{\alpha},\hat{\theta}),\hat{\alpha},\hat{\theta}) + \int_{\hat{\alpha}}^{1} u_q(q(s,0),s,0)ds + \int_{\hat{\theta}}^{\theta} u_q(q(1,z),1,z)dz \tag{8}$$

Theorem 1, which will be discussed in more details below, establishes that every incentive compatible, individually rational mechanism uniquely induces a boundary L and allocation along it. In particular, part (i) describes the lower boundary $\underline{\theta}(.)$. Part (iii) asserts that every point in the participation region lies on an isoquant emanating from a point on the boundary L. Part (iv) links the transfer function $t(.)$ in a mechanism to the allocation along the boundary L. The final result of this section provides a converse to Theorem 1:

Theorem 2 *Suppose Assumptions 1 and 2 hold, and* $\sup \frac{u_\alpha}{u_\theta}(q,\alpha,\theta) < \infty$*. for all* $(\alpha,\theta) \in [0,1]$ *and* $q \in [0, q^*(1,1)]$.

Consider some $(\widehat{\alpha},\widehat{\theta}) \in [0,1]^2$ *s.t. either* $\widehat{\alpha} = 1$ *or* $\widehat{\theta} = 0$ *or both, and two non-decreasing, u.h.c., convex-valued, correspondences* $\underline{Q}^*(\cdot) : [0,1] \to \mathbf{R}_+$ *and* $\overline{Q}^*(\cdot) : [\widehat{\theta},1] \to \mathbf{R}_+$ *with* $\underline{Q}^*(1) = \overline{Q}^*(\widehat{\theta})$.

Then, the following direct mechanism $(q(\alpha,\theta), t(\alpha,\theta))$ *is incentive compatible and individually rational:*

(i) Take $q(\alpha,\underline{\theta}(\alpha))$ *to be some selection from* $\underline{Q}^*(\alpha)$ *and* $q(1,\theta)$ *be some selection from* $\overline{Q}^*(\theta)$*;*

(ii) $\underline{\theta}(\alpha)$ *is uniquely defined by* $\underline{\theta}(\hat{\alpha}) = \hat{\theta}$ *and equation (6) for* $\alpha \in [0,\hat{\alpha}]$ *s.t.* $\max \underline{Q}^*(\alpha) > 0$*;* $\underline{\theta}(\alpha) = 1$ *for* $\alpha \in [0,\hat{\alpha}]$ *s.t.* $\max \underline{Q}^*(\alpha) = 0$*; and* $\underline{\theta}(\alpha) = \hat{\theta}$ *for* $\alpha \in [\hat{\alpha},1]$. [8]

(iii) For every (α,θ) *s.t.* $\theta > \underline{\theta}(\alpha)$ *there exists either (a)* $\alpha' \in [0,1]$ *and* $\tilde{q} \in \underline{Q}^*(\alpha')$ *s.t.* $u_q(\tilde{q},\alpha,\theta) = u_q(q(\alpha,\theta),\alpha',\underline{\theta}(\alpha'))$*. In this case,* $q(\alpha,\theta) = \tilde{q}$*.*

(b) or $\theta' \in [\overline{\theta},1]$ *and* $\bar{q} \in \overline{Q}(\theta')$ *s.t.* $u_q(\bar{q},\alpha,\theta) = u_q(\bar{q},1,\theta')$*. In this case,* $q(\alpha,\theta) = \bar{q}$*.*

(iv)

$$t(\alpha,\theta) = \begin{cases} u(q(\alpha,\theta),\alpha',\underline{\theta}(\alpha')) & \text{for all } (\alpha,\theta) \text{ s.t. } q(\alpha,\theta) \in [0, \max \underline{Q}^*(\hat{\alpha})], \\ u(\max \underline{Q}^*(\hat{\alpha}),\widehat{\alpha},\underline{\theta}(\widehat{\alpha})) + \int_{\max \underline{Q}^*(\hat{\alpha})}^{q} u_q(z,\alpha(z),\theta(z))dz & \text{for } (\alpha,\theta) \text{ s.t. } q(\alpha,\theta) \in [\max \underline{Q}^*(\hat{\alpha}), \max \overline{Q}^*(1)] \end{cases} \tag{9}$$

Theorems 1 and 2 establish a one-to-one relationship between the set of incentive compatible individually rational direct mechanisms, on the one hand, and, on the other hand, the set of quantity correspondences $(\underline{Q}^*(\alpha), \overline{Q}^*(\theta))$ on the boundary L and the "junction" points $(\hat{\alpha},\hat{\theta})$ at which $\underline{\theta}(\alpha)$ intersects the boundary of the type space. Per Lemma 3, the lower boundary $\underline{\theta}(\alpha)$ is uniquely determined by the choice of $(\hat{\alpha},\hat{\theta})$ and the quantity allocation $\underline{q}(.)$ on the lower boundary. Since $\underline{Q}^*(.)$ is increasing and u.h.c., $\underline{Q}^*(\alpha)$ is unique for almost all α. So, $\underline{q}(.)$ is uniquely determined by $\underline{Q}^*(.)$ for almost all α, and hence $\underline{\theta}(\alpha)$ is also uniquely determined by the choice of $(\hat{\alpha},\hat{\theta})$ and $\underline{Q}^*(.)$.

Thus, to find the solution to problem (1)-(3), we could optimize over the set of non-decreasing, u.h.c., convex-valued quantity correspondences $\underline{Q}^*(\cdot)$ and $\overline{Q}^*(\cdot)$ for the boundary L and "junction" points $(\hat{\alpha},\hat{\theta})$. We will use this method to compute the optimal mechanism. However, before we will be able to to do, we will need to tackle the following issues. First, we need to complete the reformulation of our problem to express the expected profits in terms of the $\underline{Q}^*(\cdot)$, $\overline{Q}^*(\cdot)$, and $(\hat{\alpha},\hat{\theta})$. Second, we will also have to deal with additional technical complications that arise when $\underline{Q}^*(\alpha)$ and $\overline{Q}^*(\theta)$ are sets rather than singletons, as well as when $\underline{Q}^*(.)$ is constant over some interval. These issues will be considered in the following sections.

[8] Since $\underline{Q}^*(\alpha)$ is an increasing and closed correspondence, it must be a singleton for almost all $\alpha \in [0,1]$. So, for given $\underline{Q}^*(\alpha)$, the choice of $q(\alpha,\underline{\theta}(\alpha))$ is trivial for almost all $\alpha \in [0,1]$. The same applies to $\overline{Q}^*(\theta)$. Hence, $\underline{\theta}(\alpha)$ is uniquely defined by specifying $\underline{Q}^*(\alpha)$ and $(\hat{\alpha},\hat{\theta})$.

4 The Reformulated Problem

In the previous section we have established an isomorphy between the set of incentive compatible individually rational mechanism and the set an element of which consists of a point $(\widehat{\alpha}, \widehat{\theta})$ (s.t. either $\widehat{\alpha} = 1$ or $\widehat{\theta} = 0$) and two nondecreasing, u.h.c, convex-valued correspondences $\underline{Q}^*(\cdot) : [0,1] \rightrightarrows \mathbf{R}_+$, $\overline{Q}^*(\cdot) : [\hat{\theta}, 1] : [0,1] \rightrightarrows \mathbf{R}_+$ and $\underline{Q}^*(1) = \overline{Q}^*(\hat{\theta})$. This will allow us to simplify the problem (1)-(3). However, maximizing over the set of correspondences is analytically cumersome. So we now proceed with an additional simplification. Instead of assigning quantities to types, as in the standard mechanism setting, we will proceed in the opposite direction and will be assigning types to quantities. Specifically, suppose that we are given two nondecreasing, u.h.c, convex-valued correspondences $\underline{Q}^*(\cdot)$, $\overline{Q}^*(\cdot)$ s.t. $\underline{Q}^*(1) = \overline{Q}^*(\hat{\theta})$ and a point $(\widehat{\alpha}, \widehat{\theta})$ s.t. either $\widehat{\alpha} = 1$ or $\widehat{\theta} = 0$.

Then, let us define a pair of functions $(\alpha(q), \theta(q))$ as follows. For $q \in [0, \underline{Q}^*(1)]$, let $\alpha(q) = \max\{\alpha | q \in \underline{Q}^*(\alpha)\}$, $\theta(q) = \underline{\theta}(\alpha(q))$ where the function $\underline{\theta}(.)$ is defined according to (ii) in Theorem 2. For $q \geq \max \underline{Q}^*(1)$ let $\alpha(q) = 1$, $\theta(q) = \max\{\theta | q \in \overline{Q}^*(\theta)\}$. Note that intervals where $\underline{Q}^*(.)$ or $\overline{Q}^*(.)$ are constant correspond to the discontinuities in the functions $(\alpha(\cdot), \theta(\cdot))$. Furthermore, if $\alpha(.)$ is constant on some interval in $[0, \max \underline{Q}^*(1)]$, then $\theta(.)$ must also be constant on this interval by the definition of $\underline{\theta}(.)$. For otherwise, $\underline{\theta}(.)$ would not be a function.

Given the above definitions and for fixed $(\hat{\alpha}, \hat{\theta}) \in [0,1]^2, \bar{q} \in \mathbf{R}_+$ with either $\hat{\alpha} = 1$ or $\hat{\theta} = 0$, there is a 1-to-1 relationship between the set of nondecreasing, u.h.c, convex-valued, bounded correspondences $\underline{Q}^*(\cdot)$, $\overline{Q}^*(\cdot)$ s.t. $\max \overline{Q}^*(1) = \bar{q}$ and the set of functions $(\alpha(.), \theta(.))$ continuous a.e. on $[0, \bar{q}]$ and such that $\alpha(.)$ is increasing and $\theta(.)$ is decreasing on $[0, \max \underline{Q}^*(1)]$, while on $[\max \underline{Q}^*(1), \bar{q}]$ $\alpha(.)$ is equal to 1 and $\theta(.)$ is increasing, and $\theta(q) = \underline{\theta}(\alpha(q))$ for q s.t. $q \leq \min q | \alpha(q) \geq 1$, where $\underline{\theta}(.)$ is given by the solution to (ii) in Theorem 2.

Note that any pair of admissible functions $(\theta(q), \alpha(q))$ together with $(\hat{\alpha}, \hat{\theta})$ uniquely determines $\max \underline{Q}^*(1)$ via the condition $\underline{Q}^*(1) = \max\{q | \alpha(q) = 1, \theta(q) \leq \hat{\theta}\}$. We will henceforth define $\underline{q}(1) = \underline{Q}^*(1) = \min\{q | \alpha(q) = 1\}$. Also, $\hat{q}$ can be set to be any solution to the equations $\hat{\alpha} = \alpha(q)$ and $\theta(q) \leq 1$. Specifically, we will set $\hat{q} = \min\{q | \alpha(q) = \hat{\alpha}, \theta(q) \leq 1\}$. Also, because $\theta(.)$ must be constant on any interval where $\alpha(.)$ is constant, for any pair of admissible functions $(\alpha(.), \theta(.))$, $\underline{\theta}(\alpha)$ is well-defined by

$$\underline{\theta}(\alpha) = \theta(q'), \text{ where } q' \text{ saisfies } \alpha(q') = \alpha \tag{10}$$

Using this isomorphism, we will now proceed to reformulate the problem (1)-(3) in terms of an optimal choice of a 5-tuple $(\alpha(.), \theta(.), \widehat{\alpha}, \widehat{\theta}, \bar{q})$ which we will henceforth refer to as a mechanism (Recall that $\bar{q} = \max \overline{Q}^*(1)$).

In order to complete this reformulation, we need to define a probability measure on the set of quantities induced by a mechanism $(\alpha(.), \theta(.), \widehat{\alpha}, \widehat{\theta}, \bar{q})$. For this, we need another piece of notation. For any point (α, θ) let the function $\sigma(q, \alpha, \theta, \cdot.)$ represent the iso-price curve through (α, θ) at the quantity q. That is, for any $q \in \mathbf{R}_+$, $(\theta, \alpha) \in (0,1] \times [0,1]$ and $a \in [0,1]$, the function $\sigma(q,\alpha,\theta,a)=\sigma$ where σ is defined as follows:

$$\begin{aligned}&u_q(q,\sigma,a)=u_q(q,\theta,\alpha), \text{ if } u_q(q,1,a)\geq u_q(q,\theta,\alpha)\geq u_q(q,0,a)\\&\sigma=1, \text{ if } u_q(q,1,a)<u_q(q,\theta,\alpha)\\&\sigma=0, \text{ if } u_q(q,0,a)>u_q(q,\theta,\alpha)\end{aligned} \tag{11}$$

Observe that, whenever $\sigma(q,\alpha,\theta,a)\in(0,1)$, we have:

$$\sigma_q(q,\alpha,\theta,a)=\frac{u_{qq}(q,\alpha,\theta)-u_{qq}(q,a,\sigma)}{u_{q\theta}(q,a,\sigma)} \tag{12}$$

$$\sigma_\theta(q,\alpha,\theta,a)=\frac{u_{q\theta}(q,\alpha,\theta)}{u_{q\theta}(q,a,\sigma)} \tag{13}$$

$$\sigma_\alpha(q,\alpha,\theta,a)=\frac{u_{q\alpha}(q,\alpha,\theta)}{u_{q\theta}(q,a,\sigma)} \tag{14}$$

Further, let $\underline{\alpha}(q,\alpha,\theta)$ be defined as the solution in a to the equation $\sigma(q,\alpha,\theta,a)=1$, that is $u_q(q,\alpha,\theta)=u_q(q,\underline{\alpha}(q,\alpha,\theta),1)$ if such a solution exists, and $\underline{\alpha}(q,\alpha,\theta)=0$, otherwise i.e. if $u_q(q,\alpha,\theta)\leq u_q(q,0,1)$. (In the latter case, there exists $\theta'\in[0,1)$ such that $u_q(q,\alpha,\theta)=u_q(q,0,\theta'))$.

Next, define

$$H(q,\alpha,\theta)=\int_{\underline{\alpha}(q,\alpha(q),\theta(q))}^{1}\int_{\max\{\sigma(q,\alpha,\theta,a),\underline{\theta}(a)\}}^{1}f(a,t)dtda \tag{15}$$

Then the probability measure of the set of types assigned quantities that do not exceed q is equal to

$$1-H(q,\alpha(q),\theta(q)).$$

The points of discontinuity of $\alpha(q)$ correspond to atoms of the probability distribution $1-H(q,\alpha(q),\theta(q))$. Particularly, the size of an atom at a quantity $\tilde{q}$ is equal to:

$$\lim_{q\uparrow\tilde{q}}H(q,\alpha(q),\theta(q))-H(\tilde{q},\alpha(\tilde{q}),\theta(\tilde{q}))$$

When $\alpha(.)$ and $\theta(.)$ are differentiable, then the density of the quantity q is equal to:

$$\hat{h}(q)\equiv\tilde{h}(q,\alpha(q),\theta(q),\alpha'(q),\theta'(q))=\int_0^{\alpha}f(\sigma(q,\alpha,\theta,a),a)[\sigma_q(q,\alpha,\theta,a)+\sigma_\theta(q,\alpha,\theta,a)\theta'+\sigma_\alpha(q,\alpha,\theta,a)\alpha']da$$

Thus the expected seller's revenue in the mechanism is equal to

$$ER=\int_0^{\bar{q}}t(q)d\left(1-H(q,\alpha(q),\theta(q)\right) \tag{16}$$

Since,

$$t(q)=t(\alpha(q),\theta(q))=\begin{cases}u(q,\alpha(q),\theta(q)), & \text{for all } q\in[0,\hat{q}]\\ u(\hat{q},\alpha(\hat{q}),\theta(\hat{q}))+\int_{\hat{q}}^{q}u_q(z,\alpha(z),\theta(z))dz, & \text{for all } q\in[\hat{q},\bar{q}]\end{cases},$$

(16) can be rewritten as follows:

$$ER = \int_0^{\hat{q}} u(q,\alpha(q),\theta(q))d\left(1 - H(q,\alpha(q),\theta(q))\right) + u(\hat{q},\alpha(\hat{q}),\theta(\hat{q}))H(\hat{q},\alpha(\hat{q}),\theta(\hat{q})) + \int_{\hat{q}}^{\bar{q}} \int_{\hat{q}}^{q} u_q(z,\alpha(z),\theta(z))dzd\left(1 - H(q,\alpha(q),\theta(q))\right) \tag{17}$$

We will henceforth assume that the functions $\alpha(q)$ and $\theta(q))$ are piecewise continuously differentiable on the interval $(q^0, \bar{q}]$ where $q^0 = \min \underline{Q}^*(0)$. This assumption can be justified as follows. Suppose there exists a solution to problem (1) when $(\alpha(\cdot), \theta(\cdot))$ is piecewise continuously differentiable. Because piecewise continuously differentiable functions are dense in the set of measurable functions, such a solution must also be a solution to the unrestricted problem. Below we will identify conditions under which the restricted problem has such a solution. Under those existence conditions the quantity allocation along L is increasing and hence 'ironing' will not be needed. When 'ironing' is necessary, a more complicated approach using impulse control is appropriate (see Deneckere and Severinov, 2009c). We may now state:

Theorem 3 *Suppose Assumptions 1 and 2 hold, and consider mechanism* $(\alpha(.), \theta(.), \widehat{\alpha}, \widehat{\theta}, \bar{q})$ *s.t.* $(\alpha(\cdot), \theta(\cdot))$ *are piecewise continuous and piecewise continuously differentiable. Let* $\hat{q} = \min\{q|\alpha(q) = \hat{\alpha}, \theta(q) \leq 1\}$

Then the monopolist's profits are given by

$$\int_0^{\hat{q}} u(q,\alpha(q),\theta(q))\tilde{h}(q,\alpha(q),\theta(q),\alpha'(q),\theta'(q))dq + u(\hat{q},\widehat{\alpha},\widehat{\theta})H(\hat{q},\widehat{\alpha},\widehat{\theta}) + \int_{\hat{q}}^{\bar{q}} H(q,\alpha(q),\theta(q))u_q(q,\alpha(q),\theta(q))dq \tag{18}$$

Theorem 3 says that the monopolist's profits consists of two parts. The first part depends only upon the allocation $(\alpha(q), \theta(q))$ for $q \leq \hat{q}$ on the lower boundary, with the associated transfer equal to the gross utility of the type $(\alpha(q), \theta(q))$ (as the types on the lower boundary earn zero surplus) and the density of types from which this transfer is collected, $\tilde{h}(q, \alpha(q), \theta(q), \alpha'(q), \theta'(q))$ i.e., the types located on the isoquant through the point $(\alpha(q), \theta(q))$.

The second part depends only upon the allocation $(\alpha(q), \theta(q))$ for $q \geq \hat{q}$. Over this interval, the firm's profits equals the sum of two terms. First, from each type that consumes more than $\hat{q}$ the monopolist collects $u(\underline{q}(\widehat{\alpha}), \widehat{\alpha}, \widehat{\theta})$, the price paid by type $(\widehat{\alpha}, \widehat{\theta})$. The probability measure of these types is $H(\underline{q}(\widehat{\alpha}), \widehat{\alpha}, \widehat{\theta})$. Second, the monopolist collects the marginal price $u_q(q, \alpha(q), \theta(q))$ from each type that consumes more than q, of which there are $H(q, \alpha(q), \theta(q))$.

Consequently, the monopolist's optimization problem can be split into the following three subproblems.

Subproblem (i). For fixed $\widehat{q} \in \mathbf{R}_+$ and $(\widehat{\alpha}, \widehat{\theta}) \in [0,1]^2$ such that either $\widehat{\alpha} = 1$ or $\widehat{\theta} = 0$, choose functions $\alpha(q)$ and $\theta(q)$ to solve

$$W(\widehat{q}, \widehat{\alpha}, \widehat{\theta}) = \max \int_0^{\hat{q}} u(q,\alpha(q),\theta(q))\tilde{h}(q,\alpha(q),\theta(q),\alpha'(q),\theta'(q))dq \tag{19}$$

subject to the following constraints:

$$\begin{aligned}\alpha(0) \geq 0,\ \alpha(\widehat{q}) = \widehat{\alpha},\ \theta(\widehat{q}) = \widehat{\theta} \\ \alpha'(q) \geq 0 \\ \theta'(q) = -\frac{u_\alpha(q, \theta(q), \alpha(q))}{u_\theta(q, \theta(q), \alpha(q))}\alpha'(q)\end{aligned} \tag{20}$$

Subproblem (ii). Given $\widehat{q} \in \mathbf{R}_+$ and $(\widehat{\alpha}, \widehat{\theta}) \in [0,1]^2$ such that either $\widehat{\alpha} = 1$ or $\widehat{\theta} = 0$, choose $\underline{q}(1)$ and $\bar{q} \in \mathbf{R}_+$, and functions $\alpha \in Lip([\widehat{q}, \underline{q}(1)])$ and $\theta \in Lip([\underline{q}(1), \bar{q}(1)])$ to solve:

$$Z(\widehat{q}, \widehat{\alpha}, \widehat{\theta}) = \max \int_{\widehat{q}}^{\bar{q}} H(q, \alpha(q), \theta(q)) u_q(q, \alpha(q), \theta(q)) dq \tag{21}$$

subject to the following constraints:

$$\begin{gathered}\widehat{q} \leq \underline{q}(1) \leq \bar{q}, \\ \alpha(q) \ \text{ and } \ \theta(q) \text{ are nondecreasing} \\ \alpha(\hat{q}) = \hat{\alpha}, \quad \alpha(q) = 1 \text{ for } q \geq \underline{q}(1), \\ \theta(q) = \hat{\theta} \text{ for } q \in [\widehat{q}, \underline{q}(1)], \quad \theta(\bar{q}) = 1.\end{gathered}$$

Subproblem (iii). Finally, select $\widehat{q} \in \mathbf{R}_+$, and $(\widehat{\alpha}, \widehat{\theta}) \in [0,1]^2$ such that either $\widehat{\alpha} = 1$ or $\widehat{\theta} = 0$ to solve

$$V(\widehat{q}, \widehat{\alpha}, \widehat{\theta}) = \max_{\widehat{q}, \widehat{\alpha}, \widehat{\theta}} W(\widehat{q}, \widehat{\alpha}, \widehat{\theta}) + u(\widehat{q}, \widehat{\alpha}, \widehat{\theta}) H(\widehat{q}, \widehat{\alpha}, \widehat{\theta}) + Z(\widehat{q}, \widehat{\alpha}, \widehat{\theta}) \tag{22}$$

4.1 Solution to Subproblem (i)

First, consider the density function $\tilde{h}(q, \alpha, \theta, \alpha', \theta')$ in the integrand of (19). According to (20), along the lower boundary $\theta'(q)$ is uniquely determined by the value of the four-tuple $(q, \alpha(q), \theta(q), \alpha'(q))$. Therefore, the 5-tuple of arguments $(q, \alpha, \theta, \alpha', \theta')$ of $\tilde{h}(.)$ can be replaced with a 4-tuple $(q, \alpha, \theta, \alpha')$ and, with a slight abuse of notation, we obtain:

$$\tilde{h}(q, \alpha, \theta, \alpha', \theta') \equiv h(q, \alpha, \theta, \dot{\alpha}) = \int_{\underline{\alpha}(q,\alpha,\theta)}^{\alpha} f(\sigma(q, \alpha, \theta, a), a) \left\{\sigma_q - \sigma_\theta g \dot{\alpha} + \sigma_\alpha \dot{\alpha}\right\} da. \tag{23}$$

Next, note that

$$h(q, \alpha, \theta, \dot{\alpha}) = h_0 + (h_2 - g h_1)\dot{\alpha}. \tag{24}$$

where:

$$h_0(q, \alpha, \theta) = \int_{\underline{\alpha}(q,\alpha,\theta)}^{\alpha} f(\sigma(q, \alpha, \theta, a), a) \sigma_q(q, \sigma(q, \alpha, \theta, a), a) da \tag{25}$$

$$h_1(q, \alpha, \theta) = \int_{\underline{\alpha}(q,\alpha,\theta)}^{\alpha} f(\sigma(q, \alpha, \theta, a), a) \sigma_\theta(q, \sigma(q, \alpha, \theta, a), a) da \tag{26}$$

$$h_2(q, \theta, \alpha) = \int_{\underline{\alpha}(q,\alpha,\theta)}^{\alpha} f(\sigma(q, \alpha, \theta, a), a) \sigma_\alpha(q, \sigma(q, \alpha, \theta, a), a) da. \tag{27}$$

Recall that $\underline{\alpha}(q, \alpha, \theta)$ in (25)-27) is the solution in a to the equation $\sigma(q, \alpha, \theta, a) = 1$, if such a solution exists, and $\underline{\alpha}(q, \alpha, \theta) = 0$, otherwise.

Next, let $q_0 \equiv \inf_q \alpha(q) > 0$. Note that each of (25),(26) and (27) and hence (23) is equal to zero at any point $(q, \alpha(q), \theta(q))$ s.t. $\alpha(q) = 0$. Therefore, $\int_0^{q_0} u(q, \alpha(q), \theta(q))h(q, \alpha(q), \theta(q), \dot{\alpha}(q))dq = 0$, and so we can rewrite (19) as follows:

$$W_0(\widehat{q}, \widehat{\alpha}, \widehat{\theta}) = \max_{q_0 \geq 0, \dot{\alpha}(\cdot) \geq 0} \int_{q_0}^{\widehat{q}} u(q, \alpha(q), \theta(q))h(q, \alpha(q), \theta(q), \dot{\alpha}(q))dq. \tag{28}$$

Now, we are ready to state Subproblem (i) as an optimal control problem. To this end, let us form the Hamiltonian:

$$J(q, \alpha, \theta, \dot{\alpha}, \mu, \lambda) = uh + \mu\dot{\alpha} - \lambda g\dot{\alpha} = uh_0 + [u(h_2 - gh_1) + (\mu - \lambda g)]\,\dot{\alpha}, \tag{29}$$

where μ and λ are the multipliers on the state evolution equations for α and θ, respectively.

Equation (24) makes it apparent that the integrand of (22) is linear in $\dot{\alpha}$. Equation (20) for the evolution of θ is also linear in $\dot{\alpha}$ and the expression for the evolution of α is an identity and, therefore, is also linear in $\dot{\alpha}$. So, the Hamiltonian (29) is also linear in the control $\dot{\alpha}$.

This linearity creates certain technical difficulties for solving subproblem (i) as it implies that $\dot{\alpha}$ cannot be solved for directly from the standard first-order conditions of optimality for an optimal control problem. Also, recall that $\dot{\alpha}$ may exhibit discontinuities. Nevertheless, the optimal control theory can still be used to solve this problem, albeit with some intricacies. Notably, by Pontryagin's Maximum principle, it remains true that the optimal control $\dot{\alpha} \geq 0$ maximizes the Hamiltonianm (29). So, let

$$S(q, \alpha(q), \theta(q), \mu(q), \lambda(q)) = u(h_2 - gh_1) + (\mu - \lambda g) \tag{30}$$

Then optimality requires the following:

$$\begin{aligned} S(q, \alpha(q), \theta(q), \mu(q), \lambda(q)) &< 0 \Rightarrow \dot{\alpha} = 0 \\ S(q, \alpha(q), \theta(q), \mu(q), \lambda(q)) &= 0 \Rightarrow \dot{\alpha} \geq 0 \end{aligned} \tag{31}$$

The function $S(q, \alpha(q), \theta(q), \mu(q), \lambda(q))$ is called the *switching function.* Note that it can never be strictly positive, since then the value of the objective would be infinite and optimal control $\dot{\alpha}$ would be undefined.

An interval of q on which S vanishes ($S = 0$) is called a *singular arc.* Note that on a singular arc, the optimality conditions do not pin down the value of the optimal control $\dot{\alpha}$. An interval of q on which $S < 0$ is called a *nonsingular arc.* A point q at which a singular and a nonsingular arc meet is called a *junction point.* It is apparent from the switching conditions that at a junction point the optimal control may be discontinuous.

To recover the optimal control along a singular arc, we proceed as follows. Considering $S(q, \alpha(q), \theta(q), \mu(q), \lambda(q))$ as a function of q along a singular arc, we will choose the control $\dot{\alpha}$ to maintain $S(q, \alpha(q), \theta(q), \mu(q), \lambda(q)) \equiv 0$. For this let us define:

$$\psi(q, \alpha, \theta) \equiv u_q(q, \alpha, \theta)u_\theta(q, \alpha, \theta) \int_{\underline{\alpha}(q,\alpha,\theta)}^{\alpha} \frac{f(\sigma(q, \alpha, \theta, a), a)}{u_{q\theta}(q, a, \sigma(q, \alpha, \theta, a))} da. \tag{32}$$

Then we have:

Lemma 5 *The optimal solution $(q, \alpha(q), \theta(q))$ satisfies:*

$$\dot{S} = g_q(\psi - uh_1 - \lambda) \tag{33}$$

Along a non-singular arc:

$$\ddot{S} = \frac{dg_q}{dq}(\psi - uh_1 - \lambda) + g_q\left(\psi_q + u_\theta h_0 - u_q h_1\right)$$

On a singular arc:

$$\ddot{S} = g_q\left\{[\psi_q + u_\theta h_0 - u_q h_1] + [\psi_\alpha - \psi_\theta g - uf(\theta, \alpha) + u_\theta(h_2 - gh_1) - \psi g_\theta]\,\dot{\alpha}\right\}, \tag{34}$$

Lemma 5 allows us to solve for the optimal control $\dot{\alpha}$ by setting (34) equal to zero, since along a singular arc $\ddot{S} = 0$. Applying Pontryagin's maximum principle we also obtain the optimal solutions to (28) on non-singular arcs. The result is provided in the following Theorem.

Theorem 4 *The solution to the maximization problem (28) has the following properties:*
(i) Over any interval where $\alpha(q)$ is strictly increasing and hence $\theta(q)$ is strictly decreasing, we have:

$$\begin{aligned}
\dot{\alpha}(q) &= \frac{u_\theta h_0 - u_q h_1 + \psi_q}{uf + \psi g_\theta + \psi_\theta g - \psi_\alpha - u_\theta(h_2 - gh_1)} && (35)\\
\dot{\theta}(q) &= -g\dot{\alpha}. && (36)\\
\lambda(q) &= \psi - uh_1 && (37)\\
\mu(q) &= \psi g - uh_2 && (38)
\end{aligned}$$

(ii) Over any interval on which α, and hence θ, are constant, we have:

$$\dot{\mu} = -u_\alpha h_0 - u\frac{\partial h_0}{\partial \alpha} \tag{39}$$

$$\dot{\lambda} = -u_\theta h_0 - u\frac{\partial h_0}{\partial \theta} \tag{40}$$

(iii) The functions $\mu(q)$ and $\lambda(q)$ are continuous.
(iv) We have: $\alpha(q_0)q_0 = 0$.

Theorem 4 provides the optimal solution on every singular and non-singular arc. It also gives a partial answer regarding the location of such arcs. Particularly, by Part (iii) of this Theorem, the juncture points between singular and non-singular arcs must be chosen so that the Lagrange multipliers λ and δ remain continuous throughout. Below, we will explore this property further to provide a more detailed characterization of the solution. Before doing this, we provide additional details of the solution in the following two Lemmas.

Lemma 6 ***(Generalized Legendre-Clebsch):*** *Let $D(q,\alpha,\theta)$ be equal to the denominator of (35) i.e.,*

$$D(q,\alpha,\theta) \equiv uf + \psi g_\theta + \psi_\theta g - \psi_\alpha - u_\theta(h_2 - gh_1) \tag{41}$$

The solution to the maximization problem (28) is such that $D(q,\alpha,\theta) \leq 0$ along any optimal singular (sub) arc.

Next, recall that $q_0 \equiv \inf_{\alpha(q)>0} q$. Also, let q^{**} be such that $\sigma(q^{**},\alpha(q^{**}),\theta(q^{**}),0) = 1$. In words, the isoquant corresponding to q^{**}, $I(q^{**},\alpha(q^{**}),\theta(q^{**}))$, hits the "northwest" corner $(0,1)$ of the type space, and consequently, the density function $h(q,\alpha(q),\theta(q),\dot{\alpha}(q))$ is discontinuous at q^{**}. To see this, consider the lower limit of the integrals in (25)- (27), $\underline{\alpha}(q,\alpha(q),\theta(q))$. It is fairly apparent that it is not continuously differentiable at q^{**} since its total derivative from the left is zero, while its right-hand side derivative is strictly positive.

Using Lemma 6 we can establish the following properties of the optimal q_0 and q^{**}:

Lemma 7 *Suppose that $u - \frac{u_\theta u_q}{u_{q\theta}} > 0$ for all (q,α,θ) s.t. $q > 0$. Then*
(i) $q_0 = 0$ and $\alpha(q_0) > 0$.
(ii) $\theta(q^{**}) < 1$.

Note that the condition, $u - \frac{u_\theta u_q}{u_{q\theta}} > 0$ for all (q,α,θ), holds for most commonly specified utility functions, and is satisfied whenever $u - \frac{u_\theta u_q}{u_{q\theta}}$ is strictly increasing in q. A sufficient condition for the latter property is that $u_{qq\theta} \geq 0$. (Indeed, we have $\frac{\partial}{\partial q}(u_{q\theta}u - u_\theta u_q) = u_{qq\theta}u - u_\theta u_{qq} > 0$ whenever $u_{qq\theta} > 0$, since $u_{qq} < 0$.)

4.2 Solution to subproblems (ii)-(iii)

Consider subproblem (ii). It represents a maximization problem with fixed left hand and right hand boundaries, fixed initial "time" $\widehat{q}$, and free right hand "time" $\bar{q}$. The next theorem describes its solution. To state it we need to introduce some additional notation. Let

$$\phi(q,\theta) = u_q(q,1,\theta)H_\theta(q,1,\theta) + u_{\theta q}(q,1,\theta)H(q,1,\theta) \tag{42}$$

$$\varkappa(q,\alpha) = u_q(q,\alpha,0)H_\alpha(q,\alpha,0) + u_{\alpha q}(q,\alpha,0)H(q,\alpha,0) \tag{43}$$

Let $\theta^\phi(q)$ be the solution to $\phi(q,\theta) = 0$, when such exists; $\theta^\phi(q) = 0$ if $\phi(q,\theta) > 0$ for all $\theta \in [0,1]$; $\theta^\phi(q) = 1$ if $\phi(q,\theta) < 0$ for all $\theta \in [0,1]$.

Also, let $\alpha^\varkappa(q)$ denote the solution in α to the equation $\varkappa(q,\alpha) = 0$ when such solution exists, $\alpha^\varkappa(q) = 0$ if for all $\alpha \in [0,1]$ $\varkappa(q,\alpha) < 0$ and $\alpha^\varkappa(q) = 1$ if $\varkappa(q,\alpha) > 0$ for all $\alpha \in [0,1]$.

Theorem 5 *Suppose that $\phi(.)$ is increasing in q and decreasing in θ. Also, suppose that $\varkappa(q,\alpha)$ is increasing in q and decreasing in α. Let $\underline{q}(1)$ be defined by $\phi(\underline{q}(1),0) = 0$. Then $\underline{q}(1)$ also satisfies $\varkappa(\underline{q}(1),1) = 0$. Also, let $\bar{q}$ be defined by $\phi(\bar{q},1) = 0$.*

Then the solution to problem (21) is as follows.

If $\widehat{\alpha} = 1$, then $\theta(\widehat{q}) = [\widehat{\theta}, \max\{\widehat{\theta},\theta^\phi(\widehat{q})\}]$, $\theta(q) = \max\{\theta^\phi(q),\widehat{\theta}\}$ for $q \in (\hat{q},\bar{q})$, so that $\theta(\bar{q}) = 1$.

If $\widehat{\alpha} < 1$, then $\alpha(\widehat{q}) = [\hat{\alpha}, \max\{\hat{\alpha},\alpha^\varkappa(\widehat{q})\}]$, $\alpha(q) = \max\{\hat{\alpha},\alpha^\varkappa(q)\}$ for all $q \in (\widehat{q}, \max\{\underline{q}(1),\widehat{q}\}]$, $\theta(\max\{\underline{q}(1),\widehat{q}\}) = [0, \theta^\phi(\max\{\underline{q}(1),\widehat{q}\})]$, $\theta(q) = \theta^\phi(q)$ for $q \in (\max\{\underline{q}(1),\widehat{q}\},\bar{q}]$, so that $\theta(\bar{q}) = 1$.

Under the conditions of Theorem 5 the monotonicity constraints $\alpha'(q) \geq 0$ and $\theta'(q) \geq 0$ can be ignored, and problem (21) is solved by pointwise maximization under the integrand. The conditions $\varkappa(q, \alpha) = 0$ and $\phi(q, \theta) = 0$ are a multi-dimensional version of a condition familiar from the one-dimensional type case, that at the optimum marginal virtual surplus must be equal to zero.[9]

Since $\lim_{q \to \bar{q}(1)} H(q, \alpha(q), \theta(q)) = 0$ and $\lim_{q \to \bar{q}(1)} H_\theta(q, \alpha(q), \theta(q)) < 0$, we also obtain the familiar condition that the allocation of the "top" type $(1, 1)$ is undistorted i.e.,

$$u_q(\bar{q}(1), 1, 1) = 0.$$

Note that when $\widehat{\theta} > \theta^\phi(\widehat{q})$, or when $\widehat{\theta} = 0$ and $\widehat{\alpha} > \alpha^\varkappa(\widehat{q})$, then there is a non-empty right neighborhood of $\widehat{q}$ over which all isoquants emanate from $(\widehat{\alpha}, \widehat{\theta})$. We shall show in Theorem 6 that it is never optimal to do so.

When the monotonicity constraints are binding, then as in subproblem (i) we must associate Lagrange multipliers $\mu(q) \geq 0$ and $\delta(q) \geq 0$ with the constraints $\alpha'(q) \geq 0$ and $\theta'(q) \geq 0$, respectively. Our next result describes how to obtain a solution in this case.

Lemma 8 *The solution to subproblem (ii) satisfies $\mu(q)\alpha'(q) = 0$ and $\delta(q)\theta'(q) = 0$. If $\widehat{\alpha} < 1$ and $\widehat{q} < \underline{q}(1)$, then $\mu'(q) = \varkappa(q, \alpha(q))$ over any interval in $(\widehat{q}, \underline{q}(1)]$ on which $\mu(q) > 0$. Also, $\delta'(q) = \phi(q, \theta(q))$ over any interval in $[\underline{q}(1), \bar{q}]$ on which $\delta(q) > 0$.*

4.3 Transversality Conditions for $(\widehat{\alpha}, \widehat{\theta})$ and $\widehat{q}$.

It remains to combine the solutions to subproblems (i) and (ii) and establish the transversality conditions for $(\widehat{\alpha}, \widehat{\theta})$ and $\widehat{q}$. Recall that there are only two free variables, since either $\widehat{\alpha} = 1$ or $\widehat{\theta} = 0$. Our first result establishes that at the optimum there is a one to one relationship between $(\widehat{\alpha}, \widehat{\theta})$ and $\widehat{q}$, effectively reducing our optimization problem to the determination of a single parameter.

Theorem 6 *Suppose that the functions $\phi(q, \theta)$ and $\varkappa(q, \alpha)$ are increasing in q and decreasing in α and θ. Then at the optimum, $\widehat{\theta} = \theta^\phi(\widehat{q})$ whenever $\widehat{\alpha} = 1$, and $\widehat{\alpha} = \alpha^\varkappa(\widehat{q})$ whenever $\widehat{\theta} = 0$.*

Define q^0 to be the unique solution to the equation $\phi(q, 0) = 0$. According to Theorem 6 whenever $\widehat{q} \geq q^0$ it must be that $\widehat{\theta} = \theta^\phi(\widehat{q})$, and whenever $\widehat{q} \leq q^0$we have $\widehat{\alpha} = \alpha^\varkappa(\widehat{q})$.

Now, let us derive the optimal value of $\widehat{q}$. The next Theorem addresses this issue, using the notation $\mu_-(q^{**}) = \lim_{q \uparrow q^{**}} \mu(q)$ and $\mu_+(q^{**}) = \lim_{q \downarrow q^{**}} \mu(q)$, and similarly for $\mu_-(q^{**})$ and $\mu_+(q^{**})$:

Theorem 7 *Suppose that the functions $\phi(q, \theta)$ and $\varkappa(q, \alpha)$ are increasing in q and decreasing in α, respectively. Then:*

(i) If $\bar{\alpha}(\widehat{q}, \widehat{\alpha}, \widehat{\theta}) > 0$ and $\alpha'(\widehat{q}) > 0$ the following transversality condition must hold:

$$[\mu_+(q^{**}) - \mu_-(q^{**})] \frac{d\alpha^{**}}{d\widehat{q}} + [\lambda_+(q^{**}) - \lambda_-(q^{**})] \frac{d\theta^{**}}{d\widehat{q}} = 0;$$

[9]More explicitly, letting t denote the type parameter in the one-dimensional screening model, and letting $F(t)$ denote its distribution function, the optimality condition is $u_q F' + u_{qt}(1 - F(t)) = 0$.

(ii) If $\bar{\alpha}(\widehat{q}, \widehat{\alpha}, \widehat{\theta}) = 0$, or if $\alpha'(q) = 0$ for all $q \leq \widehat{q}$, the following transversality condition must hold:[10]

$$\begin{aligned}
\lambda(\widehat{q}) &= \psi(\widehat{q}, 1, \theta^{\phi}(\widehat{q})) - u(\widehat{q}, 1, \theta^{\phi}(\widehat{q}))h_{\theta'}(\widehat{q}, 1, \theta^{\phi}(\widehat{q})), \text{ if } \theta^{\phi}(\widehat{q}) > 0 \\
\mu(\widehat{q}) &= \psi(\widehat{q}, \alpha^{\varkappa}(\widehat{q}), 0)g(\widehat{q}, \alpha^{\varkappa}(\widehat{q}), 0) - u(\widehat{q}, \alpha^{\varkappa}(\widehat{q}), 0)h_{\alpha'}(\widehat{q}, \alpha^{\varkappa}(\widehat{q}), 0), \text{ if } \alpha^{\varkappa}(\widehat{q}) < 1
\end{aligned}$$

To interpret Theorem 7 note that in case (i) the transversality condition only has bite if $\alpha'(q) > 0$ in a right neighborhood of q^{**} but $\alpha'(q) = 0$ in a left neighborhood of q^{**}. Theorem 7 therefore suggests that at the optimum, whenever $\widehat{q} > 0$ or $\underline{\alpha}(\widehat{q}, \widehat{\alpha}, \widehat{\theta}) > 0$, there will be an interval of quantities (a left neighborhood of q^{**} in case (i), and a left neighborhood of $\widehat{q}$ in case (ii)) for which isoquants emanate from the same point (the point $(\alpha^{**}, \theta^{**})$ in case (i), and the point $(\widehat{\alpha}, \widehat{\theta})$ in case (ii)). Our next Lemma simplifies the task of applying Theorem 7, under an additional regularity condition:

Lemma 9 *Suppose that $\frac{\partial}{\partial q}\{u_\theta h_0 - u_q h_{\theta'} + \psi_q\} < 0$, and suppose that $\alpha'(q) = 0$. Then $\lambda(q) = -\int_0^q \frac{\partial(uh_0)}{\partial \theta}(z, \alpha(z), \theta(z))dz$ and $\mu(q) = -\int_0^q \frac{\partial(uh_0)}{\partial \alpha}(z, \alpha(z), \theta(z))dz$.*

Note that the condition, $u - \frac{u_\theta u_q}{u_{q\theta}} > 0$ for all (q, α, θ), holds for most commonly specified utility functions, and is satisfied whenever $u - \frac{u_\theta u_q}{u_{q\theta}}$ is strictly increasing in q. A sufficient condition for the latter property is that $u_{qq\theta} \geq 0$. [11]

5 Qualitative Properties of a Mechanism.

5.1 Singular and Non-singular Arcs

In this section, we characterize the regions where the solution to subproblem (i) consists of a singular arc and where it consists of a non-singular arc. To this purpose, let $N(q, \alpha, \theta)$ denote the numerator in the differential equation (35) governing $\dot{\alpha}$ along a singular arc. That is,

$$N(q, \alpha, \theta) = \psi_q - u_q h_1 + u_\theta h_0 \tag{44}$$

The following assumption imposes regularity conditions on $N(q, \alpha, \theta)$

Assumption 3 *For all (q, α, θ), we have:*
(i) $N_{qq} > 0$;
(ii) $N_{q\alpha\alpha} > 0$ and $N_{q\theta} \leq 0$.

[10] The proof of the Theorem states a more complicated condition that must hold if $\alpha'(\widehat{q}) = 0$, but $\alpha'(q) > 0$ for some $q < \widehat{q}$.

[11] We have $\frac{\partial}{\partial q}(u_{q\theta}u - u_\theta u_q) = u_{qq\theta}u - u_\theta u_{qq} > 0$ whenever $u_{qq\theta} > 0$, since $u_{qq} < 0$.

Under this assumption, the solution to subproblem (i) takes on a particularly simple form:

Suppose that Assumption 3 holds. Then for any $\widehat{q}$, there exists a unique junction point $\widetilde{q} \in (0, \widehat{q}]$ such that the solution to subproblem (i) is a nonsingular arc on $[0, \widetilde{q}]$ and a singular arc on $(\widetilde{q}, \widehat{q}]$.

Theorem 5.1 saya that the optimal solution on the interval $[0, \widetilde{q}]$ is a nonsingular, and thus the isoquants for all quantities $q \in [0, \widetilde{q}]$ emanate from the single point $(\alpha(\widetilde{q}), \theta(\widetilde{q}))$. Hence, the type $(\alpha(\widetilde{q}), \theta(\widetilde{q}))$ is indifferent between all quantities $q \in [0, \widetilde{q}]$. Importantly, all isoquants associated with quantities in the interval $q \in [0, \widetilde{q}]$ emanate from the same point $(\alpha(\widetilde{q}), \theta(\widetilde{q}))$ on the lower boundary.

In other words, there is a discontinuity in the allocation assigned to types on the lower boundary at the point $(\alpha(\widetilde{q}), \theta(\widetilde{q}))$. Unlike in the one-dimensional type case, this discontinuity is not associated with gaps in the consumption schedule.[12]

5.2 Exclusion and Other Properties

We can use Theorem 7 to derive necessary conditions for the demand profile approach to yield the correct optimal screening mechanism:

Theorem 8 *Suppose that the functions $\phi(q, \theta)$ and $\varkappa(q, \alpha)$ are increasing in q and decreasing in α and θ. Then for the demand profile approach to yield the optimal screening mechanism it is necessary and sufficient that $\widehat{q} = 0$ in the optimal mechanism.*

The conditions of Theorem 8 are extremely stringent, as our example below will illustrate.

We can use Theorem 5 to establish necessary and sufficient conditions for the absence of exclusion in the optimal screening mechanism:

Theorem 9 *Suppose the conditions of Theorem 5 hold. Then almost all types get assigned a strictly positive quantity if and only if there exist $\mathring{\alpha} > 0$ such that $u_{q\alpha}(0, \alpha, 0) = 0$ for all $\alpha \in [0, \mathring{\alpha}]$ and*

$$- u_q(0, \mathring{\alpha}, 0) \int_0^{\mathring{\alpha}} \frac{f(a, 0)}{u_{q\theta}(q, a, 0)} da + H(0, \mathring{\alpha}, 0) = 0 \tag{45}$$

Theorem 9 sheds light on how multi-dimensionality of customer types affects the monopolist's incentive to exclude some customers from the market. To this effect, let us provide an economic interpretation of equation (45). Consider the aggregate demand for the first increment of quantity, $N(p, 0) = \#\{t : u_q(0, t) \geq p\}$. Let $\alpha(p)$ be such that the demand price of type $(\alpha(p), 0)$ equals p, i.e. $u_q(0, \alpha(p), 0) = p$. Then we have $N(p, 0) = H(0, \alpha(p), 0)$. Thus equation (45) says that the marginal price for the first increment must maximize the profits from that increment, i.e.. it is equivalent to:

$$p \frac{\partial N}{\partial p}(p, 0) + N(p, 0) = 0$$

[12] In the one-dimensional type case, if there is an interval of $[q_1, q_2]$ on which $t(q)$ is constant, then no consumer other than $t(q_1)$ purchases quantities in the interval $[q_1, q_2]$.

Exclusion will occur if and only if at the monopoly price for this increment some consumers decide not to purchase the increment, i.e. if there are some types $\alpha < \mathring{\alpha}$ for which $u_q(0, \alpha, 0) < u_q(0, \mathring{\alpha}, 0)$. The condition $u_{q\alpha}(0, \alpha, 0) = 0$ for all $\alpha \in [0, \mathring{\alpha}]$ rules this out.

As in the one-dimensional type case, absence of exclusion requires the demand curve for the first increment to be perfectly inelastic at a price equal to marginal cost. Indeed, if there is no "gap" between the lowest demand price for the first increment and marginal cost, i.e. if $u_q(0, \alpha, 0) = 0$, then equation (45) cannot hold since it requires $\int_0^\alpha f(a, 0)da = \infty$. So, exclusion occurs. On the other hand, if the gap between the lowest demand price for the first increment and marginal cost is sufficiently large, then like in the one-dimensional type case there can be no exclusion, provided $u_{q\alpha}(q, \alpha, 0) = 0$ for all $\alpha \in [0, \mathring{\alpha}]$. Our next example shows that this can indeed happen.

Example 3: Let $u(q, \alpha, \theta) = \frac{1+\theta+k}{2} q - \frac{b-\alpha}{2} q^2$ for some $k \geq 0$ and $b \geq 1$, and let $f(\alpha, \theta) = 1$ for all $(\alpha, \theta) \in [0, 1]^2$. Then (45) becomes $-(1 + k)\alpha + 1 = 0$, so we have $\mathring{\alpha} = \frac{1}{1+k} \in (0, 1]$.

Armstrong (1996) has argued that exclusion necessarily occurs when types are multi-dimensional. Since our other assumptions are consistent with Armstrong's[13], Theorem 9 indicates that Armstrong's conclusion is specific to cases where the allocation space and the type space have the same dimensionality. Nevertheless, our theorem also demonstrates that there is a sense in which non-exclusion is harder to obtain when the type space is multi-dimensional. When the monopolist raises the marginal price for the first increment above the level where all consumers are included, she trades off the extra dollar gained on all existing customers (measured by the term $H(0, \alpha, 0)$) against the loss in revenue caused by some consumers dropping out of the market (measured by the term in (45)). The number of lost customers is measured (roughly) by the length of the isoquant emanating from the point $(\mathring{\alpha}, 0)$. If $\mathring{\alpha} = 0$, then the number of customers dropping out would be negligible, and exclusion would always pay. This is essentially the effect identified by Armstrong. On the other hand, if $\mathring{\alpha} > 0$, then for no customer to be excluded at the price $u_q(0, \mathring{\alpha}, 0)$ the isoquant through $(\mathring{\alpha}, 0)$ at the quantity $q = 0$ must be flat. If $u_{q\alpha}(0, \alpha, \theta) > 0$ for all (α, θ), then exclusion would necessarily occur.

6 A linear-quadratic example

In this section, we derive an explicit solution for a parametrically specified example. Let

$$u(q, \alpha, \theta) = \theta q - \frac{b - \alpha}{2} q^2 \tag{46}$$

where $b \geq 1$. Furthermore, let (α, θ) be uniformly distributed on the unit square $I = [0, 1] \times [0, 1]$:

$$f(\alpha, \theta) = 1 \text{ for all } (\alpha, \theta) \in I. \tag{47}$$

Note that since $u_q(0, \alpha, 0) = 0$, by Theorem 9 there will always be exclusion in the optimal mechanism. The solution to this example takes on a different qualitative form depending

[13]In particular, our type space is a strictly convex set with non-empty interior. Also, the utility function in Example 3 is convex and homogeneous of degree one in types.

upon whether $b \geq \frac{3}{2}$ or $b < \frac{3}{2}$. We start with the case $b \geq \frac{3}{2}$, which was previously analyzed by Laffont, Maskin and Rochet (1987).

Theorem 10 *The optimal screening mechanism for the linear-quadratic uniformly distributed example (46)-(47) with $b \geq 3/2$ is as follows:*

Let $q^* = \frac{2}{2b+1}$, $\theta^* = \frac{2b-1}{2b+1}$, and $\bar{q} = \frac{1}{b-1}$. Then $\alpha(q) = 1$ for all $q \in [0, \bar{q}]$ and

$$\theta(q) = \begin{cases} \frac{1+2(b-1)q}{3}, & \text{for } q \in [q^*, \bar{q}] \\ \frac{1+(b-\frac{3}{2})q}{2}, & \text{for } q \in [0, q^*]. \end{cases}$$

Thus the optimal nonlinear tariff is given by:

$$P(q) = \begin{cases} \frac{1}{6(2b+1)} + \frac{q(2-(b-1)q)}{6}, & \text{for } q \in [q^*, \bar{q}] \\ \frac{q}{8}(4 - (2b-1)q), & \text{for } q \in [0, q^*]. \end{cases}$$

When $b \geq \frac{3}{2}$, we have $q^{**} = \widehat{q} = 0$ so the region associated with 19 is empty. All isoquants therefore emanate from the portion of right hand boundary $\alpha = 1$ above $\theta = \frac{1}{2}$, i.e. the interval of points $\{(1, \theta) : \theta \in [\frac{1}{2}, 1]$. Note in particular that the isoquant associated with $q = 0$ is a flat line segment at $\theta = \frac{1}{2}$, i.e. the collection of points $\{(\alpha, \frac{1}{2}) : \alpha \in [0, 1]\}$. Figure 2 illustrates the isoquants for this case. None of the iso-price lines associated with this mechanism intersect each other in the type space. As a consequence, the demand profile approach properly identifies the optimal mechanism. Since α varies from $b - 1$ to b, large values of b are associated with low variability in the slope parameter. Thus, one way to interpret this result is that when the uncertainty is (sufficiently) close to one dimensional, the demand profile approach is valid. We now turn to the significantly more complicated case where $b < 3/2$.

Theorem 11 *The optimal screening mechanism for the linear-quadratic uniformly distributed example (46)-(47) with $b < 3/2$ is as follows:*

Let $\alpha^{**} = \frac{2b}{3}$, $\theta^{**} = 1 - \frac{2bq^{**}}{3}$, q^{**} the unique non-negative root to the equation

$$(1 + bq - bq^2(\frac{3}{2} + 2b))^2 = (1 - bq)(1 + bq - bq^2(\frac{5}{2} + b))^3, \tag{48}$$

and

$$\widehat{\theta} = \frac{1 + \sqrt{(1 - bq)(4(1 + bq) - 2bq^2(5 + 2b)}}{3},$$

and $\widehat{q} = \frac{3\widehat{\theta}-1}{2(b-1)}$. Then

$$\alpha(q) = \begin{cases} \alpha^{**}, & \textit{for } q \in [0, q^{**}] \\ c_0 + \frac{c_1}{27}(2 - \sqrt{1 - \frac{6}{c_1 q}})(1 + \sqrt{1 - \frac{6}{c_1 q}})^2, & \textit{for } q \in [q^{**}, \widehat{q}] \\ 1, & \textit{for } q \in [\widehat{q}, \bar{q}] \end{cases}$$

and

$$\theta(q) = \begin{cases} \theta^{**}, & \textit{for } q \in [0, q^{**}] \\ \frac{2}{3} - \frac{1}{3}\sqrt{1 - \frac{6}{c_1 q}}, & \textit{for } q \in [q^{**}, \widehat{q}] \\ 1, & \textit{for } q \in [\widehat{q}, \bar{q}] \end{cases}$$

where the constants c_0 and c_1 are related to $\widehat{\theta}$ as follows:

$$c_1 = \frac{4(b-1)}{(1-\widehat{\theta})(3\widehat{\theta}-1)^2}, \textit{ and} \tag{49}$$

$$c_0 = \frac{\widehat{\theta}^2(5+4b) - \widehat{\theta}(2+4b)+1}{(3\widehat{\theta}-1)^2} < 0 \tag{50}$$

Thus the optimal nonlinear tariff is given by:

$$P(q) = \begin{cases} u(q, \alpha(q), \theta(q)), & \textit{for } q \in [0, \widehat{q}] \\ \frac{(b-1)\widehat{q}^2}{3} + \frac{q}{6}(2-(b-1)q), & \textit{for } q \in [0, q^*]. \end{cases}$$

For $b < \frac{3}{2}$, in the optimal screening mechanism the isoquants for $q \in [0, q^{**}]$ all emanate from the point $(\alpha^{**}, \theta^{**})$ on the lower boundary. In particular, for $q = 0$ the isoquant is the flat segment at the level $\theta = \theta^{**}$ with $\alpha = \alpha^{**}$, i.e. the collection of points $\{(\alpha, \theta^{**}) : \alpha \in [0, \alpha^{**}]\}$. For $q \in [q^{**}, \widehat{q}]$ the lower boundary is strictly decreasing, and given by the equation $\alpha = c_0 + c_1\theta(1-\theta)^2$. For this segment of q values there is a unique isoquant associated with every point on the lower boundary. Note that since all types (α, θ^{**}) along the lower boundary with $\alpha \leq \alpha^{**}$ are assigned a quantity 0, and since all types along the lower boundary with $\theta > \theta^{**}$ are assigned a quantity $q \geq q^{**}$, there is a discontinuity in the optimal quantity assignment along the lower boundary. Finally, for $q \geq \widehat{q}$, all isoquants emanate from the portion of the right hand boundary $\alpha = 1$ with $\theta \geq \widehat{\theta}$. Figure 3 illustrates the lower boundary and the isoquants for the case $b < \frac{3}{2}$. It is important to observe that while the isoquants associated with the optimal mechanism never intersect in the interior of the participation region, the corresponding price lines would intersect in the region of non-participation. Thus in accordance with Theorem 8 for every value of the parameter b with $b < 3/2$, the demand profile is incapable of correctly identifying the optimal mechanism.

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7 Conclusions

In this paper, we have shown that the traditional method for identifying an optimal screening mechanism, the demand profile approach, generally fails when there is multi-dimensional uncertainty. Only under rather extreme conditions on the type distribution, essentially reducing the problem to one with single dimensional uncertainty, will the chosen mechanism be optimal. We identified the reason for this failure: with multi-dimensional uncertainty, a consumer's demand schedule must generally intersect the optimal marginal price schedule multiple times, thereby wreaking havoc with the global incentive compatibility requirement.

We introduced a novel condition, termed single crossing of demand (SCD), under which global incentive compatibility can nevertheless be assured. This condition guarantees that if a quantity $q > 0$ solves the surplus maximization problem of an agent of type (α, θ), then q must also be a global optimum for any type on the portion of the iso-price curve at the quantity q going through the point (α, θ) that lies to the northwest of this point. As a consequence, isoquants are the portions of isoprice curves that lie above a lower boundary defined by the individual rationality constraint.

Correct identification of these isoquants then allowes us to reduce the problem to a one-dimensional screening problem, all be it a rather complicated one. We were able to reduce the resulting optimization problem to an optimal control problem, and identify its solution. We also illustrated application of our methodology to an example with quadratic demand and uniformly distributed types.

Our methodology has already identified some relatively robust properties of optimal screening mechanism with multidimensional types. In particular, the allocation to an agent may be discontinuous in type along the boundary of the participation region. We also showed that the optimal mechanism may or may not exclude some types from participation. We hope that our paper will stimulate new research into several of the applications cited in the introduction.

While the present analysis was confined to the case where the (physical) allocation space is one-dimensional, our approach should prove useful in analyzing more general screening problems in which the dimensionality of the type space exceeds the dimensionality of the allocation space.

8 Appendix A

In the subsequent proofs, we will make use of the following technical Lemma:

Lemma 10 *Suppose that Assumption 2 holds. Then* $\frac{u_{q\alpha}(q,\alpha,\theta)}{u_{q\theta}(q,\alpha,\theta)} - \frac{u_{\alpha}(q,\alpha,\theta)}{u_{\theta}(q,\alpha,\theta)} > 0$

Proof: Fix $(\alpha, \theta) \in [0,1]^2$ and define $\varphi(q) = \frac{u_{q\alpha}}{u_{q\theta}}(q,\alpha,\theta) - \frac{u_{\alpha}}{u_{\theta}}(q,\alpha,\theta)$. Then, $\varphi'(q) = \frac{d}{dq}\left(\frac{u_{q\alpha}}{u_{q\theta}}\right) - \varphi(q)\frac{u_{q\theta}}{u_{\theta}}$. Assumption 2 implies that for any $q > 0$ s.t. $\varphi(q) \leq 0$ we have $\varphi'(q) > 0$. Thus, if $\varphi(q) \leq 0$ for some $q > 0$, then $\varphi(q') < \varphi(q)$ for all $q' < q$, and so $\lim_{q' \to 0} \varphi(q') < 0$. But since u_α and u_θ both converge to zero as $q \to 0$, it follows from l'Hospital's rule that $\lim_{q\to 0} \varphi(q) = 0$, a contradiction. Hence, $\varphi(q) > 0$ if $q > 0$. *Q.E.D.*

Proof of Lemma 1: Observe that, for a fixed q the relation $u_q(q, \theta', \alpha') = u_q(q, \theta, \alpha)$ implicitly defines a function $\widetilde{\theta}(\alpha)$. Note that $u_{q\theta}(q, \widetilde{\theta}(\alpha), \alpha)\frac{d\widetilde{\theta}}{d\alpha} + u_{q\alpha}(q, \widetilde{\theta}(\alpha), \alpha) = 0$. Hence $u_{qq}(q, \theta', \alpha') - u_{qq}(q, \theta, \alpha) = \int_{\alpha}^{\alpha'} [u_{qq\theta}(q, \widetilde{\theta}(a), a)\frac{d\widetilde{\theta}}{d\alpha} + u_{qq\alpha}(q, \widetilde{\theta}(a), a)]da = \int_{\alpha}^{\alpha'} [-u_{qq\theta}\frac{u_{q\alpha}}{u_{q\theta}} + u_{qq\alpha}]da > 0$, proving the desired result. *Q.E.D.*

Proof of Lemma 2: (i) Let t_1 be the transfer associated with quantity q_1 in the mechanism (i.e., there is a type $(\tilde{\alpha}, \tilde{\theta})$ s.t. $q_1 = q(\tilde{\alpha}, \tilde{\theta})$, $t_1 = t(\tilde{\alpha}, \tilde{\theta})$). Since $q_1 \in Q^*(\alpha_1, \theta_1)$, $u(q_1, \alpha_1, \theta_1) - t_1 \geq u(q(\alpha', \theta'), \alpha_1, \theta_1) - t(\alpha', \theta')$ for all $(\alpha', \theta'^2$. Rearranging, we have

$$t(\alpha', \theta') - t_1 \geq u(q(\alpha', \theta'), \alpha_1, \theta_1) - u(q_1, \alpha_1, \theta_1) \tag{51}$$

Next, note that by assumption of the Lemma, $u_q(q_1, \alpha_1, \theta_1) - u_1(q_1, \alpha_2, \theta_2) = 0$ i.e., q_1 is a stationary point of $u(q, \alpha_1, \theta_1) - u(q, \alpha_2, \theta_2)$ viewed as a function of q. Further, Lemma 1 implies that q_1 is a unique stationary point and is, in fact, a global minimum of of $u(q, \alpha_1, \theta_1) - u(q, \alpha_2, \theta_2)$. Hence, $u(q_1, \alpha_1, \theta_1) - u(q_1, \alpha_2, \theta_2) < u(q(\alpha', \theta'), \alpha_1, \theta_1) - u(q(\alpha', \theta'), \alpha_2, \theta_2)$. Combining this inequality with inequality (51), we obtain:

$$t(\alpha', \theta') - t_1 > u(q(\alpha', \theta'), \alpha_2, \theta_2) - u(q_1, \alpha_2, \theta_2)$$

Since $(q(\alpha', \theta'), t(\alpha', \theta'))$ was chosen arbitrarily, the pair (q_1, t_1) is the unique optimal choice for type (α_2, θ_2) i.e., $Q^*(\alpha_2, \theta_2) = q_1$.

(ii) Let t_i be the transfer associated with quantity q_i in the mechanism, for $i \in \{1, 2\}$ (i.e., there is a type $(\tilde{\alpha}_i, \tilde{\theta}_i)$ s.t. $q_i = q(\tilde{\alpha}_i, \tilde{\theta}_i)$, $t_i = t(\tilde{\alpha}_i, \tilde{\theta}_i)$).

For any $\alpha' \in [\alpha, 1]$, let $\widetilde{\theta}(\alpha')$ solve the equation:

$$u(q_1, \alpha', \widetilde{\theta}(\alpha')) - t_1 = u(q_2, \alpha', \widetilde{\theta}(\alpha')) - t_2.$$

Thus, type $(\alpha', \widetilde{\theta}(\alpha'))$ is indifferent between (q_1, t_1) and (q_2, t_2). Suppose that $q_1 < q_2$. The proof for the opposite case is symmetric. By the implicit function theorem,

$$\frac{d\widetilde{\theta}}{d\alpha}_{|\alpha=\alpha'} = -\frac{u_\alpha(q_1, \alpha', \widetilde{\theta}(\alpha')) - u_\alpha(q_2, \alpha', \widetilde{\theta}(\alpha'))}{u_\theta(q_1, \alpha', \widetilde{\theta}(\alpha')) - u_\theta(q_2, \alpha', \widetilde{\theta}(\alpha'))} = -\frac{\int_{q_1}^{q_2} u_{q\alpha}(z, \alpha', \widetilde{\theta}(\alpha'))dz}{\int_{q_1}^{q_2} u_{q\theta}(z, \alpha', \widetilde{\theta}(\alpha'))dz}$$

It follows from Assumption 2 that

$$-\frac{u_{q\alpha}(q_1, \alpha', \widetilde{\theta}(\alpha'))}{u_{q\alpha}(q_1, \alpha', \widetilde{\theta}(\alpha'))} < \frac{d\widetilde{\theta}}{d\alpha}_{|\alpha=\alpha'} \tag{52}$$

On the other hand, letting $\hat{\theta}(\alpha')$ solve

$$u_q(q_1, \alpha, \theta) = u_q(q_1, \alpha', \hat{\theta}(\alpha'))$$

we obtain:

$$\frac{d\hat{\theta}}{d\alpha}_{|\alpha=\alpha'} = -\frac{u_{q\alpha}(q_1, \alpha', \hat{\theta}(\alpha'))}{u_{q\alpha}(q_1, \alpha', \hat{\theta}(\alpha'))} \tag{53}$$

Combining (52) and (53), we conclude that $\hat{\theta}(\alpha') > \widetilde{\theta}(\alpha')$ for $\alpha' > \alpha$. However, since type $(\alpha', \widetilde{\theta}(\alpha'))$ is indifferent between (q_1, t_1) and (q_2, t_2), $q_1 > q_2$, and $u(.)$ is supermodular, type $(\alpha', \hat{\theta}(\alpha'))$ must strictly prefer (q_2, t_2) to (q_1, t_1) i.e., $q_1 \notin Q^*(\alpha', \hat{\theta}(\alpha'))$. *Q.E.D.*

Proof of Lemma 3: The continuity of $s(\alpha, \theta)$ follows from the continuity of the function $u(q, \alpha, \theta)$ in (α, θ). Note that in the optimal mechanism $s(\alpha, \theta) = 0$ if $q(\alpha, \theta) = 0$, for otherwise the firm can increase its profits by setting to zero the transfer paid by the types who get zero quantity in the mechanism. So, $s(\alpha, \theta) = 0$ if $q(\alpha, \theta) = 0$. Further, if (θ', α) is such that $\theta' > \underline{\theta}(\alpha)$, then $q(\alpha, \theta'') > 0$ for any $\theta'' \in (\underline{\theta}(\alpha), \theta')$. Since $s(\alpha, \theta'') \geq 0$, strict monotonicity of $u(q(\alpha, \theta''), \alpha, \theta)$ in θ implies that $s(\alpha, \theta') > 0$.

The continuity of $\underline{\theta}(\alpha)$ then follows from the continuity of $s(\alpha, \theta)$.

To establish that $\underline{\theta}(\alpha)$ is monotonically decreasing, recall that $u(.)$ is supermodular in (α, θ). Hence, $Q^*(.)$ is an increasing correspondence. Therefore, if $q(\alpha, \theta) > 0$ for some type (α, θ) and $\alpha' > \alpha$, then $q(\alpha', \theta) > 0$. Hence, $\underline{\theta}(\alpha') \leq \underline{\theta}(\alpha)$.

Let us now show that $\underline{\theta}(.)$ is strictly decreasing at α if $q(\alpha, \underline{\theta}(\alpha)) > 0$. Indeed, in this case, the monotonicity of $u(.)$ in θ implies that $s(\alpha', \underline{\theta}(\alpha)) > 0$ and hence $\underline{\theta}(\alpha') < \underline{\theta}(\alpha)$ for all $\alpha' > \alpha$.

To show that $Q^*(\alpha, \underline{\theta}(\alpha))$ is increasing in α, suppose otherwise. Then there exist $\alpha_1, \alpha_2 \in [0, 1]$, $\alpha_1 > \alpha_2$, with $q_i \in Q^*(\alpha_i, \underline{\theta}(\alpha_i))$ for $i \in \{1, 2\}$ s.t. $q_1 < q_2$. Consider the left isoprice line $I(q_1, \alpha_1, \underline{\theta}(\alpha_1))$ and some $(\alpha', \theta') \in I(q_1, \alpha_1, \underline{\theta}(\alpha_1)) \setminus (\alpha_1, \underline{\theta}(\alpha_1))$ By Lemma 2, $q_1 = Q^*(\alpha', \theta')$. Since without loss of generality the allocation $(q = 0, t = 0)$ is one of the choices offered in the optimal mechanism, the fact that $\{0\} \notin Q^*(\alpha', \theta')$ implies that $s(\alpha', \theta') > 0$ and, hence, $\theta' > \underline{\theta}(\alpha')$.

Since (α', θ') is an arbitrary point in $I(q_1, \alpha_1, \underline{\theta}(\alpha_1)) \setminus (\alpha_1, \underline{\theta}(\alpha_1))$, it follows that $I(q_1, \alpha_1, \underline{\theta}(\alpha_1)) \cap \{(\alpha, \underline{\theta}(\alpha)) | \alpha \in [0, 1]\} = (\alpha_1, \underline{\theta}(\alpha_1))$. Combining the latter fact with Assumption 1 (iii) we conclude that there exists $\alpha'' \in (\alpha_2, \alpha_1)$ s.t. $(\alpha'', \underline{\theta}(\alpha_2)) \in I(q_1, \alpha_1, \underline{\theta}(\alpha_1))$ and thus $Q^*(\alpha'', \underline{\theta}(\alpha_2)) = q_1$. But since $q_1 < q_2 \in Q^*(\alpha_2, \underline{\theta}(\alpha_2))$, this contradicts the fact that the correspondence $Q^*(.)$ is increasing. Since $Q^*(\alpha, \underline{\theta}(\alpha))$ is monotonically increasing and bounded on $[0, 1]$ it follows that, except at a countably many points, $Q^*(\alpha, \underline{\theta}(\alpha))$ is a singleton i.e. $Q^*(\alpha, \underline{\theta}(\alpha)) = q(\alpha, \underline{\theta}(\alpha))$ and $q(\alpha, \underline{\theta}(\alpha))$ is continuous.

To prove the absolute continuity of $\underline{\theta}(\alpha)$ under the additional assumption of the Lemma, consider some $\alpha, \alpha' \in [0,1]$ s.t. $\alpha > \alpha'$. From $s(\alpha, \underline{\theta}(\alpha)) = s(\alpha', \underline{\theta}(\alpha')) = 0$ and incentive compatibility, it follows that

$$\begin{aligned} 0 &= u(q(\alpha,\underline{\theta}(\alpha)),\alpha,\underline{\theta}(\alpha)) - t(\alpha,\underline{\theta}(\alpha)) \geq u(q(\alpha',\underline{\theta}(\alpha')),\alpha,\underline{\theta}(\alpha)) - t(\alpha',\underline{\theta}(\alpha')) \\ 0 &= u(q(\alpha',\underline{\theta}(\alpha')),\alpha',\underline{\theta}(\alpha')) - t(\alpha',\underline{\theta}(\alpha')) \geq u(q(\alpha,\underline{\theta}(\alpha)),\alpha',\underline{\theta}(\alpha')) - t(\alpha,\underline{\theta}(\alpha)) \end{aligned}$$

Consequently, we have

$$u(q(\alpha,\underline{\theta}(\alpha)),\alpha,\underline{\theta}(\alpha)) - u(q(\alpha,\underline{\theta}(\alpha)),\alpha',\underline{\theta}(\alpha')) \geq 0 \geq u(q(\alpha',\underline{\theta}(\alpha')),\alpha,\underline{\theta}(\alpha)) - u(q(\alpha',\underline{\theta}(\alpha')),\alpha',\underline{\theta}(\alpha')) \tag{54}$$

Using the mean value theorem, we obtain

$$u_\theta(q(\alpha',\underline{\theta}(\alpha')),\alpha_0,\underline{\theta}(\alpha_0))(\underline{\theta}(\alpha) - \underline{\theta}(\alpha')) + u_\alpha(q(\alpha',\underline{\theta}(\alpha')),\alpha_0,\underline{\theta}(\alpha_0))(\alpha - \alpha') \leq 0 \tag{55}$$

$$u_\theta(q(\alpha,\underline{\theta}(\alpha)),\alpha_1,\underline{\theta}(\alpha_1))(\underline{\theta}(\alpha) - \underline{\theta}(\alpha')) + u_\alpha(q(\alpha,\underline{\theta}(\alpha)),\alpha_1,\underline{\theta}(\alpha_1))(\alpha - \alpha') \geq 0 \tag{56}$$

for some α_0 and α_1 s.t. $\alpha_0, \alpha_1 \in [\alpha', \alpha]$.

The inequalities (55) and (56) imply

$$-\frac{u_\alpha}{u_\theta}(q(\alpha,\underline{\theta}(\alpha)),\alpha_1,\underline{\theta}(\alpha_1)) \leq \frac{\underline{\theta}(\alpha) - \underline{\theta}(\alpha')}{\alpha - \alpha'} \leq -\frac{u_\alpha}{u_\theta}(q(\alpha',\underline{\theta}(\alpha')),\alpha_0,\underline{\theta}(\alpha_0)) \tag{57}$$

Let $T = \max_{(q,\alpha,\theta)} \frac{u_\alpha}{u_\theta}(q,\alpha,\theta) < \infty$. From (57) it follows that the function $\underline{\theta}(\alpha)$ is Lipschitz continuous with Lipschitz constant T, and hence it is absolutely continuous.

Since $\underline{\theta}(\alpha)$ is monotonically decreasing, it is differentiable almost everywhere. Taking limits in (57), we obtain that at any continuity point of $q(\alpha, \underline{\theta}(\alpha))$ we have:

$$\underline{\theta}'(\alpha) = -\frac{u_\alpha}{u_\theta}(q(\alpha,\underline{\theta}(\alpha)),\alpha,\underline{\theta}(\alpha)). \tag{58}$$

Q.E.D.

Proof of Lemma 4: (i) Rochet and Stole (2003) and Basov (2001) have shown that the optimal allocation $q(\alpha, \theta)$ must satisfy an elliptical partial differential equation. It is well-known that solutions to elliptical partial differential equations on a domain with a piecewise smooth boundary are continuous on the interior of that domain. (ii) Note that without loss of generality, the optimal direct mechanism $(t(\alpha,\theta), q(\alpha,\theta))$ can be represented as a set of quantity transfer pairs $\{(t,q)\}_{q\in[0,q^*(1,1)}$, where $q^{1,1} = \arg\max_q u(q,1,1)$ and t is continuous in q. Then, it follows from the Generalized Theorem of the Maximum (Ausubel and Deneckere, 1993) that $Q^*(.)$ is a non-empty closed-valued u.h.c. correspondence.

To show that $Q^*(\alpha,\theta)$ is convex-valued for any (α,θ) s.t. either $\theta < 1$ or $\alpha < 1$, suppose that q_1, $q_2 \in Q^*(\alpha,\theta)$ with $q_1 < q_2$, and let $q \in (q_1, q_2)$. Choose some $\varepsilon > 0$ s.t. $\theta + \varepsilon \leq 1$. Then by Lemma 2, for any $\theta' \in (\theta, \theta+\varepsilon)$ there exists α_1 and α_2 such that $(\alpha_1, \theta') \in I(q_1, \alpha, \theta)$ and $(\alpha_2, \theta') \in I(q_2, \alpha, \theta)$, and so $q(\alpha_1, \theta') = q_1$ and $(\alpha_2, \theta') = q_2$.

Further, by continuity of $q(.)$, there exists $\alpha'(\theta') \in (\alpha_1, \alpha_2)$ such that $q = q(\alpha'(\theta'), \theta')$. Since this is true for all $\theta' \in (\theta, \theta + \varepsilon)$, it follows from u.h.c. of the correspondence Q^* that $q \in Q^*(\alpha, \theta)$.

(iii) Part (ii) of this Lemma implies that that the correspondence $w : L \to R$ defined by $w(\alpha,\theta) = \{u_q(q,\alpha,\theta) : q \in Q^*(\alpha,\theta)\}$ is convex valued and hence that $w(L)$ is a closed interval.

Let us now show that for each $(\alpha,\theta) \notin L$ s.t. $q(\alpha,\theta) > 0$, there exists some $(\alpha',\theta') \in L$ and $q' \in Q^*(\alpha',\theta')$ such that $u_q(q',\alpha,\theta) = u_q(q',\alpha',\theta')$. Since $u_{q\theta} > 0$, we have $u_q(q(\alpha,\underline{\theta}(\alpha)),\alpha,\underline{\theta}(\alpha)) < u_q(q(\alpha,\underline{\theta}(\alpha)),\alpha,\theta)$. Also, since $u_{q\alpha} > 0$, we have $u_q(q(1,\theta),1,\theta)) > u_q(q(1,\theta),\alpha,\theta)$. Hence there exists (α',θ') on the segment of L connecting $(\alpha,\underline{\theta}(\alpha))$ to $(1,\theta)$ and $q' \in Q^*(\alpha',\theta')$ such that $u_q(q',\alpha,\theta) = u_q(q',\alpha',\theta')$ i.e., $(\alpha,\theta) \in I(q',\alpha',\theta')$. From Lemma 2

Finally, if (α,θ) can lie on at most one isoquant emanating from L, for otherwise isoquants would intersect in the interior of the participation region. *Q.E.D.*

Proof of Theorem 1: (ii) First, we shall argue that the correspondence Q^* is nondecreasing along L. To this effect, define an artificial type $\lambda \in [0, 2-\widehat{\theta}]$ along L such that $\lambda = \alpha$ if $\alpha < 1$ and $\lambda = 1 + (\theta - \widehat{\theta})$ if $\alpha = 1$. Lemma 10 implies that $u_{q\lambda} = u_{q\theta}\underline{\theta}'(\alpha) + u_{q\alpha} > 0$ for $\lambda < 1$ and $q > 0$, and $u_{q\lambda} = u_{q\alpha} > 0$ for $\lambda \geq 1$. This supermodularity implies that every selection from $Q^*(\lambda)$ must be non-decreasing. Hence $Q^*(\lambda)$ is single-valued for almost all λ, and any selection from $Q^*(\lambda)$ is a non-decreasing function. Since changing the allocation on a set of measure zero of λ does not alter the monopolist's expected profits, we may select $\underline{q}(\alpha) = \min Q^*(\alpha,\underline{\theta}(\alpha))$ and $\underline{q}(\theta) = \min Q^*(1,\theta)$.

(iii) Let $q \in Q^*(\alpha,\underline{\theta}(\alpha))$ and $q' \in Q^*(\alpha',\underline{\theta}(\alpha'))$. Since the mechanism $(q(.),t(.))$ is incentive compatible,

$$u(q,\alpha,\underline{\theta}(\alpha)) - u(q',\alpha,\underline{\theta}(\alpha)) \leq t(q) - t(q') \leq u(q,\alpha',\underline{\theta}(\alpha')) - u(q',\alpha',\underline{\theta}(\alpha'))$$

Using the mean value theorem, we obtain:

$$u_q(z_0,\underline{\theta}(\alpha),\alpha)(q-q') \leq t(q) - t(q') \leq u_q(z_1,\underline{\theta}(\alpha'),\alpha')(q-q') \tag{59}$$

for some z_0 and $z_1$between q and q'. Let $M = \max_{(\theta,\alpha)\in[0,1]^2} u_q(0,\theta,\alpha)$. Then it follows from (59) that P is Lipschitz continuous with Lipschitz constant M, and hence absolutely continuous.

If $q \in [\underline{q}(0), \underline{q}(\widehat{\alpha})]$, then $q \in Q^*(\alpha,\underline{\theta}(\alpha))$ for some α. Hence we have $s(\alpha,\underline{\theta}(\alpha)) = 0$, implying $u(q,\alpha,\underline{\theta}(\alpha)) - P(q) = 0$.

Next, suppose that $\widehat{\alpha} < 1$, and $q \in [\underline{q}(\widehat{\alpha}), \underline{q}(1)]$. First, consider any $\alpha \in [\widehat{\alpha}, 1]$ at which $\underline{q}(\alpha)$ is discontinuous, so that $Q^*(\alpha,\underline{\theta}(\alpha))$ is multi-valued. Let $q_1 = \min Q^*(\underline{\theta}(\alpha),\alpha)$ and $q_2 = \max Q^*(\underline{\theta}(\alpha),\alpha)$. By (ii) we have $u(q,\alpha,\underline{\theta}(\alpha)) - P(q) = u(q_1,\alpha,\underline{\theta}(\alpha)) - P(q_1)$ for all $q \in [q_1,q_2]$, implying $u_q(q,\alpha,\underline{\theta}(\alpha)) = P'(q)$ for all $q \in (q_1,q_2)$. Next, consider any $\alpha \in [\widehat{\alpha}, 1]$ at which $\underline{q}(\alpha)$ is continuous and strictly increasing. Dividing (59) by $(q'-q)$ and taking limits as $\alpha' \downarrow \alpha$, it follows that $P'(q) = u_q(q,\alpha,\underline{\theta}(\alpha))$. We conclude that the Stieltjes integral $P(q) = u(\underline{q}(\widehat{\alpha}),\widehat{\alpha},\underline{\theta}(\widehat{\alpha})) + \int_{\underline{q}(\widehat{\alpha})}^{q} u_q(z,\underline{\theta}(\alpha(z)),\alpha(z))dz$ holds. For $q \in (\underline{q}(1), \bar{q}(1)]$ the argument is analogous. *Q.E.D.*

Proof of Theorem 2: First, we establish that the allocation $q(\alpha,\theta)$ is incentive compatible along L. It follows from (ii) that $q(\alpha,\underline{\theta}(\alpha)) = \underline{q}(\alpha)$ for all $\alpha \in [0,1]$, and $q(1,\theta) = \bar{q}(\theta)$

for all $\theta \in [\widehat{\theta}, 1]$. Hence the allocation is nondecreasing along L. Since $u_{q\lambda} > 0$, it follows that the allocation is incentive compatible along L. Lemma 2 then implies that $q(\alpha, \theta)$ is incentive compatible for all (α, θ) in the participation region. It remains to be shown that $Q^*(\alpha, \theta) = \{0\}$ for all (α, θ) such that $\theta < \underline{\theta}(\alpha)$. Note that for any $q > 0$, we have $u(q, \theta, \alpha) - P(q) < u(q, \underline{\theta}(\alpha), \alpha) - P(q) \leq s(\underline{\theta}(\alpha), \alpha)) = 0$. Thus for any such type it is uniquely optimal to select $q = 0$. That (iii) holds follows from the proof of part (iii) of Theorem 1. *Q.E.D.*

Proof of Theorem 3: Theorem 2 implies that the set of types who are assigned quantities that strictly exceed q is given by:

$$\{(a, \theta) \in [0, 1]^2 : a \leq \alpha(q),\ \theta > \sigma(q, \alpha(q), \theta(q), a)\} \cup \{(a, \theta) \in [0, 1]^2 : a > \alpha(q),\ \theta \geq \underline{\theta}(a)\}$$

Therefore, the probability measure of the set of types who are assigned quantities above or equal to q is

$$H(q, \alpha(q), \theta(q) = \int_0^{\alpha(q)} \int_{\sigma(q,\alpha(q),\theta(q),a)}^1 f(a, \theta) d\theta da + \int_{\alpha(q)}^1 \int_{\underline{\theta}(a)}^1 f(a, \theta) d\theta da = \int_0^1 \int_{\max\{\sigma(q,\alpha(q),\theta(q),a), \underline{\theta}(a)\}}^1 f(a, \theta) d\theta da$$

Correspondingly, the measure of the set of types who are assigned quantities that do not exceed q is equal to:

$$1 - H(q, \alpha(q), \theta(q)).$$

Differentiation yields the density, $\tilde{h}$, of this probability measure over the interval $[0, \widehat{q}]$:

$$\tilde{h}(q, \alpha(q), \theta(q), \alpha'(q), \theta'(q)) = \int_0^{\alpha(q)} f(\sigma(q, \alpha(q), \theta(q), a) \frac{d}{dq} \sigma(q, \alpha(q), \theta(q), a) da$$

Combining this with (7) allows us to rewrite the monopolist's profits as follows:

$$\int_0^{\hat{q}} u(q, \alpha(q), \theta(q)) \tilde{h}(q, \alpha(q), \theta(q), \alpha'(q), \theta'(q)) dq + \int_{\hat{q}}^{\bar{q}} \left\{ u(\hat{q}, \widehat{\alpha}, \widehat{\theta}) + \int_{\hat{q}}^q u_q(z, \alpha(z), \theta(z)) dz \right\} d(1 - H(q, \alpha(q), \theta(q)))$$

Since $H(\bar{q}, \alpha(\bar{q}), \theta(\bar{q})) = 0$, integration by parts of the second integral yields:

$$H(\hat{q}, \widehat{\alpha}, \widehat{\theta}) u(\hat{q}, \widehat{\alpha}, \widehat{\theta}) + \int_{\hat{q}}^{\bar{q}} H(q, \alpha(q), \theta(q)) u_q(q, \alpha(q), \theta(q)) dq$$

Q.E.D.

Proof of Lemma 5:
Differentiating $S \equiv \frac{\partial J}{\partial \dot{\alpha}}$ yields:

$$\dot{S} = \frac{d}{dq} \frac{\partial J}{\partial \dot{\alpha}} = \frac{d\,(u(h_2 - g h_1))}{dq} + \dot{\mu} - \dot{\lambda} g - \lambda \frac{dg}{dq} \tag{60}$$

By Pontryagin's Maximum Principle the costate equations corresponding to Hamiltonian (29) are:

$$\dot{\mu} = -\frac{\partial J}{\partial \alpha} = -\frac{\partial(uh)}{\partial \alpha} - \frac{\partial(\mu - \lambda g)}{\partial \alpha} \dot{\alpha} = -\frac{\partial(u h_0)}{\partial \alpha} - \frac{\partial(u(h_2 - g h_1)}{\partial \alpha} \dot{\alpha} + \lambda g_\alpha \dot{\alpha} \tag{61}$$

$$\dot{\lambda} = -\frac{\partial J}{\partial \theta} = -\frac{\partial(uh)}{\partial \theta} - \frac{\partial(\mu - \lambda g)}{\partial \theta} \dot{\alpha} = -\frac{\partial(u h_0)}{\partial \theta} - \frac{\partial(u(h_2 - g h_1)}{\partial \theta} \dot{\alpha} + \lambda g_\theta \dot{\alpha} \tag{62}$$

Then we have:

$$\dot{S} = \frac{d\left(u(h_2 - gh_1)\right)}{dq} - \frac{\partial(uh_0)}{\partial\alpha} - \frac{\partial(u(h_2 - gh_1)}{\partial\alpha}\dot{\alpha} + g\frac{\partial(uh_0)}{\partial\theta} + g\frac{\partial(u(h_2 - gh_1)}{\partial\theta}\dot{\alpha} - \lambda g_q \tag{63}$$

$$= \frac{\partial(u(h_2 - gh_1))}{\partial q} - \frac{\partial(uh_0)}{\partial\alpha} + g\frac{\partial(uh_0)}{\partial\theta} - \lambda g_q = \frac{\partial(u(h_2 - gh_1))}{\partial q} - u\left(\frac{\partial h_0}{\partial\alpha} - g\frac{\partial h_0}{\partial\theta}\right) - \lambda g_q \tag{64}$$

where the first equality is obtained by substituting (61) and (62) into (60) and using $\frac{dg}{dq} = g_q + g_\alpha\dot{\alpha} - g_\theta g\dot{\alpha}$ to cancel terms. The second equality holds because $\frac{d(u(h_2-gh_1))}{dq} = \frac{\partial(u(h_2-gh_1))}{\partial q} + \frac{\partial(u(h_2-gh_1))}{\partial\alpha}\dot{\alpha} - g\frac{\partial(u(h_2-gh_1))}{\partial\theta}\dot{\alpha}$. The third equality in (63) holds because, by definition, $u_\alpha - gu_\theta = 0$. The fourth equality follows because $\frac{\partial h_2}{\partial q} = \frac{\partial h_0}{\partial\alpha}$ and $\frac{\partial h_1}{\partial q} = \frac{\partial h_0}{\partial\theta}$. The fifth (last) equality in (63) holds because, by definition of ψ, $g_q\psi = u_q(h_2 - gh_1)$.

Our next step is to compute $\ddot{S}$. For this, we need the following intermediate result:

Lemma 11

$$\textit{We have: } \frac{d}{dq}h_1 - h_\theta = f(\theta,\alpha)\dot{\alpha} + g_\theta h_1\alpha \tag{65}$$

We will use Lemma 11 in computing $\ddot{S}$ below. First, fully differentiating (33) with respect to q, we obtain:

$$\ddot{S} = \frac{dg_q}{dq}(\psi - uh_1 - \lambda) + g_q\left(\frac{d\left(\psi - uh_1 - \lambda\right)}{dq}\right) \tag{66}$$

Let us consider the second term of (66). We obtain:

$$\begin{aligned}
&\frac{d\left(\psi - uh_1 - \lambda\right)}{dq} = \psi_q + \psi_\alpha\dot{\alpha} - \psi_\theta g\dot{\alpha} - u_q h_1 - u\frac{dh_1}{dq} + \frac{\partial(uh_0)}{\partial\theta} + \frac{\partial(uh_2 - gh_1)}{\partial\theta}\dot{\alpha} - \lambda g_\theta\dot{\alpha} = \\
&\psi_q + \psi_\alpha\dot{\alpha} - \psi_\theta g\dot{\alpha} - u_q h_1 - u\frac{dh_1}{dq} + u\frac{\partial h}{\partial\theta} + u_\theta h_0 + u_\theta(h_2 - gh_1)\dot{\alpha} - \lambda g_\theta\dot{\alpha} = \\
&\psi_q + \psi_\alpha\dot{\alpha} - \psi_\theta g\dot{\alpha} - u_q h_1 - uf(\theta,\alpha)\dot{\alpha} - ug_\theta h_1\dot{\alpha} + u_\theta h_0 + u_\theta(h_2 - gh_1)\dot{\alpha} - \lambda g_\theta\dot{\alpha} = \\
&\psi_q + u_\theta h_0 - u_q h_1 + \left[\psi_\alpha - \psi_\theta g - uf(\theta,\alpha) + u_\theta(h_2 - gh_1) - (\lambda + uh_1)g_\theta\right]\dot{\alpha}
\end{aligned} \tag{67}$$

The first equality in (67) is obtained by differentiating, substituting in the expression (62) for $\dot{\lambda}$ and also cancelling terms using the identity $u_\alpha = gu_\theta$. The second equality uses $h = h_0 + (h_2 - gh_1)\dot{\alpha}$. The third equality holds by Lemma 11. The fourth equality holds by rearrangement. Substituting (67) into (66) yields:

$$\ddot{S} = \frac{dg_q}{dq}(\psi - uh_1 - \lambda) + g_q\left(\psi_q + u_\theta h_0 - u_q h_1 + \left[\psi_\alpha - \psi_\theta g - uf(\theta,\alpha) + u_\theta(h_2 - gh_1) - (\lambda + uh_1)g_\theta\right]\dot{\alpha}\right) \tag{68}$$

On a non-singular arc we have $\dot{\alpha} = 0$, using which in (68) yields (**??**).

Further, on a singular arc we have $\dot{S} = 0$, so by (33) $\lambda = \psi - uh_1$. Using this in (68) yields (34). *Q.E.D.*

Proof of Lemma 11: Taking a partial derivative of $h(q,\alpha,\theta,\dot{\alpha})$ in (23) yields:

$$h_\theta = -\underline{\alpha}_\theta f\left(\sigma_q - g\sigma_\theta\alpha' + \sigma_\alpha\alpha'\right)|_{a=\underline{\alpha}} + \int_{\underline{\alpha}(q,\alpha,\theta)}^{\alpha} f_\theta\sigma_\theta\left(\sigma_q - g\sigma_\theta\alpha' + \sigma_\alpha\alpha'\right) + f\left(\sigma_{q\theta} - g\sigma_{\theta\theta}\alpha' + \sigma_{\alpha\theta}\alpha'\right) - fg_\theta(\theta,\alpha)\sigma_\theta\alpha' da \tag{69}$$

On the other hand, fully differentiating (26) with respect to q we obtain:

$$\frac{dh_1}{dq} = \dot{\alpha}f\sigma_\theta\,|_{a=\alpha} - \left(\underline{\alpha}_q - \underline{\alpha}_\theta g\dot{\alpha} + \underline{\alpha}_\alpha\dot{\alpha}\right)f\sigma_\theta\,|_{a=\underline{\alpha}} + \int_{\underline{\alpha}(q,\alpha,\theta)}^{\alpha} f_\theta\left(\sigma_q - g\sigma_\theta\dot{\alpha} + \sigma_\alpha\dot{\alpha}\right)\sigma_\theta + f\left(\sigma_{q\theta} - g\sigma_{\theta\theta}\dot{\alpha} + \sigma_{\alpha\theta}\dot{\alpha}\right)da \tag{70}$$

Combining (69) and (70) yields:

$$\frac{dh_1}{dq} - h_\theta = \dot{\alpha}f\sigma_\theta\,|_{a=\alpha} + f(\sigma,a)\left(\underline{\alpha}_\theta\sigma_q - \underline{\alpha}_q\sigma_\theta + \left(\underline{\alpha}_\theta\sigma_\alpha - \underline{\alpha}_\alpha\sigma_\theta\right)\dot{\alpha}\right)|_{a=\underline{\alpha}} + g_\theta(\alpha,\theta)\int_{\underline{\alpha}(q,\alpha,\theta)}^{\alpha} f\sigma_\theta\dot{\alpha}da \tag{71}$$

Note that $\sigma_\theta\,|_{a=\alpha}= 1$ by (13). Then a comparison of (71) with (65) reveals that (65) holds if and only if

$$\underline{\alpha}_\theta\sigma_q - \underline{\alpha}_q\sigma_\theta + \left(\underline{\alpha}_\theta\sigma_\alpha - \underline{\alpha}_\alpha\sigma_\theta\right)\dot{\alpha}_{a=\underline{\alpha}} = 0 \tag{72}$$

Recall that, by definition, $\underline{\alpha} = 0$ if $\sigma(q,\alpha,\theta,0) < 1$, and otherwise $\underline{\alpha}$ solves the equation $\sigma(q,\theta,\alpha,a) = 1$ in a. So, (72) holds for all (q,α,θ) such that $\underline{\alpha} = 0$, because there $\sigma_q(.) = \sigma_\alpha(.) = \sigma_\theta(.)$.

Next, if $\underline{\alpha} > 0$, then the partial derivatives of the function $\underline{\alpha}(.)$ can be computed by differentiating $\sigma(q,\theta,\alpha,a) = 1$. In particular, differentiating the latter equation we get $\sigma_q(q,\theta,\alpha,\underline{\alpha}) + \sigma_a(q,\theta,\alpha,\underline{\alpha})\underline{\alpha}_q = 0$, $\sigma_\theta(q,\theta,\alpha,\underline{\alpha}) + \sigma_a(q,\theta,\alpha,\underline{\alpha})\underline{\alpha}_\theta = 0$, and $\sigma_\alpha(q,\theta,\alpha,\underline{\alpha}) + \sigma_a(q,\theta,\alpha,\underline{\alpha})\underline{\alpha}_\alpha = 0$. The first two of these equations imply that $\underline{\alpha}_\theta\sigma_q - \underline{\alpha}_q\sigma_\theta = 0$ and the second and the third equations imply that $\underline{\alpha}_\theta\sigma_\alpha - \underline{\alpha}_\alpha\sigma_\theta = 0$. Thus, (72) holds in this case also. *Q.E.D.*

Proof of Theorem 4 :

(i) An interval on which $\alpha(q)$ is strictly increasing must be a singular (aub)arc where $S = \dot{S} = \ddot{S} = 0$. Then setting (34) to zero yields the differential equation (35). Equation (36) is identical to (20).

Further, setting (33) to zero yields (37). Then setting (30) to zero and combining this with (37) yields (38).

(ii) Equations (39) and (40) follow immediately from "co-state evolution" equations for our optimal control problem, (61) and (62), respectively, when we set $\dot{\alpha} = \dot{\theta} = 0$.

(iv) To prove part (iv), note that there is nothing to prove when $q_0 = 0$. If $q_0 > 0$, then the transversality condition associated with free left time q_0 is that the Hamiltonian $J(q_0,\alpha(q_0),\theta(q_0),\alpha'(q_0),\mu(q_0),\lambda(q_0))$ defined in (29) is equal to zero.

Note that the linearity of (29) in α' imply that $S\dot{\alpha} = 0$ for all q and so

$$J(q_0,\alpha(q_0),\theta(q_0),\alpha'(q_0),\mu(q_0),\lambda(q_0)) = u(q_0,\alpha(q_0),\theta(q_0))h_0(q_0,\alpha(q_0),\theta(q_0))$$

Inspecting the definition of h_0 in equation (25) is easy to see that $J(q_0,\alpha(q_0),\theta(q_0),\alpha'(q_0),\mu(q_0),\lambda(q_0)) = 0$ is equivalent to either $q_0 = 0$ or $\alpha(q_0) = 0$. Q.E.D.

Proof of Lemma 6: Let $J_{\dot{\alpha}}$ be the partial derivative of the Hamiltonian (29) with respect to $\dot{\alpha}$. The Generalized Legendre Clebsch condition requires that if p is the smallest number such that $\frac{d^{2p}J_{\dot{\alpha}}}{dq^{2p}} \neq 0$ at some point on the optimal singular arc, then:

$$(-1)^p \frac{d^{2p}J_{\dot{\alpha}}}{dq^{2p}} \leq 0.$$

In our case, $p = 1$, and $\frac{d^{2p}J_{\dot{\alpha}}}{dq^{2p}} = -D$. So we must have $D \leq 0$. *Q.E.D.*

Proof of Lemma 7: (i) We will prove that $\alpha(q_0) > 0$. From this, by Theorem 4 (iv) it follows that $q_0 = 0$.

The proof is by contradiction. So suppose that $\alpha(q_0) = 0$. Then by definition of q_0, for every $\varepsilon > 0$ there exists $q \in [q_0, q_0 + \varepsilon)$ such that $\alpha(q) > 0$. But since $\alpha(q_0) = 0$, the optimal solution must include a singular arc in a right neighborhood of q_0.

We will next show that $D(q, \alpha, \theta) > 0$ for all sufficiently small α, which would contradict Lemma 6. To this end, let us first show the following:

$$\begin{aligned} D &= \left(u - \frac{u_\theta u_q}{u_{q\theta}}\right) f + 2(u_{q\theta}u_\alpha - u_{q\alpha}u_\theta) \int_0^\alpha \frac{f(a,\sigma)}{u_{q\theta}(q,a,\sigma)} da \\ &\quad + u_q(u_{q\theta}u_\alpha - u_{q\alpha}u_\theta) \int_0^\alpha \frac{f_\theta u_{q\theta} - f u_{q\theta\theta}}{u_{q\theta}^3}(q,a,\sigma) da \end{aligned} \tag{73}$$

To see this, let us consider the terms in the definition of D in (41) one by one. Note that because $\alpha(q_0) = 0$, we have $\underline{\alpha}(q, \alpha(q), \theta(q)) = 0$ for all q in some right neighborhood of q_0. We will use this to set the lower limit of the integrals below.

First, by definition of g (the first equation)and by (26), (27), (13) and (14) (second equation), we have:

$$-u_\theta(h_2 - gh_1) = u_\alpha h_1 - u_\theta h_2 = (u_\alpha u_{q\theta} - u_{q\alpha}u_\theta) \int_0^\alpha \frac{f(a,\sigma)}{u_{q\theta}(q,a,\sigma)} da. \tag{74}$$

Next, combining the definitions of ψ in (32) and $g = \frac{u_\alpha}{u_\theta}$ yields: $\psi(q,\alpha,\theta)g(q,\alpha,\theta) = u_q(q,\alpha,\theta)u_\alpha(q,\alpha,\theta) \int_0^\alpha \frac{f(a,\sigma)}{u_{q\theta}(q,a,\sigma)} da$ Differentiating, we obtain:

$$\psi_\theta g + \psi g_\theta = (u_q u_{\alpha\theta} + u_{q\theta}u_\alpha) \int_0^\alpha \frac{f(a,\sigma)}{u_{q\theta}(q,a,\sigma)} da + u_q u_\alpha \int_0^\alpha \left(\frac{f_\theta(\sigma,a)}{u_{q\theta}(q,\sigma,a)} - \frac{f(\sigma,a)u_{q\theta\theta}(q,\sigma,a)}{u_{q\theta}^2(q,\sigma,a)} \right) \sigma_\theta da \tag{75}$$

Finally,

$$\psi_\alpha = f\frac{u_\theta u_q}{u_{q\theta}}(q,\alpha,\theta) + (u_{q\alpha}u_\theta + u_q u_{\theta\alpha}) \int_0^\alpha \frac{f(\sigma,a)}{u_{q\theta}(q,\sigma,a)} da + u_q u_\theta \int_0^\alpha \left(\frac{f_\theta(\sigma,a)}{u_{q\theta}(q,\sigma,a)} - \frac{f(\sigma,a)u_{q\theta\theta}(q,\sigma,a)}{u_{q\theta}^2(q,\sigma,a)} \right) \sigma_\alpha da \tag{76}$$

Substituting (74), (75) and (76) into (41) yields (73).

Since $\left(u - \frac{u_\theta u_q}{u_{q\theta}}\right) > 0$ by the Assumption of the Lemma and $u_{q\theta}$ is bounded away from zero, equation (73) implies that $D(q, \alpha, \theta) > 0$ if α is sufficiently small. Therefore, $D(q, \theta(q), \alpha(q)) > 0$ for all $q \in (0, \varepsilon)$ for some $\varepsilon > 0$, contradicting Lemma 6.

(ii) Suppose to the contrary that $\theta(q^{**}) = 1$. Then $\underline{\alpha}(q^{**}, \alpha(q^{**}), \theta(q^{**})) = \alpha(q^{**})$, and $\underline{\alpha}(q, \alpha(q), \theta(q)) > 0$ for all $q \geq q^{**}$. Therefore, we need to use $\underline{\alpha}(q, \alpha(q), \theta(q))$ explicitly as the lower limit of the integrals that enter terms of $D(q, \theta(q), \alpha(q)))$. Specifically, performing the same derivations as in part (i) we obtain that for q in some neighborhood of q^{**} we have:

$$D(q, \theta(q), \alpha(q))) = \left(u - \tfrac{u_\theta u_q}{u_{q\theta}}\right) f + 2(u_{q\theta} u_\alpha - u_{q\alpha} u_\theta) \int_{\underline{\alpha}(q,\alpha(q),\theta(q))}^{\alpha} \tfrac{f(a,\sigma)}{u_{q\theta}(q,a,\sigma)} da + \tag{77}$$
$$u_q(u_{q\theta} u_\alpha - u_{q\alpha} u_\theta) \int_{\underline{\alpha}(q,\alpha(q),\theta(q))}^{\alpha} \tfrac{f_\theta u_{q\theta} - f u_{q\theta\theta}}{u_{q\theta}^3}(q, a, \sigma) da - u_q u_\alpha \underline{\alpha}_\theta \tfrac{f(\underline{\alpha},1)}{u_{q\theta}(q,\underline{\alpha},1)} + u_q u_\theta \underline{\alpha}_\alpha \tfrac{f(\underline{\alpha},1)}{u_{q\theta}(q,\underline{\alpha},1)}$$

Evaluating (77) at $q = q^{**}$, we obtain:

$$D(q^{**}, \theta(q^{**}), \alpha(q^{**})) = \left(u + \frac{u_\theta u_q}{u_{q\theta}} (\underline{\alpha}_\alpha - 1) - \frac{u_\alpha u_q}{u_{q\theta}} \underline{\alpha}_\theta\right) f(\underline{\alpha}, 1) \tag{78}$$

Since by definition $\underline{\alpha}(q, \alpha(q), \theta(q))$ solves $\sigma(q, \alpha(q), \theta(q), a) \equiv 1$ for a, it follows that

$$\underline{\alpha}_\alpha = -\frac{\sigma_\alpha}{\sigma_a} \text{ and } \underline{\alpha}_\theta = -\frac{\sigma_\theta}{\sigma_a}. \tag{79}$$

Then substituting (12), (14) and (13) into (79) yields:

$$\underline{\alpha}_\alpha = \frac{u_{q\alpha}(q, \alpha, \theta)}{u_{q\alpha}(q, a, \sigma)} \text{ and } \underline{\alpha}_\theta = \frac{u_{q\theta}(q, \alpha, \theta)}{u_{q\alpha}(q, a, \alpha)}$$

At $q = q^{**}$ we have $\underline{\alpha}_\alpha = 1$ and $\underline{\alpha}_\theta = \frac{u_{q\alpha}}{u_{q\theta}}$. Using this in (78) yields:

$$D(q^{**}, \theta(q^{**}), \alpha(q^{**})) = \left(u - \frac{u_\alpha u_q}{u_{q\alpha}}\right) f \tag{80}$$

Finally, since $u - \frac{u_\theta u_q}{u_{q\theta}} > 0$ by assumption of the Lemma, and since $\frac{u_\theta}{u_{q\theta}} > \frac{u_\alpha}{u_{q\alpha}}$, the value of (80) is strictly positive. But this contradicts that q^{**} must lie on a singular arc, since by definition q^{**} is the largest quantity such that $(\alpha, 1)$ lies on an isoquant emanating from $(q^{**}, \alpha(q^{**}), \theta(q^{**}))$. *Q.E.D.*

Proof of Theorem 5: Let

$$G(q, \alpha, \theta) \equiv u_q(q, \alpha, \theta) H(q, \alpha, \theta).$$

Also, let $\bar{q}$ be defined by $\phi(\bar{q}, 1) = 0$. Note that $\bar{q} > \underline{q}(1)$.

First, let us consider the case when $\widehat{\alpha} = 1$. The first-order condition for (21) is:

$$G_\theta(q, 1, \theta) = \phi(q, \theta) = 0 \tag{81}$$

Let $\theta^\phi(q)$ be the solution to (81), when such exists. Also, let $\theta^\phi(q) = 1$ if $\phi(q, \theta) < 0$ for all $\theta \in [0, 1]$ and let $\theta^\phi(q) = 0$ if $\phi(q, \theta) > 0$ for all $\theta \in [0, 1]$. Since $\phi_q > 0$ and $\phi_\theta < 0$, $\theta^\phi(q)$ is increasing in θ.

If $\theta^\phi(\widehat{q}) > \widehat{\theta}$, then the boundary condition $\theta(\widehat{q}) = \widehat{\theta}$ implies that it is optimal to set $\theta(\widehat{q}) = [\widehat{\theta}, \theta^\phi(\widehat{q})]$, $\theta(q) = \theta^\phi(q)$ for all $q \in (\widehat{q}, \bar{q}]$.

On the other hand, if $\theta^\phi(\widehat{q}) \leq \widehat{\theta}$, then set $\theta(q) = \widehat{\theta}$ for q s.t. $\theta^\phi(q) \leq \widehat{\theta}$, $\theta(q) = \theta^\phi(q)$ for q s.t. $\theta^\phi(q) > \widehat{\theta}$, and $\theta(\bar{q}) = 1$.

To summarize, $\theta(\widehat{q}) = [\widehat{\theta}, \max\{\widehat{\theta}, \theta^\phi(\widehat{q})\}]$, $\theta(q) = \max\{\theta^\phi(q), \widehat{\theta}\}$ for $q \in (\hat{q}, \bar{q})$, $\theta(\bar{q}) = 1$.

Now suppose that $\widehat{\alpha} < 1$ and $\widehat{\theta} = 0$. In this case, the first-order condition for (21) with respect to α is:

$$G_\alpha(q, \alpha, 0) = \varkappa(q, \alpha) = 0 \tag{82}$$

Let q^s solve $\varkappa(q^s, \hat{\alpha}) = 0$ and let $\underline{q}(1)$ solve $\phi(\underline{q}(1), 0) = 0$. Note that $q^s < \underline{q}(1) < \bar{q}$. Below, we will show that $\underline{q}(1)$ also solves $\varkappa(\underline{q}(1), 1) = 0$.

If $\widehat{q} \leq q^s$, then it is optimal to set $\alpha(q) = \max\{\hat{\alpha}, \alpha^{\varkappa}(q)\}$ for $q \in [\widehat{q}, \underline{q}(1)]$, $\theta(q) = \theta^{\phi}(q)$ for $q \in [\underline{q}(1), \bar{q}]$.

If $q^s < \hat{q} < \underline{q}(1)$, then set $\alpha(\widehat{q}) = [\hat{\alpha}, \alpha^{\varkappa}(\widehat{q})]$, $\alpha(q) = \alpha^{\varkappa}(q)$ for $q \in (\widehat{q}, \underline{q}(1)]$, $\theta(q) = \theta^{\phi}(q)$ for $q \in [\underline{q}(1), \bar{q}]$.

If $q^s < \underline{q}(1) \leq \widehat{q} \leq \bar{q}$, then set $\alpha(\widehat{q}) = [\widehat{\alpha}, 1]$, $\theta(\widehat{q}) = [0, \theta^{\phi}(\widehat{q})]$, $\theta(q) = \theta^{\phi}(q)$ for all $q \in (\widehat{q}, \bar{q}]$.

To summarize, we have $\alpha(\widehat{q}) = [\hat{\alpha}, \max\{\hat{\alpha}, \alpha^{\varkappa}(\widehat{q})\}]$, $\alpha(q) = \max\{\hat{\alpha}, \alpha^{\varkappa}(q)\}$ for all $q \in (\widehat{q}, \max\{\underline{q}(1), \widehat{q}\}]$, $\theta(q) = \theta^{\phi}(q)$ for $q \in [\max\{\underline{q}(1), \widehat{q}\}, \bar{q}]$.

Let us now show that $\phi(q, 0) = 0$ if and only if $\varkappa(q, 1) = 0$, and consequently $\alpha^{\varkappa}(\underline{q}(1)) = 1$ implies that $\theta^{\phi}(\underline{q}(1)) = 0$. In fact, we will show that $H_\theta(q, 1, 0) = \frac{u_{q\theta}(q,1,0)}{u_{q\alpha}(q,1,0)} H_\alpha(q, 1, 0)$, so that $\phi(1, 0) = \frac{u_{q\theta}(q,1,0)}{u_{q\alpha}(q,1,0)} \varkappa(q, 1)$, implying the desired result.

Finally, differentiating (15) we obtain:

$$H_\theta(q, 1, \theta) = -\int_{\bar{\alpha}(q,1,\theta)}^{1} f(a, \sigma(q, 1, \theta, a)\sigma_\theta(q, 1, \theta, a))da \tag{83}$$

$$H_\alpha(q, \alpha, 0) = -\int_{\bar{\alpha}(q,\alpha,0)}^{1} f(a, \sigma(q, \alpha, 0, a)\sigma_\alpha(q, \alpha, 0, a)da \tag{84}$$

where $\bar{\alpha}(q, \alpha, \theta)$ is the solution in a to the equation $\sigma(q, \alpha, \theta, a) = 1$ if such a solution exists and is nonnegative, and 0, otherwise. Then, combining (13), (14), (83) and (84), then yields $H_\theta(q, 1, 0) = \frac{u_{q\theta}(q,1,0)}{u_{q\alpha}(q,1,0)} H_\alpha(q, 1, 0)$. *Q.E.D.*

Proof of Lemma 8: We can formulate the maximization problem (21) on the interval $(\max\{\widehat{q}, \underline{q}(1)\}, \bar{q}(1)]$ via the Lagrangian:

$$\max L(q, 1, \theta, \theta') = G(q, 1, \theta) + \delta\theta' \tag{85}$$

The first-order conditions associated with maximizing (85) are:

$$\begin{aligned} L_\theta - \frac{d}{dq} L_{\theta'} &= \phi(q, \theta) - \delta' = 0 \\ \delta(q)\theta'(q) &= 0 \\ \delta(q) &\geq 0 \\ \theta'(q) &\geq 0 \end{aligned} \tag{86}$$

In addition, the transversality condition for the free 'terminal time' $\bar{q}(1)$ is:

$$L - L_{\theta'}\theta' = \phi(\bar{q}(1), 1) = 0$$

Since $H(\bar{q}(1), 1, 1) = 0$, and since $H_\theta(\bar{q}(1), 1, 1) < 0$, the transversality condition yields $u_q(\bar{q}(1), 1, 1) = 0$. From (86), we have $\delta'(q) = G_\alpha(q, \alpha(q), 0) = \varkappa(q, \alpha(q))$. The proof for $\mu(q)$ is analogous. *Q.E.D.*

Proof of Theorem 6: We need to consider several cases depending upon whether $\widehat{\alpha} = 1$ or $\widehat{\theta} = 0$. Here, we will consider the case $\widehat{\alpha} = 1$; the proof for the other cases is similar.

Suppose now that contrary to the statement of the theorem we had $\widehat{\theta} > \theta^{\phi}(\widehat{q})$. Then we may rewrite (22) as follows:

$$V(\widehat{q},1,\widehat{\theta}) = W(\widehat{q},1,\widehat{\theta}) + Z(\widehat{q},1,\widehat{\theta})$$

where

$$Z(\widehat{q},1,\widehat{\theta}) = u(\widehat{q},1,\widehat{\theta})H(\widehat{q},1,\widehat{\theta}) + \int_{\widehat{q}}^{q^{\phi}(\widehat{\theta})} H(q,1,\widehat{\theta})u_q(q,1,\widehat{\theta})dq + \int_{q^{\phi}(\widehat{\theta})}^{\bar{q}(1)} H(q,1,\theta(q))u_q(q,1,\theta(q))dq,$$

and $q^{\phi}(\widehat{\theta})$ is the unique solution to the equation $\theta^{\phi}(q) = \widehat{\theta}$. We will show that marginally lowering $\widehat{\theta}$ raises $V(\widehat{q},1,\widehat{\theta})$.

The partial derivative w.r.t. $\widehat{\theta}$ of the value function $W(\widehat{q},1,\widehat{\theta})$ is given by (Seierstad and Sydsaeter, p. 213):

$$\frac{\partial W}{\partial \widehat{\theta}} = -\lambda(\widehat{q})$$

Furthermore, we may calculate

$$\frac{\partial Z}{\partial \widehat{\theta}} = u_{\theta}(\widehat{q},1,\widehat{\theta})H(\widehat{q},1,\widehat{\theta}) + u(\widehat{q},1,\widehat{\theta})H_{\theta}(\widehat{q},1,\widehat{\theta}) + \int_{\widehat{q}}^{q^{\phi}(\widehat{\theta})} \phi(q,\widehat{\theta})dq$$

where

$$\phi(q,\widehat{\theta}) = u_q(q,1,\widehat{\theta})H_{\theta}(q,1,\widehat{\theta}) + u_{\theta q}(q,1,\widehat{\theta})H(q,1,\widehat{\theta})$$

Combining, we obtain:

$$\begin{aligned}
\frac{\partial V}{\partial \widehat{\theta}}(\widehat{q},1,\widehat{\theta}) &= -\lambda(\widehat{q}) + u_{\theta}(\widehat{q},1,\widehat{\theta})H(\widehat{q},1,\widehat{\theta}) + u(\widehat{q},1,\widehat{\theta})H_{\theta}(\widehat{q},1,\widehat{\theta}) + \int_{\widehat{q}}^{q^{\phi}(\widehat{\theta})} \phi(q,\widehat{\theta})dq \\
&= -\psi(\widehat{q}) + u_{\theta}(\widehat{q},1,\widehat{\theta})H(\widehat{q},1,\widehat{\theta}) + \int_{\widehat{q}}^{q^{\phi}(\widehat{\theta})} \phi(q,\widehat{\theta})dq \\
&\leq \int_{\widehat{q}}^{q^{\phi}(\widehat{\theta})} \phi(q,\widehat{\theta})dq < 0
\end{aligned}$$

The second equality follows because $\lambda(q) = -uh_1 + \psi$, and because by definition, we have

$$H_{\theta}(\widehat{q},1,\widehat{\theta}) = -\int_{\overline{\alpha}(\widehat{q},1,\widehat{\theta})}^{1} f(a,\sigma((\widehat{q},1,\widehat{\theta},a))\sigma_{\theta}(q,1,\theta,a)da = -h_1(\widehat{q},1,\widehat{\theta})$$

where the second equality follows from we used $\lambda(q) = -uh_{\theta'} + \psi$, and the penultimate inequality follows because $\phi(q,\theta)$ is decreasing in θ yields:

$$\phi(\widehat{q},\widehat{\theta}) = u_q(\widehat{q},1,\widehat{\theta})H_{\theta}(\widehat{q},1,\widehat{\theta}) + u_{\theta q}(\widehat{q},1,\widehat{\theta})H(\widehat{q},1,\widehat{\theta}) \leq \phi(\widehat{q},\theta^{\phi}(\widehat{q})) = 0$$

and so

$$u_{\theta}(\widehat{q},1,\widehat{\theta})H(\widehat{q},1,\widehat{\theta}) \leq \frac{u_q u_{\theta} h_1}{u_{q\theta}}(\widehat{q},1,\widehat{\theta}) = \psi(\widehat{q},1,\widehat{\theta})$$

It follows that the perturbation is profitable, showing that in an optimal solution to (22) we cannot have $\widehat{\theta} > \theta^{\phi}(\widehat{q})$.

Next, suppose that contrary to the statement of the theorem we had $\widehat{\theta} < \theta^{\phi}(\widehat{q})$. Then since by Theorem 5 all types $(1, \theta)$ with $\theta \in [\widehat{\theta}, \theta^{\phi}(\widehat{q})]$ are assigned the quantity $\widehat{q}$, and hence pay $t(\widehat{q})$ we may rewrite (22) as follows:

$$V(\widehat{q}, 1, \widehat{\theta}) = W(\widehat{q}, 1, \widehat{\theta}) + Z(\widehat{q}, 1, \widehat{\theta})$$

where

$$Z(\widehat{q}, 1, \widehat{\theta}) = u(\widehat{q}, 1, \widehat{\theta})H(\widehat{q}, 1, \widehat{\theta}) + \int_{\theta^{\phi}(\widehat{q})}^{\bar{q}(1)} H(q, 1, \theta^{\phi}(q))u_q(q, 1, \theta(q))dq,$$

We may then calculate

$$\begin{aligned} \frac{\partial V}{\partial \widehat{\theta}}(\widehat{q}, 1, \widehat{\theta}) &= -\lambda(\widehat{q}) + u_{\theta}(\widehat{q}, 1, \widehat{\theta})H(\widehat{q}, 1, \widehat{\theta}) + u(\widehat{q}, 1, \widehat{\theta})H_{\theta}(\widehat{q}, 1, \widehat{\theta}) \\ &= -\psi(\widehat{q}) + u_{\theta}(\widehat{q}, 1, \widehat{\theta})H(\widehat{q}, 1, \widehat{\theta}) \\ &> 0 \end{aligned}$$

The second equality follows because $\lambda(q) = -uh_1 + \psi$ because $H_{\theta}(\widehat{q}, 1, \widehat{\theta}) = -h_1(\widehat{q}, 1, \widehat{\theta})$. The final inequality follows because $\widehat{\theta} > \theta^{\phi}(\widehat{q})$ implies $\phi(\widehat{q}, \widehat{\theta}) > \phi(\widehat{q}, \theta^{\phi}(\widehat{q})) = 0$, and so

$$u_{\theta}(\widehat{q}, 1, \widehat{\theta})H(\widehat{q}, 1, \widehat{\theta}) > \frac{u_{\theta}(\widehat{q}, 1, \widehat{\theta})}{u_{\theta q}(\widehat{q}, 1, \widehat{\theta})} u_q(\widehat{q}, 1, \widehat{\theta})h_1(\widehat{q}, 1, \widehat{\theta}) = \psi(\widehat{q}).$$

We conclude that raising $\widehat{\theta}$ marginally is profitable, showing that in an optimal solution to (22) we cannot have $\widehat{\theta} < \theta^{\phi}(\widehat{q})$ either.

Q.E.D.

Q.E.D.

Proof of Theorem 7: (i) We need to consider two cases: $\widehat{\alpha} = 1$ and $\widehat{\theta} = 0$. We give a proof for the case $\widehat{\alpha} = 1$. The proof for the case $\widehat{\theta} = 0$ is analogous. According to Theorem 6 we have $\widehat{\theta} = \theta^{\phi}(\widehat{q})$. Hence

$$W(\widehat{q}, \theta^{\phi}(\widehat{q})) = u(\widehat{q}, 1, \theta^{\phi}(\widehat{q}))H(\widehat{q}, 1, \theta^{\phi}(\widehat{q})) + \int_{\widehat{q}}^{\bar{q}(1)} H(q, 1, \theta^{\phi}(q))u_q(q, 1, \theta^{\phi}(q))dq$$

Thus we have

$$\begin{aligned} \frac{d}{d\widehat{q}} W(\widehat{q}, \theta^{\phi}(\widehat{q})) &= uH_q + (u_{\theta}H + uH_{\theta})\frac{d\theta^{\phi}}{dq} \\ &= -uh_0 + (\frac{u_{\theta}u_q}{u_{q\theta}} - u)h_{\theta'}\frac{d\theta^{\phi}}{dq} \end{aligned}$$

where the final equality follows from $\phi(\widehat{q},\widehat{\theta}) = u_q H_\theta + u_{\theta q} H = 0$, and the definition $\psi = \frac{u_q u_\theta}{u_{q\theta}} h_{\theta'}$. Furthermore, we have

$$\frac{\partial W_2}{\partial q} + \frac{\partial W_2}{\partial \theta}\frac{d\theta^\phi}{dq} = uh_0 - \lambda \frac{d\theta^\phi}{dq}$$

Hence

$$\left(\frac{\partial W_2}{\partial q} + \frac{\partial W_2}{\partial \theta}\frac{d\theta^\phi}{dq}\right) + \frac{d}{d\widehat{q}} W(\widehat{q},\theta^\phi(\widehat{q})) = (\psi - uh_{\theta'} - \lambda)\frac{d\theta^\phi}{dq} = 0 \tag{87}$$

where the ultimate equality follows from the assumption that $\alpha'(\widehat{q}) > 0$ and the expression for λ in Theorem 4.

Next, observe that

$$\frac{\partial W_1}{\partial q^{**}} + \frac{\partial W_2}{\partial q^{**}} = 0$$

and

$$\begin{aligned}\frac{\partial W_1}{\partial \alpha^{**}} + \frac{\partial W_2}{\partial \alpha^{**}} &= \mu_+(q^{**}) - \mu_-(q^{**}) \\ \frac{\partial W_1}{\partial \theta^{**}} + \frac{\partial W_2}{\partial \theta^{**}} &= \lambda_+(q^{**}) - \lambda_-(q^{**})\end{aligned}$$

Since at the optimum we must have

$$\frac{d}{d\widehat{q}} V(\widehat{q},1,\theta^\phi(\widehat{q})) = 0$$

it follows that we must have

$$[\mu_+(q^{**}) - \mu_-(q^{**})]\frac{d\alpha^{**}}{d\widehat{q}} + [\lambda_+(q^{**}) - \lambda_-(q^{**})]\frac{d\theta^{**}}{d\widehat{q}} = 0$$

(ii) Again, there are two cases to be considered, $\widehat{\alpha} = 1$ and $\widehat{\theta} = 0$. We give a proof for the case $\widehat{\alpha} = 1$. The proof for the case $\widehat{\theta} = 0$ is analogous. If $\alpha'(\widehat{q}) = 0$ then (87) still holds, but we no longer have $\lambda(\widehat{q}) = \psi - uh_{\theta'}$. Hence if $\underline{\alpha}(\widehat{q},\widehat{\alpha},\widehat{\theta}) \leq 0$ then

$$\frac{d}{d\widehat{q}} V(\widehat{q},1,\theta^\phi(\widehat{q})) = (-\psi - uh_{\theta'} - \lambda)\frac{d\theta^\phi}{dq}$$

It follows that we must have $\lambda(\widehat{q}) = \psi - uh_{\theta'}$. If $\underline{\alpha}(0,\alpha^{**}(0),\underline{\theta}(0)) > 0$, then $V(\widehat{q},1,\theta^\phi(\widehat{q})) = W_2(q^{**},\alpha^{**},\theta^{**},\widehat{q},\theta^\phi(\widehat{q})) + W(\widehat{q},\theta^\phi(\widehat{q})$ we have

$$\frac{d}{d\widehat{q}} V(\widehat{q},1,\theta^\phi(\widehat{q})) = (\psi - uh_{\theta'} - \lambda)\frac{d\theta^\phi}{dq} + \mu(0)\frac{d\alpha^{**}}{d\widehat{q}} + \lambda(0)\frac{d\theta^{**}}{d\widehat{q}} - uh_0\frac{dq^{**}}{d\widehat{q}}$$

The desired conclusion then follows from the transversality condition for problem (28): $\mu(0) = \lambda(0) = 0$, and the fact that $u(0,\alpha,\theta) = 0$ for all (α,θ). We are therefore left with the case where $\underline{\alpha}(\widehat{q},\widehat{\alpha},\widehat{\theta}) > 0$ and $\underline{\alpha}(0,\alpha^{**}(0),\theta(0)) = 0$. We then have

$$\frac{d}{d\widehat{q}} V(\widehat{q},1,\theta^\phi(\widehat{q})) = (\psi - uh_{\theta'} - \lambda)|_{q=\widehat{q}}\frac{d\theta^\phi}{dq}(\widehat{q}) + [\mu_+(q^{**}) - \mu_-(q^{**})]\frac{d\alpha^{**}}{d\widehat{q}} + [\lambda_+(q^{**}) - \lambda_-(q^{**})]\frac{d\theta^{**}}{d\widehat{q}}$$

The desired result then follows because $\alpha'(\widehat{q}) = 0$ and the regularity assumption $\frac{\partial}{\partial q}\{u_\theta h_0 - u_q h_{\theta'} + \psi_q\} < 0$ imply $\alpha'(q) = 0$ for all $q \in [0,\widehat{q}]$. In that case μ and λ are continuous on the interval $[0,\widehat{q}]$, and so the last two terms in the above expression equal zero. Q.E.D.

Proof of Theorem 5.1: Consider subproblem (i), and assume that $\widehat{q} > 0$. Along the solution to this problem, there could be two types of junction points. One is associated with a transition from a nonsingular region to a singular region; let us call this a type I junction point. The other one is associated with a transition from a singular region to a nonsingular region. We call this a type II junction point. First we establish that there cannot be a type II junction point on the interval $(0, \widehat{q})$.

Lemma 12 *Suppose Assumption 3 holds. Then along the solution to subproblem (i) there is no type II junction point* $q^{\#} \in (0, \widehat{q})$.

Proof of Lemma 12: We proceed in two steps. First, we establish that if there existed a type II junction point $q^{\#} \in (0, \widehat{q})$, then there cannot be a type I junction point greater than $q^{\#}$. This then implies that we must have $\dot{\alpha} = 0$ for all $q \in (q\#, \widehat{q})$.

Suppose to the contrary that q' is the first type I junction point greater than $q\#$. Because there is a left neighborhood of $q\#$ and a right neighborhood of q' on which the solution is singular, it follows from equation (35) and Lemma 6 that we have $N(q\#) \leq 0$ and $N(q') \leq 0$, where

$$N(q) = \psi_q + u_\theta h_0 - u_q h_1.$$

By Assumption 3(i) we have $N_{qq} > 0$; it follows that $N < 0$ for all $q \in (q\#, q')$.

Equation (33) and the fact that $g_q > 0$ imply that over the interval $(q\#, q')$ the sign of $\dot{S}$ equals the sign of the term $\psi - uh_1 - \lambda$. From the proof of Lemma 5, on this interval we have

$$\frac{\partial}{\partial q}(\psi - uh_1 - \lambda) = \psi_q + u_\theta h_0 - u_q h_1 = N < 0$$

Since at $q = q\#$ we have $\psi - uh_1 - \lambda = 0$ (see equation (37), this implies that $\psi - uh_1 - \lambda < 0$ that on $[q\#, q']$, and hence that $\dot{S} < 0$ on $[q\#, q']$. Since we have $S(q\#) = 0$, this then implies that $S(q') < 0$. But this contradicts that q' is a type I junction point, establishing step 1. We conclude that if there exists a type II junction point $q\# \in (\widetilde{q}, \widehat{q})$, then the solution must be nonsingular for $q \in (q\#, \widehat{q})$, i.e. that we must have $\alpha(q\#) = \alpha(\widehat{q}) = 1$.

Next, it now follows from Theorem 6 that we must have $q\# = \widehat{q}$, so there cannot be a type II junction point on the interval $(\widetilde{q}, \widehat{q})$. Thus on this interval, the solution must be singular.

We now show that Assumption 3(ii) implies that there exists a type I junction point $\widetilde{q} > 0$.

Lemma 13 *Suppose Assupmption 3 holds. Then the solution to subproblem (i) contains a unique type I junction point* $\widetilde{q} \in (0, \widehat{q})$.

Proof : Suppose to the contrary that the solution to subproblem (i) is a singular arc on $[0, \widehat{q}]$. We will show that in this case we would have $\alpha(q) \to 0$ as $q \to 0$, which contradicts Lemma 7.

To establish the claim, recall that

$$\begin{aligned} N &= u_\theta \left\{ \int_0^\alpha f(a,\sigma) \frac{2u_{qq}(q,\alpha,\theta) - u_{qq}(q,a,\sigma)}{u_{q\theta}(q,a,\sigma)} da + u_q \int_0^\alpha \frac{f_\theta u_{q\theta} - f u_{q\theta\theta}}{u_{q\theta}^2} \frac{u_{qq}(q,\theta,\alpha) - u_{qq}(q,\sigma,a)}{u_{q\theta}(q,\sigma,a)} da \right\} \\ &\quad - u_\theta \left\{ u_q \int_0^\alpha \frac{f(a,\sigma)}{u_{q\theta}(q,a,\sigma)^2} u_{qq\theta}(q,a,\sigma) da \right\} \end{aligned} \quad (88)$$

and

$$D = \left(u - \frac{u_\theta u_q}{u_{q\theta}}\right) f + (u_{q\theta} u_\alpha - u_{q\alpha} u_\theta) \left\{ 2 \int_0^\alpha \frac{f(a,\sigma)}{u_{q\theta}(q,a,\sigma)} da + u_q \int_0^\alpha \frac{f_\theta u_{q\theta} - f u_{q\theta\theta}}{u_{q\theta}^3}(q,a,\sigma) da \right\}$$

Since $u(0,\alpha,\theta) = u_\alpha(0,\alpha,\theta) = u_\theta(0,\alpha,\theta) = 0$, a Taylor series expansion shows that

$$u_\theta(q,\alpha,\theta) = q u_{q\theta}(0,\alpha,\theta) + o(q)$$

Similarly, a second order Taylor series expansion shows that

$$u - \frac{u_\theta u_q}{u_{q\theta}} = q^2 \left(-u_{qq} + \frac{u_q u_{qq\theta}}{u_{q\theta}}\right)|_{(0,\alpha,\theta)} + o(q^2)$$

and that

$$u_{q\theta} u_\alpha - u_{q\alpha} u_\theta = (u_{qq\theta} u_{q\alpha} - u_{qq\alpha} u_{q\theta})|_{(0,\alpha,\theta)} + o(q^2)$$

Hence we have

$$\dot{\alpha} = \frac{1}{q} \kappa(q,\alpha,\theta)$$

where $\kappa(q,\alpha,\theta) = \frac{(N/q)}{(D/q^2)}$.

We will show below that there exists $\varepsilon_1 > 0$ such that $|N/q| \geq \varepsilon_1$ in a neighborhood of $q = 0$. Since

$$\lim_{q \to 0} \frac{N}{q}(q,\alpha,\theta) = N_q(0,\alpha,\theta)$$

it will suffice to show that $N_q(0,\alpha,\theta)$ is bounded away from zero.

To establish this, observe that since the solution is singular on $[0,\widehat{q}]$, it follows from equation (35) and Lemma 6 that $N(q,\alpha(q),\theta(q)) \leq 0$ for all $q \in (0,\widehat{q})$. Meanwhile, recall that by definition we have:

$$\psi = \frac{u_q u_\theta}{u_{q\theta}} h_1.$$

Using the above expression in (44) yields:

$$N(q,\alpha,\theta) = u_\theta \left(h_1 \frac{\partial}{\partial q}\left(\frac{u_q}{u_{q\theta}}\right) + \frac{u_q}{u_{q\theta}} \frac{\partial}{\partial q} h_1 + h_0 \right). \tag{89}$$

Since $u_\theta(0,\alpha,\theta) = 0$, we have $N(0,\alpha(q),\theta(q)) = 0$. Since $N(q,\alpha(q),\theta(q)) \leq 0$ and $N(0,\alpha(q),\theta(q)) \leq 0$, it follows from Assumption 3(i) that $N(q',\alpha(q),\theta(q)) < 0$ for all $q' \in (0,q)$. We shall now show that this implies that $N_q(0,\alpha(q),\theta(q)) < 0$ for all $q \in (0,\widehat{q})$.

For suppose instead we had $N_q(0,\alpha(q),\theta(q)) \geq 0$ for some $q \in (0,\widehat{q})$. The strict convexity of $N(x,\alpha(q),\theta(q))$ in x then implies that $N(x,\alpha,\theta) > 0$ for all $x > 0$. Combining this with the fact that $N(0,\alpha(q),\theta(q)) = 0$ implies that $N(q',\alpha(q),\theta(q)) > 0$ for all $q' \in (0,q)$, contradicting what we established above. Hence we have $N_q(0,\alpha(q),\theta(q)) < 0$ for all $q \in (0,\widehat{q})$.

Because $N_{q\theta} \leq 0$ by Assumption 3, and because $\theta(0) > \theta(q)$, it follows that $N_q(0,\alpha(q),\theta(0)) \leq N_q(0,\alpha(q),\theta(q)) < 0$. Since $N_q(0,0,\theta(q)) = 0$, and since by Lemma 7 we have $\alpha(0) > 0$, the convexity of N_q in α then implies that $N_q(0,\alpha(0),\theta(0)) < 0$, finally establishing that $N_q(0,\alpha(q),\theta(q))$ is bounded away from zero in a neighborhood of $q = 0$.

Since $|D/q^2|$ remains bounded above in a neighborhood of $q = 0$, $|N/q| \geq \varepsilon_1$ in a neighborhood of $q = 0$, it follows that there exists $\varepsilon_2 > 0$ such that $\kappa(q,\alpha,\theta) \geq \varepsilon_2$ over this neighborhood. Hence we have

$$\dot{\alpha} \geq \frac{\varepsilon}{q},$$

implying

$$\alpha(q) \leq \alpha(q_1) + \varepsilon \ln q$$

Thus $\alpha(q) = 0$ for some $q > 0$, thereby establishing the required contradiction. *Q.E.D.*
Q.E.D.

Proof of Theorem 8: First, let us establish that it is necessary that $\alpha'(q) = 0$ for all $q \leq \widehat{q}$. Suppose instead that in the optimal mechanism there existed an interval $[q_-, q_+]$ of $q < \widehat{q}$ on which $\alpha'(q) > 0$. Then for any $q \in [q_-, q_+)$ we have $\theta > 0$. It follows that the iso-price line $\sigma(q, \alpha(q), \theta(q), a)$ trough the point $(\alpha(q), \theta(q))$ at the level q contains points (those with coordinates $a \in (\alpha, \alpha(\widehat{q})])$ which violate the individual rationality condition. Types $(\sigma(q, \alpha(q), \theta(q), a), a)$ with $a \in (\alpha, \alpha(\widehat{q})]$ will therefore not consume the the increment q, or any of the increments $z < q$, as is assumed in the demand profile approach.

Next, we establish the necessity of $\widehat{q} = 0$. Suppose to the contrary that we had $\widehat{q} > 0$. Let us now assume that $\theta^{\phi}(0) > 0$; and entirely analogous argument treats the case where $\theta^{\phi}(q) = 0$ for some $q > 0$. It follows from Theorem 6 that $\widehat{\theta} = \theta^{\phi}(\widehat{q})$, and so we have $\phi(\widehat{q}, \widehat{\theta}) = 0$. Furthermore, since ϕ is decreasing in q, we have $\phi(q, \widehat{\theta}) > 0$ for all $q < \widehat{q}$, and so

$$u_q(q, 1, \widehat{\theta}) H_{\theta}(q, \widehat{\theta}) + u_{q\theta}(q, 1, \widehat{\theta}) H(q, \widehat{\theta}) > 0. \tag{90}$$

Now recall that $N(p, q)$ is be the measure of types (α, θ) for whom $u_q(q, \alpha, \theta) \geq p$. Thus, letting $\widetilde{\theta}(p, q)$ be the solution to $u_q(q, 1, \theta) = p$, we have $N(p, q) = H(q, \widetilde{\theta}(p, q))$. The optimality condition for the problem $\max_p pN(p, q)$ can thus be written as

$$N(p, q) + p \frac{\partial N}{\partial p}(p, q) = 0, \tag{91}$$

or equivalently that

$$u_{q\theta}(q, 1, \theta) H(q, \theta) + u_q(q, 1, \theta) H_{\theta}(q, \theta) = 0 \text{ at } \theta = \widetilde{\theta}(p, q). \tag{92}$$

It follows from (90), (92) and the fact that ϕ is increasing in θ that $\widehat{\theta} > \widetilde{\theta}(p, q)$. Consequently, the optimal mechanism must differ from the mechanism selected by the demand profile approach.

Next, let us establish sufficiency. If $\widehat{q} = 0$, then in the optimal mechanism we have $\phi(q, \theta^{\phi}(q)) = 0$ for all $q \in [0, \bar{q}(1)]$, implying that (91) holds at $p = u_q(q, 1, \theta^{\phi}(q))$. Furthermore, the monotonicity of ϕ in θ implies that there is no $\theta \neq \theta^{\phi}(q)$ for which (92) holds, so $p = u_q(q, 1, \theta^{\phi}(q))$ is a global optimizer of (91), and so the demand profile approach identifies the optimal mechanism. *Q.E.D.*

Proof of Theorem 9: Note that (13), (15), (43) and (83) can be combined to show that $\phi(q, \theta) = u_{q\theta}(q, 1, \theta) v(q, \theta)$, where

$$v(q, \theta) = -u_q(q, 1, \theta) \int_{\alpha_-(q,\theta)}^{1} \frac{f(a, \sigma(q, 1, \theta, a)}{u_{q\theta}(q, a, \sigma(q, 1, \theta, a))} da + H(q, 1, \theta)$$

Also, it follows from (14), (15), (43) and (84) that $\varkappa(q, \alpha) = u_{q\alpha}(q, \alpha, 0)\rho(q, \alpha)$, where

$$\rho(q, \alpha) = -u_q(q, \alpha, 0) \int_{\alpha_-(q,\alpha)}^{\alpha} \frac{f(a, \sigma(q, \alpha, 0, a))}{u_{q\theta}(q, a, \sigma(q, \alpha, 0, a))} da + H(q, \alpha, 0)$$

Next, we show the necessity of condition (45). Suppose first that $\rho(0, \mathring{\alpha}) = 0$ for some $\mathring{\alpha} \in (0, 1]$ but $u_{q\alpha}(0, \alpha, 0) > 0$ for some $\alpha \in (0, \mathring{\alpha})$. Since $\sigma_\alpha(0, \alpha, 0, a) = u_{q\alpha}(0, \alpha, 0)/u_{q\theta}(0, a, \sigma(0, \alpha, 0, a))$, it then follows that $\sigma(0, \alpha, 0, 0) > 0$. Thus a positive measure of types must receive a zero quantity. Next, suppose that $u_{q\alpha}(0, \alpha, 0) = 0$ for all α, but there exists no $\mathring{\alpha} \in (0, 1]$ such that $\rho(0, \alpha) = 0$. Since $\rho(0, 0) = 1$, we must then have $\rho(0, \alpha) > 0$ for all α. Because $\phi(0, 0) = u_{q\theta}(0, 1, 0)\rho(0, 1) > 0$, and because by assumption we have $\phi_\theta < 0$, it then follows that $\phi(0, \theta(0)) = 0$ for some $\theta(0) > 0$. Therefore all consumer types (α, θ) with $\theta < \theta(0)$ are excluded.

Next, we will argue that if the conditions of the Theorem hold, then the associated mechanism is optimal and no consumer is excluded. Let the monopolist select $(\widehat{\alpha}, \widehat{\theta}) = (\mathring{\alpha}, 0)$ and $\widehat{q} = 0$.

Theorem 5 implies that in this case $\underline{q}(1)$ is the solution to the equation $\varkappa(q, 1) = 0$, on the interval $[0, \underline{q}(1)]$ $\alpha(q)$ is defined as a solution to $\varkappa(q, \alpha) = 0$, while on the interval $[\underline{q}(1), \bar{q}(1)]$ $\theta(q)$ is defined as a solution to $\phi(q, \theta) = 0$. There is no exclusion in this mechanism, because the isoquant emanating from the point $(\mathring{\alpha}, 0)$ is flat, i.e. $\sigma_a(0, \mathring{\alpha}, 0, a) = -\frac{u_{q\alpha}(0,a,0)}{u_{q\theta}(0,a,0)} = 0$. *Q.E.D.*

9 Appendix B

In this appendix, we prove Theorems 10 and 11. Observe that $u - \frac{u_\theta u_q}{u_{q\theta}} = \frac{(b-\alpha)}{2} q^2 > 0$ for all $q > 0$. It then follows from Lemma 7(i) that $\underline{q}(0) = 0$. We start by considering the solution to Subproblem (ii) in (21) for an arbitrary fixed $\hat{q} > 0$.

Lemma 14 *Suppose that $u(q, \theta, \alpha) = \theta q - \frac{b-\alpha}{2} q^\gamma$, and $F(\theta, \alpha)$ is uniform on $[0, 1]^2$. Then in the optimal solution $\alpha^* = \frac{2b}{3}$ and the interval $[0, q^*]$ forms a non-singular arc where $\alpha' = \theta' = 0$, while the interval $[q^*, \hat{q}]$ forms a singular arc where $\alpha' > 0$.*

Proof of Lemma 14:

Step 1. Preliminary Computations.

Note the following simple results:

$$\sigma = \theta + \frac{\gamma}{2}(\alpha - a)q^{\gamma - 1} \tag{93}$$

$$\sigma_\theta = 1 \tag{94}$$

$$\sigma_\alpha = \frac{\gamma}{2} q^{\gamma - 1} \tag{95}$$

$$\sigma_q = \frac{\gamma(\gamma - 1)}{2} q^{\gamma - 2} \tag{96}$$

$$-\frac{d\theta}{d\alpha} = g \equiv \frac{u_\alpha}{u_\theta} = \frac{q^{\gamma - 1}}{2} \tag{97}$$

Let us first focus on the case $q \in [0, q^*]$. Then $\underline{\alpha} = 0$, and so we have:

$$h_0(q) = \frac{\alpha^2\gamma(\gamma-1)}{4} q^{\gamma-2} \tag{98}$$
$$h_1(q) = \alpha \tag{99}$$
$$h_2(q) = \frac{\alpha\gamma}{2} q^{\gamma-1} \tag{100}$$
$$\psi(q) = \alpha\left(\theta q - \frac{\gamma}{2}(b-\alpha)q^\gamma\right) \tag{101}$$

Using (98)-(101), we obtain:

$$D(q,\alpha,\theta) \equiv uf + \psi g_\theta + \psi_\theta g - \psi_\alpha - u_\theta(h_2 - gh_1) = \frac{\gamma-1}{2}(b-3\alpha)\, q^\gamma \tag{102}$$
$$N(q,\alpha,\theta) \equiv u_\theta h_0 - u_q h_1 + \psi_q = \frac{\alpha\gamma(\gamma-1)\left(\frac{3\alpha}{2} - b\right)}{2} q^{\gamma-1} \tag{103}$$

Combining (102) and (103), we obtain:

$$\alpha' = \frac{N}{D} = \frac{\gamma\alpha\left(\frac{3\alpha}{2} - b\right)}{q(b-3\alpha)} \tag{104}$$

Next, consider $q \in [q^*, \hat{q}]$. In this case, $\underline{\alpha} = \alpha - \frac{2(1-\theta)}{\gamma q^{\gamma-1}}$, and so we have:

$$h_0(q) = \frac{(1-\theta)^2(\gamma-1)}{\gamma} q^{-\gamma} \tag{105}$$
$$h_1(q) = \frac{2(1-\theta)}{\gamma q^{\gamma-1}} \tag{106}$$
$$h_2(q) = (1-\theta) \tag{107}$$
$$\psi(q) = \frac{2\theta(1-\theta)}{\gamma} q^{2-\gamma} - (b-\alpha)(1-\theta)q \tag{108}$$

Using (105)-(108), we obtain that for $q \in [q^*, \hat{q}]$:

$$D(q,\alpha,\theta) \equiv uf + \psi g_\theta + \psi_\theta g - \psi_\alpha - u_\theta(h_2 - gh_1) = (3\theta-2)\left(1 - \frac{1}{\gamma}\right) q \tag{109}$$
$$N(q,\alpha,\theta) \equiv u_\theta h_0 - u_q h_1 + \psi_q = (1-\theta)(1-3\theta)\frac{\gamma-1}{\gamma} q^{1-\gamma} \tag{110}$$

Combining (109) and (110), we obtain for $q \in [q^*, \hat{q}]$::

$$\alpha'(q) = \frac{N}{D} = \frac{(1-\theta)(1-3\theta)}{q^\gamma(3\theta-2)} \tag{111}$$

Step 2. Let us introduce some additional notation. Let $q_{b/3} = d\sup\{q : \alpha(q) \le \frac{b}{3}\}$, $q_{2b/3} = \sup\{q : \alpha(q) \le \frac{2b}{3}\}$.

Let us show that for any $q \in [q_0, \min\{q^*, q_{b/3}\})$, we have $\alpha'(q) = 0$.

Proof: The proof is by contradiction so suppose that there exists some $q' \in [q_0, \min\{q^*, q_{b/3}\})$ s.t. $\alpha'(q') > 0$. Then q' belong to a singular arc and from equation (102), $D(q', \alpha(q'), \theta(q')) > 0$. But this contradicts Generalized Legendre-Clebsch condition of Lemma 6.

Step 3. Suppose that $q_{b/3} < \min\{q_{2b/3}, q^*\}$. Then any $q \in [q_{b/3}, \min\{q_{2b/3}, q^*\}]$ lies on a singular arc.

By step 1, $S(q) \leq 0$ and $\alpha'(q) = 0$ for any $q \in [q_0, q_{b/3}]$. So to prove this step by contradiction, assume that there exists $q'' \in [q_{b/3}, \min\{q_{2b/3}, q^*\})$ s.t. $S(q'') = 0$ and $S(q) > 0$ in any right neighborhood of q''.

By definition, $S = u(h_2 - gh_1) + (\mu - \lambda g)$. Using (97) and (99)-(100), we obtain:

$$u(h_2 - gh_1) = (\theta q - \frac{b-\alpha}{2}q^\gamma)(\frac{\alpha\gamma}{2}q^{\gamma-1} - \frac{q^{\gamma-1}}{2}\alpha) = (\theta q - \frac{b-\alpha}{2}q^\gamma)\frac{\alpha(\gamma-1)q^{\gamma-1}}{2} \tag{112}$$

Also, by Theorem 4 on the interval $[q_{b/3}, \min\{q_{2b/3}, q^*\})$ we have: $\dot{\mu} = -u_\alpha h_0 - u\frac{\partial h_0}{\partial \alpha}$, $\dot{\lambda} = -u_\theta h_0 - u\frac{\partial h_0}{\partial \theta}$. Since $\mu(0) = 0$ and $\lambda(0) = 0$, we have:

$$\begin{aligned}
\mu(q) &= \int_0^q \mu'(x)dx = \int_0^q -u_\alpha h_0 - u\frac{\partial h_0}{\partial \alpha}dx \\
&= \int_0^q -\frac{x^{2(\gamma-1)}\gamma(\gamma-1)\alpha^2}{8} - \frac{\alpha\gamma(\gamma-1)x^{\gamma-2}}{2}(\theta x - \frac{b-\alpha}{2}x^\gamma)dx = \\
&- \frac{\alpha\theta(\gamma-1)q^\gamma}{2} + \frac{\alpha\gamma(\gamma-1)(b-\frac{3}{2}\alpha)q^{2\gamma-1}}{4(2\gamma-1)}
\end{aligned} \tag{113}$$

$$\lambda(q) = \int_0^q \lambda'(x)dx = \int_0^q -u_\theta h_0 - u\frac{\partial h_0}{\partial \theta}dx = \int_0^q -\frac{\alpha^2\gamma(\gamma-1)}{4}x^{\gamma-1}dx = -\frac{\alpha^2(\gamma-1)q^\gamma}{4} \tag{114}$$

Combining (112), (113) and (114) yields:

$$\begin{aligned}
&S = u(h_2 - gh_1) + \mu - \lambda g = \\
&(\theta q - \frac{b-\alpha}{2}q^\gamma)\frac{\alpha(\gamma-1)q^{\gamma-1}}{2} - \frac{\alpha\theta(\gamma-1)q^\gamma}{2} + \frac{\alpha\gamma(\gamma-1)(b-\frac{3}{2}\alpha(q))q^{2\gamma-1}}{4(2\gamma-1)} + \frac{q^{\gamma-1}}{2}\frac{\alpha^2(\gamma-1)q^\gamma}{4} = \\
&-\frac{b-\alpha}{2}q^\gamma\frac{\alpha(\gamma-1)q^{\gamma-1}}{2} + \frac{\alpha\gamma(\gamma-1)(b-\frac{3}{2}\alpha)q^{2\gamma-1}}{4(2\gamma-1)} + \frac{q^{\gamma-1}}{2}\frac{\alpha^2(\gamma-1)q^\gamma}{4} = \\
&\left(-1 + \frac{\gamma}{2\gamma-1}\right)\frac{\alpha(\gamma-1)(b-\frac{3}{2}\alpha(q))q^{2\gamma-1}}{4} = \frac{\alpha(\gamma-1)^2(\frac{3}{2}\alpha - b)q^{2\gamma-1}}{4(2\gamma-1)}
\end{aligned} \tag{115}$$

Note that (115) is negative if $\alpha < \frac{2b}{3}$ and is positive if $\alpha < \frac{2b}{3}$ So, we cannot have $\alpha(q) > \frac{2b}{3}$ for $q < q^*$.

Step 4. The interval $[0, q^*]$ is such that $\alpha'(q) = 0$ for all $q \in [0, q^*]$. Also, $\alpha^* \geq \frac{2b}{3}$.

Step 5. Let $[0, \bar{q})$ be the non-singular arc (so that $\alpha'(q) = 0$ for all $q \in [0, \bar{q})$). Then $\bar{q} = q^*$ and $\alpha^* = \frac{2b}{3}$, $\theta^* = 1 - \alpha^*\frac{\alpha\gamma}{2}(q^*)^{\gamma-1}$.

Proof: The previous steps have established that $\bar{q} \geq q^*$. So we only need to show that we cannot have $\bar{q} > q^*$.

To complete the proof, let us compute the value of $S(\bar{q})$. By definition, we have: $S(\bar{q}) = (u(h_2 - gh_1) + \mu - g\lambda)(\bar{q})$.

By Theorem 4, on the non-singular arc $[0, \bar{q})$, $\dot{\mu} = -u_\alpha h_0 - u\frac{\partial h_0}{\partial \alpha}$ $\dot{\lambda} = -u_\theta h_0 - u\frac{\partial h_0}{\partial \theta}$, and also $\mu(0) = 0$ and $\lambda(0) = 0$. Therefore, we have:

$$
\begin{aligned}
&\mu(\bar{q}) = \int_0^{\bar{q}} \mu'(q)dq = \int_0^{\bar{q}} -u_\alpha h_0 - u\frac{\partial h_0}{\partial \alpha}dq = \int_0^{q^*} -u_\alpha h_0 - u\frac{\partial h_0}{\partial \alpha}dq + \int_{q^*}^{\bar{q}} -u_\alpha h_0 - u\frac{\partial h_0}{\partial \alpha}dq = \\
&\int_0^{q^*} -\frac{x^{2(\gamma-1)}\gamma(\gamma-1)\alpha^2}{8} - \frac{\alpha\gamma(\gamma-1)x^{\gamma-2}}{2}(\theta x - \frac{b-\alpha}{2}x^\gamma)dx + \\
&\int_{q^*}^{\bar{q}} -\frac{x^\gamma}{2}\frac{(1-\theta)^2(\gamma-1)}{\gamma}x^{-\gamma}dx = \\
&-\frac{\alpha\theta(\gamma-1)(q^*)^\gamma}{2} + \frac{\alpha\gamma(\gamma-1)(b-\frac{3}{2}\alpha)(q^*)^{2\gamma-1}}{4(2\gamma-1)} - \frac{(1-\theta)^2(\gamma-1)}{2\gamma}(\bar{q}-q^*). \qquad (116)
\end{aligned}
$$

$$
\begin{aligned}
&\lambda(\bar{q}) = \int_0^{\bar{q}} \lambda'(q)dq = \int_0^{\bar{q}} -u_\theta h_0 - u\frac{\partial h_0}{\partial \theta}dq = \int_0^{q^*} -u_\theta h_0 - u\frac{\partial h_0}{\partial \theta}dq + \int_{q^*}^{\bar{q}} -u_\theta h_0 - u\frac{\partial h_0}{\partial \theta}dq = \\
&\int_0^{q} -\frac{\alpha^2\gamma(\gamma-1)}{4}x^{\gamma-1}dx + \int_{q^*}^{\bar{q}} -\frac{(1-\theta)^2(\gamma-1)}{\gamma}x^{1-\gamma} - (\theta x - \frac{b-\alpha}{2}x^\gamma)\frac{-2(1-\theta)(\gamma-1)}{\gamma}x^{-\gamma}dx \\
&= -\frac{\alpha^2(\gamma-1)(q^*)^\gamma}{4} - \frac{(b-\alpha)(1-\theta)(\gamma-1)}{\gamma}(\bar{q}-q^*) + \frac{(3\theta-1)(1-\theta)(\gamma-1)}{\gamma}\int_{\bar{q}}^{q^*} q^{1-\gamma}dq \qquad (117)
\end{aligned}
$$

Using (97), (99)-(100), (106)-(107), (116) and (117), we can now compute $S(\bar{q})$ for $q \geq q^*$:

$$
\begin{aligned}
&S(\bar{q}) = u(h_2 - gh_1) + \mu - g\lambda = \\
&(\theta\bar{q} - \frac{b-\alpha}{2}\bar{q}^{\gamma})(1-\theta)\frac{\gamma-1}{\gamma} \\
&- \frac{(1-\theta)\theta(\gamma-1)q^*}{\gamma} + \frac{(1-\theta)^2(\gamma-1)(b-\frac{3}{2}\alpha)q^*}{\alpha\gamma(2\gamma-1)} - \frac{(1-\theta)^2(\gamma-1)}{2\gamma}\left(\bar{q}-q^*\right) \\
&+ \frac{\alpha^2(\gamma-1)(q^*)^{\gamma}}{8}\bar{q}^{\gamma-1} + \frac{(b-\alpha)(1-\theta)(\gamma-1)}{2\gamma}\left(\bar{q}-q^*\right)\bar{q}^{\gamma-1} - \frac{(3\theta-1)(1-\theta)(\gamma-1)}{2\gamma}\int_{q^*}^{\bar{q}}\left(\frac{\bar{q}}{q}\right)^{\gamma-1}dq = \\
&\frac{(3\theta-1)(1-\theta)(\gamma-1)}{2\gamma}\left(\bar{q}-q^*\right) - \frac{b-\alpha}{2}\bar{q}^{\gamma}(1-\theta)\frac{\gamma-1}{\gamma} + \frac{(1-\theta)^2(\gamma-1)(b-\frac{3}{2}\alpha)q^*}{\alpha\gamma(2\gamma-1)} \\
&+ \frac{\alpha^2(\gamma-1)(q^*)^{\gamma}}{8}\bar{q}^{\gamma-1} + \frac{(b-\alpha)(1-\theta)(\gamma-1)}{2\gamma}\left(\bar{q}-q^*\right)\bar{q}^{\gamma-1} - \frac{(3\theta-1)(1-\theta)(\gamma-1)}{2\gamma}\int_{q^*}^{\bar{q}}\left(\frac{\bar{q}}{q}\right)^{\gamma-1}dq = \\
&\frac{(3\theta-1)(1-\theta)(\gamma-1)}{2\gamma}\left(\bar{q}-q^*\right) - \frac{(b-\alpha)(1-\theta)(\gamma-1)}{2\gamma}q^*\bar{q}^{\gamma-1} + \frac{(1-\theta)^2(\gamma-1)(b-\frac{3}{2}\alpha)q^*}{\alpha\gamma(2\gamma-1)} \\
&+ \frac{\alpha^2(\gamma-1)(q^*)^{\gamma}}{8}\bar{q}^{\gamma-1} - \frac{(3\theta-1)(1-\theta)(\gamma-1)}{2\gamma}\int_{q^*}^{\bar{q}}\left(\frac{\bar{q}}{q}\right)^{\gamma-1}dq = \\
&- \frac{(3\theta-1)(1-\theta)(\gamma-1)}{2\gamma}\int_{q^*}^{\bar{q}}\left(\frac{\bar{q}}{q}\right)^{\gamma-1} - 1dq - \frac{(b-\frac{3}{2}\alpha)(1-\theta)(\gamma-1)}{2\gamma}q^*\bar{q}^{\gamma-1} + \frac{(1-\theta)^2(\gamma-1)(b-\frac{3}{2}\alpha)q^*}{\alpha\gamma(2\gamma-1)} = \\
&- \frac{(3\theta-1)(1-\theta)(\gamma-1)}{2\gamma}\int_{q^*}^{\bar{q}}\left(\frac{\bar{q}}{q}\right)^{\gamma-1} - 1dq - \frac{(b-\frac{3}{2}\alpha)(1-\theta)^2(\gamma-1)}{\alpha\gamma}q^*\left(\frac{1}{\gamma} - \frac{1}{2\gamma-1}\right)
\end{aligned}
\tag{118}
$$

If $\alpha^* < \frac{2b}{3}$, then taking into account that $\bar{q} \geq q^*$, and inspecting the last equality in (118) establishes that $S(\bar{q}) < 0$. But this contradicts the fact that $S(\bar{q}) < 0$ since $\bar{q}$ is a juncture point between a non-singular and singular arcs and hence we must have $S(\bar{q}) = 0$. So, $\alpha^* = \frac{2b}{3}$. Then, we also must have $\bar{q} = q^*$, because otherwise $S(\bar{q}) < 0$.

It remains to check that the solution with $\alpha^* = \frac{2b}{3}$ and $\bar{q} = q^*$ is consistent with the continuity of the Lagrange multipliers, particularly at the juncture point $\bar{q}$.

Since $\bar{q}$ is a juncture point, there exists $z > 0$ such that $[\bar{q}, \bar{q} + z]$ in a singular arc. By Theorem 4, on a singular arc , $\mu(\bar{q}) = (\psi g - uh_2)(\bar{q})$ and $\lambda(\bar{q}) = (\psi - uh_1)(\bar{q})$. Therefore, we must have:

$$
\begin{aligned}
\int_0^{\bar{q}} \mu'(q)dq &= \int_0^{\bar{q}} -u_{\alpha}h_0 - u\frac{\partial h_0}{\partial \alpha}dq = \mu(\bar{q}) = (\psi g - uh_2)(\bar{q}) \\
\int_0^{\bar{q}} \lambda'(q)dq &= \int_0^{\bar{q}} -u_{\theta}h_0 - u\frac{\partial h_0}{\partial \theta}dq = \lambda(\bar{q}) = (\psi - uh_1)(\bar{q}).
\end{aligned}
$$

Using (97), and (106)-(108) we may compute:

$$(\psi g - uh_2)(\bar{q}) = (1-\theta)\left(\frac{2\theta}{\gamma}q^{2-\gamma} - (b-\alpha)q\right)\frac{q^{\gamma-1}}{2} - (1-\theta)\left(\theta q - (b-\alpha)\frac{q^{\gamma}}{2}\right) = -(1-\theta)\theta\frac{\gamma-1}{\gamma}\bar{q}, \tag{119}$$

$$(\psi - uh_1)(\bar{q}) = (1-\theta)\left(\frac{2\theta}{\gamma}q^{2-\gamma} - (b-\alpha)q\right) - \left(\theta q - (b-\alpha)\frac{q^{\gamma}}{2}\right)\frac{2(1-\theta)}{\gamma q^{\gamma-1}} = -(1-\theta)(b-\alpha)\frac{\gamma-1}{\gamma}\bar{q} \tag{120}$$

To check the continuity of μ equate (116) and (119) to obtain:

$$-(1-\theta)\theta\frac{\gamma-1}{\gamma}\bar{q} = -\frac{\alpha\theta(\gamma-1)(q^*)^{\gamma}}{2} + \frac{\alpha\gamma(\gamma-1)(b-\frac{3}{2}\alpha)(q^*)^{2\gamma-1}}{4(2\gamma-1)} - \frac{(1-\theta)^2(\gamma-1)}{2\gamma}(\bar{q}-q^*). \tag{121}$$

Since α and θ are constant functions of q on the interval $[0,\bar{q}]$, and $\bar{q} \geq q^*$, from the definition of q^* it follows that $1 = \theta + \frac{\alpha\gamma}{2}(q^*)^{\gamma-1}$. Substituting this into the first and second terms on the right-hand side of (121) yields:

$$-(1-\theta)\theta\frac{\gamma-1}{\gamma}\bar{q} = -\frac{(1-\theta)\theta(\gamma-1)q^*}{\gamma} + \frac{(1-\theta)^2(\gamma-1)(b-\frac{3}{2}\alpha)q^*}{\alpha\gamma(2\gamma-1)} - \frac{(1-\theta)^2(\gamma-1)}{2\gamma}(\bar{q}-q^*). \tag{122}$$

which can be further simplified as follows:

$$\bar{q} = q^* + \frac{(2b-3\alpha)}{\alpha(2\gamma-1)}q^*,$$
$$\bar{q} = \frac{2(b+2\alpha(\gamma-2))}{\alpha(2\gamma-1)}q^* \tag{123}$$

Simple observation establishes that the equality (123) holds when $\alpha = \frac{2b}{3}$ and $\bar{q} = q^*$.

Now let us check the continuity of the Lagrange multiplier λ at $\bar{q}$. Equating (117) and (120) we obtain:

$$-(1-\theta)(b-\alpha)\frac{\gamma-1}{\gamma}\bar{q} = -\frac{\alpha^2(\gamma-1)(q^*)^{\gamma}}{4} - \frac{(b-\alpha)(1-\theta)(\gamma-1)}{\gamma}(\bar{q}-q^*) + \frac{(3\theta-1)(1-\theta)(\gamma-1)}{\gamma}\int_{\bar{q}}^{q^*} q^{1-\gamma}dq$$

Using the equation $1-\theta = \frac{\alpha\gamma}{2}(q^*)^{\gamma-1}$ and cancelling the term on the left-hand side with the first part of the expansion of the second term on the right-hand side yields:

$$0 = -\frac{\alpha(1-\theta)(\gamma-1)q^*}{2\gamma} + \frac{(b-\alpha)(1-\theta)(\gamma-1)}{\gamma}q^* + \frac{(3\theta-1)(1-\theta)(\gamma-1)}{\gamma}\int_{\bar{q}}^{q^*} q^{1-\gamma}dq$$
$$0 = \frac{(b-\frac{3\alpha}{2})(1-\theta)(\gamma-1)}{\gamma}q^* + \frac{(3\theta-1)(1-\theta)(\gamma-1)}{\gamma}\int_{\bar{q}}^{q^*} q^{1-\gamma}dq \tag{124}$$

It is easy to see that the equality (124) holds when $\alpha = \frac{2b}{3}$ and $\bar{q} = q^*$, which completes the proof. *Q.E.D.*

Next, we consider the solution to Subproblem (ii) in (21) for an arbitrary fixed $\hat{q} > 0$.

Lemma 15 *Let $u(q,\theta,\alpha) = \theta q - \frac{b-\alpha}{2} q^{\gamma}$ and suppose that the types are distributed uniformly over $[0,1]^2$. Let $\bar{q} = \left(\frac{2}{\gamma(b-1)}\right)^{\frac{1}{\gamma-1}}$. and $\tilde{q} = \left(\frac{4}{\gamma(2b+1)}\right)^{\frac{1}{\gamma-1}}$. If $b \geq \frac{3}{2}$, then the solution to subproblem (ii) in (21) is as follows:*

$$\theta^{\phi}(q) = \frac{1+\gamma(b-1)q^{\gamma-1}}{3} \quad \text{for } q \in [\widehat{q}, \bar{q}]. \tag{125}$$

If $b < 3/2$, then the solution to the problem (21) is given by:

$$\theta^{\phi}(q) = \begin{cases} \frac{1+\gamma(b-1)q^{\gamma-1}}{3}, & \text{if } q \in [\tilde{q}, \bar{q}] \\ \frac{1+\gamma q^{\gamma-1}\left(\frac{2b-3}{4}\right)}{2}, & \text{if } q \in [0, \tilde{q}]. \end{cases} \tag{126}$$

Proof: By Theorem (6), the solution to the problem (21) is found by setting (42) to zero and solving that equation i.e., $\phi(q,\theta) \equiv u_q(q,1,\theta)H_\theta(q,1,\theta) + u_{\theta q}(q,1,\theta)H(q,1,\theta) = 0$

So, first we need to compute $H(q,1,\theta) \equiv \int_{\underline{\alpha}(q,1,\theta)}^{1} \int_{\sigma(q,\theta,1,a)}^{1} f(t,a)dtda$. Recall that $\underline{\alpha}(q,1,\theta) = 1 - \frac{2(1-\theta)}{\gamma q^{\gamma-1}} > 0$ if $\theta > 1 - \frac{\gamma q^{\gamma-1}}{2}$ and $\underline{\alpha}(q,1,\theta) = 0$ otherwise, while $\sigma(q,\theta,1,a) = \min\{\theta + \frac{\gamma(1-a)}{2}q^{\gamma-1}, 1\}$ with $\sigma(q,\theta,1,a) = \theta + \frac{\gamma(1-a)}{2}q^{\gamma-1}$ for all $a \in [\underline{\alpha}(q,1,\theta), 1]$.

So, when $\underline{\alpha}(q,1,\theta) = 1 - \frac{2(1-\theta)}{\gamma q^{\gamma-1}} > 0$ we may compute:

$$\begin{aligned} H(q,1,\theta) &= \int_{\underline{\alpha}(q)}^{1} \int_{\sigma(q,\theta,1,a)}^{1} dtda = \int_{1-\frac{2(1-\theta)}{\gamma q^{\gamma-1}}}^{1} \int_{\theta+\frac{\gamma(1-a)}{2}q^{\gamma-1}}^{1} dtda = \int_{1-\frac{2(1-\theta)}{\gamma q^{\gamma-1}}}^{1} 1-\theta-\frac{\gamma(1-a)}{2}q^{\gamma-1}da \\ &= \frac{(1-\theta)^2}{\gamma q^{\gamma-1}} \end{aligned} \tag{127}$$

Then using (127) in the equation $\phi(q,\theta) \equiv u_q(q,1,\theta)H_\theta(q,1,\theta) + u_{\theta q}(q,1,\theta)H(q,1,\theta) = 0$ and solving yields:

$$\theta^{\phi}(q) = \frac{1+\gamma(b-1)q^{\gamma-1}}{3}. \tag{128}$$

Next, suppose $\underline{\alpha}(q,1,\theta) = 0$. Then we have:

$$H(q,1,\theta) = \int_0^1 \int_{\sigma(q,\theta,1,a)}^{1} dtda = \int_0^1 \int_{\theta+\frac{\gamma(1-a)}{2}q^{\gamma-1}}^{1} dtda = \int_0^1 1-\theta-\frac{\gamma(1-a)}{2}q^{\gamma-1}da = 1-\theta-\frac{\gamma}{4}q^{\gamma-1} \tag{129}$$

Then solving $\phi(q,\theta) \equiv u_q(q,1,\theta)H_\theta(q,1,\theta) + u_{\theta q}(q,1,\theta)H(q,1,\theta) = 0$ with (129) substituted in, yields:

$$\theta^{\phi}(q) = \frac{1+\gamma q^{\gamma-1}\left(\frac{2b-3}{4}\right)}{2}. \tag{130}$$

It remains to determine the intervals on which (128) and (130) hold respectively. First, note that simple monotonicity argument shows that, if (128) applies at q_1, then it applies at $q_2 > q_1$. The highest q for which (128) applies, $\bar{q}$, is implicitly and uniquely defined by setting $\theta^{\phi}(\bar{q}) = 1$, which yields $\bar{q} = \left(\frac{2}{\gamma(b-1)}\right)^{\frac{1}{\gamma-1}}$. Further, Lemma 14 establishes that $\alpha^* = \frac{2b}{3} \leq 1$ when $b \leq \frac{3}{2}$. So, in this case $\underline{\alpha}(q,1,\theta^{\phi}(q)) \geq 0$ for all $q \geq \widehat{q}$, and hence (128) applies for all $q \in [\hat{q}, \bar{q}]$.

If $b > \frac{2}{3}$, then $\alpha^* = 1$, and $\underline{\alpha}(q,1,\theta^{\phi}(q)) = 0$ for $q \in [\widehat{q}, \tilde{q}]$, where $\tilde{q} = \left(\frac{4}{\gamma(2b+1)}\right)^{\frac{1}{\gamma-1}}$ is the solution to the equation $\underline{\alpha}(q,1,\theta^{\phi}(q)) = 1 - \frac{2(1-\theta^{\phi}(q^{\gamma-1}))}{\gamma q} = 0$ for q. So, (130) applies for all $q \in [\hat{q}, \tilde{q}]$, and (128) applies for all $q \in [\tilde{q}, \bar{q}]$. Q.E.D.

The following Lemma characterizes the solution to subproblem (i), for the case $b < 32$, on its unique singular arc $[q^*, \hat{q}]$ where $\alpha' < 0$.

Lemma 16 *Suppose that* $u = q\theta - \frac{b-\alpha}{2}q^\gamma$*,* $b < \frac{3}{2}$*, and* F *is uniform on* $[0,1]^2$*. The solution to subproblem (i) on its unique singular arc* $[q^*, \hat{q}]$ *is as follows:*

$$\theta^* \equiv \theta(q^*) = 1 - \frac{b\gamma(q^*)^{\gamma-1}}{3} \tag{131}$$

$$\hat{\alpha} \equiv \alpha(\hat{q}) = 1 \tag{132}$$

$$\theta(q) = \frac{2 - \sqrt{1 - \frac{b\gamma(q^*)^\gamma(2-b\gamma(q^*)^{\gamma-1})}{q}}}{3} \quad \textit{for all } q \in [q^*, \hat{q}] \tag{133}$$

$$b(q^*)^\gamma(2 - b\gamma(q^*)^{\gamma-1}) = (b-1)\hat{q}^\gamma(2-(b-1)\gamma\hat{q}^{\gamma-1}) \tag{134}$$

$$\left(\frac{1}{2} - \frac{b}{3}\right)\left(\frac{b\gamma(q^*)^\gamma}{3}(2 - b\gamma(q^*)^{\gamma-1})\right)^{\gamma-1} = \int_{\frac{1+\gamma(b-1)\hat{q}^{\gamma-1}}{3}}^{1-\frac{b\gamma(q^*)^{\gamma-1}}{3}} ((1-\theta)(3\theta-1))^{\gamma-1}\, d\theta \tag{135}$$

Proof of Lemma 16: By Lemma 14, $\alpha^* = \frac{2b}{3}$. Substituting this into equation $u_q(q^*, \theta^*, \alpha^*) = u_q(q^*, 0, 1)$ i.e., $\theta^* + \frac{b\gamma(q^*)^{\gamma-1}}{3} = 1$ (via which the triplet $(q^*, \theta^*, \alpha^*)$ is defined) gives us (131).

Further, Lemma 15 implies that, if $b \geq \frac{3}{2}$, then $\widehat{\theta} = \theta^\phi(\widehat{q}) \geq \theta^\phi(0) = \frac{1}{2}$, and if $b < \frac{3}{2}$ then $\widehat{\theta} = \tilde{\theta} \geq \frac{1}{3}$. Hence regardless of the value of b, we have $\widehat{\alpha} = 1$, establishing (132).

Next, combining (97) and (111) yields for all $q \in [q^*, \hat{q}]$:

$$\theta'(q) = \frac{d\theta}{d\alpha}\alpha'(q) = -\frac{1}{2}\frac{(1-\theta)(1-3\theta)}{q(3\theta-2)}$$

which can be rearranged as follows:

$$\frac{(4-6\theta)d\theta}{(1-\theta)(1-3\theta)} = \frac{dq}{q}$$

which can be integrated to yield that for all $q \in [q^*, \hat{q}]$ and some constant $k > 0$:

$$q = \frac{k}{(1-\theta)(1-3\theta)}. \tag{136}$$

Evaluating (136) at q^* and making use of $\alpha^* = \frac{2b}{3}$ and (131) allows us to compute the constant of integration k as follows $k = \frac{b\gamma}{3}(q^*)^\gamma(2 - b\gamma(q^*)^{\gamma-1})$. Solving (136) for θ and using $k = \frac{b\gamma}{3}(q^*)^\gamma(2 - b\gamma(q^*)^{\gamma-1})$ yields (133).

Next, the continuity of the solution at $\hat{q}$ implies that the value of (133) at $\hat{q}$ be the same as the one given by the first expression in (126) , $\frac{1+\gamma(b-1)\hat{q}^{\gamma-1}}{3}$, or, equivalently, that the equality $\hat{q}(1-\hat{\theta})(3\hat{\theta}-1) = q^*(1-\theta^*)(3\theta^*-1)$ holds when we substitute into it $\alpha^* = \frac{2b}{3}$, (131), and $\hat{\theta} = \frac{1+\gamma(b-1)\hat{q}^{\gamma-1}}{3}$. This yields (134).

Finally, we have $1 - \frac{2b}{3} = \hat{\alpha} - \alpha^* = \int_{\hat{\theta}}^{\theta^*} \alpha'(\theta)d\theta = \int_{\hat{\theta}}^{\theta^*} -\frac{2}{q^{\gamma-1}}d\theta = \int_{\theta^*}^{\hat{\theta}} \frac{2}{q^{\gamma-1}}d\theta$. where the third equation uses (97). Using (136) with $k = \frac{b\gamma}{3}(q^*)^\gamma(2 - b\gamma(q^*)^{\gamma-1})$ in the last equation and simplifying yields (135) *Q.E.D.*

Figure 1: Typical shape of the isoquants

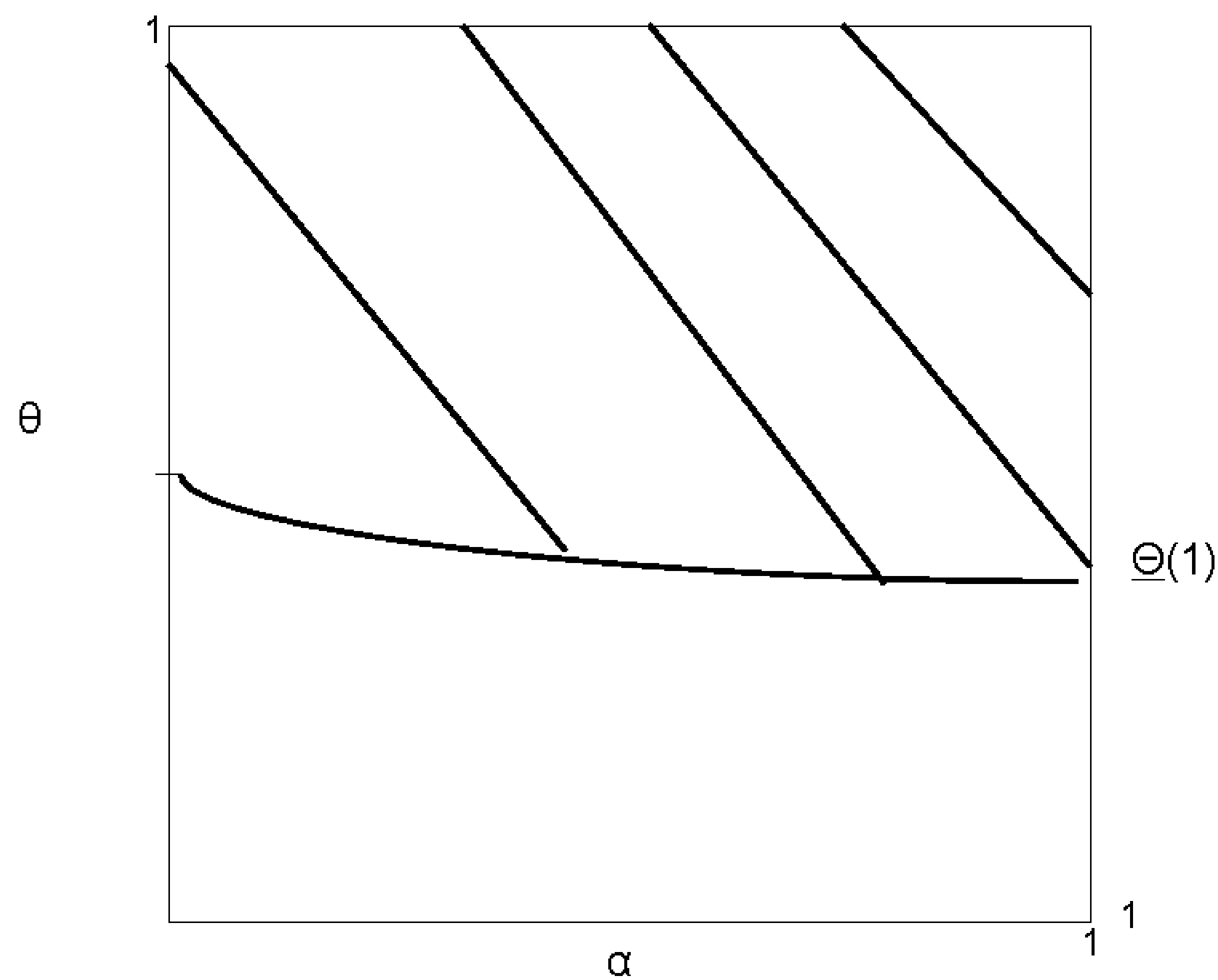

Figure 2: Linear- quadratic case with uniform distribution. Isoquants in Case A: $b > 3/2$.

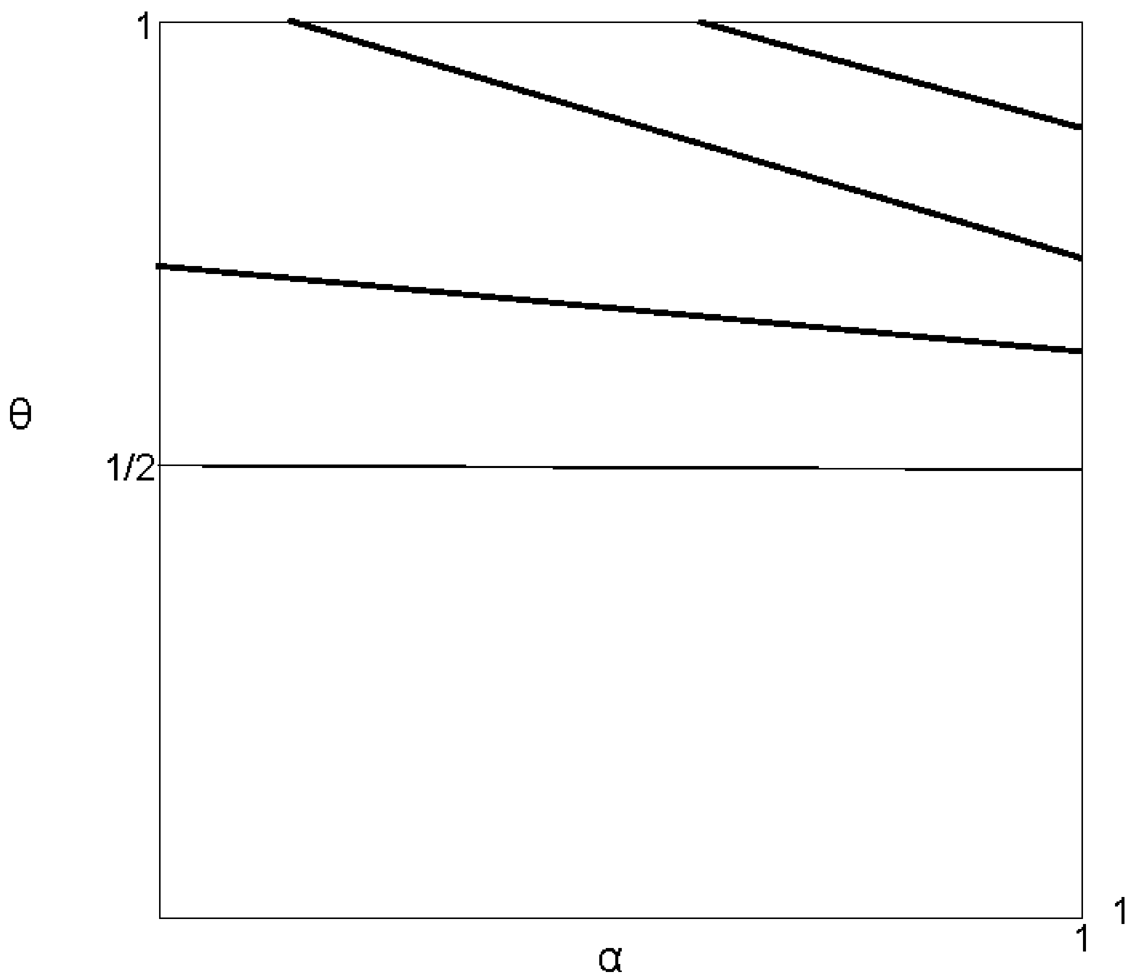

Figure 3: Linear- quadratic case with uniform distribution. Isoquants in Case C: $b < 3/2$.