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Effect of elasticity on torsional oscillator experiments probing the possible supersolidity of helium

Humphrey J. Maris

Department of Physics, Brown University, Providence, Rhode Island 02912, USA

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We consider the effect of elasticity on torsional oscillator experiments to investigate the possible supersolidity of helium. We show that in most, and possibly all torsional oscillators there is a significant and hitherto unconsidered contribution to the torsional period from the helium elasticity. This effect needs to be carefully considered before it can be concluded that a supersolid component is indeed present.

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I. INTRODUCTION

A torsional oscillator (TO) consists of a torsion rod connecting a sample cell to a fixed base. The frequency of the oscillator is given by $\omega = \sqrt{K/I}$, where K is the torsion constant of the torsion rod and I is the moment of inertia of the sample cell plus contents. Such oscillators have been used in many experiments to study the superfluidity of liquid helium.[1] At the superfluid transition a fraction of the mass of the helium decouples and ceases to move when the sample cell rotates. This leads to a frequency shift $\Delta\Omega$ from which the superfluid fraction can be estimated.

In an experiment in 2004, Kim and Chan[2] made measurements with a cell containing solid helium confined in the pores of Vycor glass. When the temperature was lowered below 200 mK, the frequency of the oscillator began to increase. The increase in frequency when the temperature was lowered to 30 mK was about 18 ppm. This increase in frequency was interpreted as a decrease in the moment of inertia of the solid helium and corresponded to a supersolid fraction of 0.25%.[3] Since this pioneering work, there have been many other torsional oscillator experiments with solid helium which have indicated nonclassical rotational inertia (NCRI).[4,5] However, different experiments have yielded substantially different magnitudes of the NCRI. The NCRI has been found to vary with ^{3}He concentration and to depend on the procedure for preparing the sample. For example, in some cases the NCRI has been reduced substantially after the sample is annealed by holding it at a higher temperature, such as 1.5 K, for a long time before making the measurements. The NCRI is reduced if the vibration amplitude of the torsional oscillator is increased; one could suppose that this is due to the velocity of the superfluid increasing above a critical value.

However, in an important paper, Day and Beamish[6] discovered that the shear modulus of solid helium also exhibits anomalous behavior below 200 mK. Their results have been confirmed and extended in a series of more recent papers.[7] The modulus has been found to increase by as much as 25% when the temperature is lowered to 50 mK. This increase is believed to result from the pinning of dislocations by ^{3}He. The temperature dependence of the shear modulus is similar to the temperature dependence of the NCRI. In addition, there is a strain amplitude dependence to the shear modulus. As a consequence, there has been speculation that there is no true transition to a supersolid state and that possibly the effects seen in the torsional oscillator experiments can somehow be explained by elasticity. However, it has not yet been shown that the effects of elasticity are large enough to explain the apparent NCRI in all experiments. In this Rapid Communication we consider one particular mechanism by which elasticity influences the frequency of a torsional oscillator.

II. DIFFERENT ELASTICITY CONTRIBUTIONS

We begin by considering the effect of elasticity for a TO with the simple geometry shown schematically in Fig. 1. Here, the cell is taken to be a cylindrical volume of inside radius r_3 with flat bottom and top plates. The cell is connected to a massive base by a torsion rod of length L, outer radius r_1, and containing a hole of radius r_2 used for filling the cell with helium. As a starting approximation one can imagine the structure to consist of two distinct components, i.e., the torsion rod and the cell body. Within this approximation the rod provides the restoring torque when the cell body rotates and the cell body and the solid helium provide all of the angular momentum. One can consider the following series of improvements to this "simplest model":

(1) The torsion constant of a rod of length L, outer diameter r_1, and inner diameter r_2 is

$$K_{\rm rod} = \frac{\pi c_m \left(r_1^4 - r_2^4\right)}{2L}, \qquad (1)$$

where c_m is the shear modulus. Reppy *et al.*[8] have noted that this is the static torsion constant and that when the rod is being twisted back and forth at a finite frequency the effective torsion constant will be modified because each part of the rod has angular momentum. They show that this changes the torsion constant by a small amount. But since this change is not affected by any change in the elastic properties of the helium, this change can generally be neglected.

(2) If the cell is not entirely rigid the effective moment of inertia will depend on frequency ω. Reppy *et al.*[8] have discussed this effect. They considered just the cylindrical outer wall of the cell and showed that when the finite stiffness is allowed for there is a small increase in the effective moment of inertia, giving a small decrease in oscillator frequency.

(3) Beamish *et al.*[9] have considered the effect of the solid helium contained in the fill line of radius r_2. This changes the effective torsion constant of the rod to

$$\tilde{K}_{\rm rod} = \frac{\pi c_m \left(r_1^4 - r_2^4\right)}{2L} + \frac{\pi c_{\rm He} r_2^4}{2L}, \qquad (2)$$

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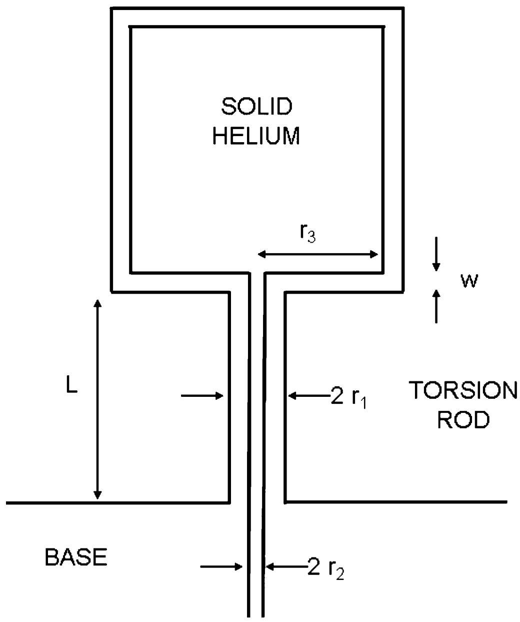

FIG. 1. Schematic diagram showing the cross section of a typical torsional oscillator.

where c_{He} is the shear modulus of the helium. Stiffening of the helium increases the torsion constant and raises the oscillator frequency, giving an apparent NCRI of the helium in the cell. Beamish *et al*. show that while this effect is large enough to explain the NCRI seen in some torsional oscillator experiments, there are many experiments in which the effect is much smaller than the measured NCRI.

(4) The finite stiffness of the solid helium in the cell modifies the effective moment of inertia of the oscillator and results in a frequency shift. It is straightforward to show that an increase in stiffness results in an increase in frequency and therefore an apparent decrease in the effective moment of inertia. This effect has been analyzed by Maris and Balibar[10] and by Reppy *et al*.[8] Maris and Balibar used a finite-difference numerical method, whereas Reppy *et al*. used an analytical approach. The conclusion from these calculations is that the effect of changes in the elasticity of the helium is too small to explain the NCRI seen in most of the TO experiments. These papers also give a discussion of the effect of helium elasticity in cells in which the helium is confined in an annular geometry, rather than within in an open volume of the type shown in Fig. 1. The annular sample space is formed between inner and outer metal cylinders. If the cell is constructed in a way such that these cylinders are not strongly connected together, then when the cell vibrates there will be relative motion of the cylinders. In this case a change in the shear modulus of the helium can have a large effect on the TO frequency and can therefore give a large apparent NCRI.

III. ANOTHER ELASTICITY CONTRIBUTION TO NCRI

Here we want to point out that there is, in fact, another way in which changes in the shear modulus can affect the frequency of a TO. It appears that in some circumstances this can lead to larger effects than can the mechanisms previously considered. We first note that most of the moment of inertia of the cell comes from the cylindrical outer wall. It follows that the effective torsion constant K_{eff} is determined not just by the torsion constant K_{rod} of the rod but also by the torsion constant K_{disk} of the disk forming the bottom of the cell. Since these are in series,

$$\frac{1}{K_{\mathrm{eff}}}=\frac{1}{K_{\mathrm{rod}}}+\frac{1}{K_{\mathrm{disk}}}. \tag{3}$$

But $K_{\mathrm{disk}} \gg K_{\mathrm{rod}}$, and so

$$K_{\mathrm{eff}} \approx K_{\mathrm{rod}}-\frac{K_{\mathrm{rod}}^2}{K_{\mathrm{disk}}}. \tag{4}$$

A change in K_{disk} will therefore cause a change $\delta\,\Omega$ in the oscillator frequency given by

$$\frac{\delta\,\Omega}{\Omega}=\frac{\delta K_{\mathrm{eff}}}{2K_{\mathrm{eff}}}=\frac{1}{2}\frac{K_{\mathrm{rod}}}{K_{\mathrm{rod}}+K_{\mathrm{disk}}}\frac{\delta K_{\mathrm{disk}}}{K_{\mathrm{disk}}}. \tag{5}$$

The solid helium inside the cell will provide an additional torsional stiffness to K_{disk} and so any change in the shear modulus will give a frequency shift.

To calculate K_{eff} we need to hold the cylindrical outer wall of the cell fixed, and calculate the torque needed to rotate the lower end of the torsion rod through a given angle. In principal, we could do this using a finite element simulation for the entire cell and helium. However, this calculation is a little delicate because we are looking for a very small effect coming from the elasticity of the helium. Let us first make a rough estimate of the magnitude of the effect that we are considering. As a first approximation, let us define the "disk" as the part of the bottom plate which extends from an inner radius of r_1 to an outer radius of r_3 (see Fig. 1). Define the "torsion rod" to include *all* of the material inside the radius r_1, so that the length of the rod is $L+w$; we discuss this in more detail later. The torsion constant of the disk is then

$$K_{\mathrm{disk}}=\frac{4\pi c_m w r_3^2 r_1^2}{r_3^2-r_1^2}, \tag{6}$$

whereas the constant for the rod is (ignoring the fill line)

$$K_{\mathrm{rod}}=\frac{\pi c_m r_1^4}{2(L+w)}. \tag{7}$$

Thus since normally $r_3 \gg r_1$ and $L \gg w$,

$$\frac{K_{\mathrm{rod}}}{K_{\mathrm{disk}}}\approx\frac{r_1^2}{8Lw}. \tag{8}$$

As a representative example, we choose the values in Fig. 1 to be $L=1$ cm, $r_1=0.1$ cm, $r_3=0.5$ cm, and $w=0.1$ cm. We will ignore the fill tube. If the torsion rod and cell are made of aluminum, the shear modulus is $c_m=3.1\times10^{11}$ cgs. With these values we find

$$K_{\mathrm{rod}}=4.43\times10^{7}\ \mathrm{cgs},\qquad K_{\mathrm{disk}}=4.06\times10^{9}\ \mathrm{cgs}, \tag{9}$$

so that K_{disk} is larger than K_{rod} by a factor of the order of 100. Note that the torsion constant of the disk, as defined here, is primarily determined by the inner radius r_1.

The delicate question now is to determine the effect of the helium stiffness on the effective torsion constant of the plate. The shear modulus of polycrystalline solid helium (when dislocations are not free to move) is 1.15×10^{8} cgs.[11] We could argue that we should add to $K_{\rm disk}$ the stiffness of a disk of helium of the same inner and outer radius. But how thick should this disk be? In fact as just noted the torsion constant of the disk is primarily determined by the inner radius; the outer radius is irrelevant. It therefore appears likely that the correct thickness of the helium disk must be of the order of r_1. Then the effective torsion constant of the disk is increased to the value

$$\tilde{K}_{\rm disk} = 4\pi c_m w r_1^2 + 4\pi c_{\rm He} r_1^3, \tag{10}$$

where again we are taking $r_3 \gg r_1$. Combining these results, we find that if the shear modulus of the helium changes by $\delta c_{\rm He}$, there is a frequency shift $\delta\,\Omega$ of the oscillator given by

$$\frac{\delta\,\Omega}{\Omega} \sim \frac{r_1^3}{16 L w^2}\frac{c_{\rm He}}{c_m}\frac{\delta c_{\rm He}}{c_{\rm He}}. \tag{11}$$

With the oscillator parameters given above, (11) gives for the change in frequency resulting from a change in the helium shear modulus, the result

$$\frac{\delta\,\Omega}{\Omega} \sim 2.3 \times 10^{-6}\frac{\delta c_{\rm He}}{c_{\rm He}}. \tag{12}$$

We now turn to a numerical calculation of the helium contribution to the effective stiffness. The equation of elasticity for purely torsional motion is

$$\rho\frac{\partial v_\phi}{\partial t} = \frac{\partial \sigma_{r\phi}}{\partial r} + \frac{\partial \sigma_{\phi z}}{\partial z} + \frac{2}{r}\sigma_{r\phi}, \tag{13}$$

where v_ϕ is the tangential velocity, the stresses $\sigma_{r\phi}$ and $\sigma_{\phi z}$ are related to the tangential displacement u_ϕ by

$$\sigma_{r\phi} = \frac{c}{2}\left(\frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\right), \qquad \sigma_{\phi z} = \frac{c}{2}\frac{\partial u_\phi}{\partial z}, \tag{14}$$

and c is the shear modulus at the point considered. Thus in static equilibrium

$$\frac{\partial^2 u_\phi}{\partial r^2} + \frac{1}{r}\frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r^2} + \frac{\partial^2 u_\phi}{\partial z^2} = 0. \tag{15}$$

In principle, the calculation of $K_{\rm eff}$ could be performed as a finite element or finite-difference calculation for the entire structure (rod, cell, and helium). However, this is a little challenging because we are looking for a very small effect due to the helium. We therefore begin by performing a simulation for just the rod and the disk. The lower end of the torsion rod (see Fig. 1) is rotated by a small angle and the outer radius of the disk is held fixed. A finite-difference relaxation method (FDRM) with 16 000 grid points is used to calculate the equilibrium configuration. The torsion constant was found to be 4.52×10^7 cgs. This is in reasonable agreement with the value of $K_{\rm eff}$ found from (3) using the values for $K_{\rm rod}$ and $K_{\rm disk}$ given in (9). In Fig. 2 we show results from the FDRM for the tangential displacement $u_\phi$and rotation angle ϕ of the top surface of the disk for a unit rotation angle of the lower end of the torsion rod. It can be seen that the center of the top surface of the disk rotates by an angle of 0.011 radians, i.e., by an angle which is smaller than the rotation of the bottom of the torsion rod by a factor of 90. This is in reasonable agreement with the estimate that $K_{\rm rod}/K_{\rm disk} \sim 100$ as given above.

To investigate more quantitatively the additional torsional stiffness provided by the helium we consider the displacement of the helium in the cell that results when the bottom surface of the cell (top surface of the disk) has the tangential displacement u_ϕ as just calculated from the FDRM program.[12] The helium at the outer walls of the cell volume is held fixed. In Fig. 3 we show the results of this calculation as a contour plot of u_ϕ. From the result we can find the torque that the helium exerts on the top surface of the disk. As a test of the calculation, we have verified that the torque exerted on the helium by the top of the disk is equal to torque by the helium on the outer wall and top wall of the cell. The result for this torque is $\tau = 3.0 \times 10^4$ cgs. This is for a rotation of the *bottom* of the torsion rod by 1 radian. Since the top surface of the disk (at the center) rotates by 0.011 radians this means that the torsion constant of the helium is

$$K_{\rm He} = 3.0 \times 10^4/0.011 = 2.7 \times 10^6 \text{ cgs}. \tag{16}$$

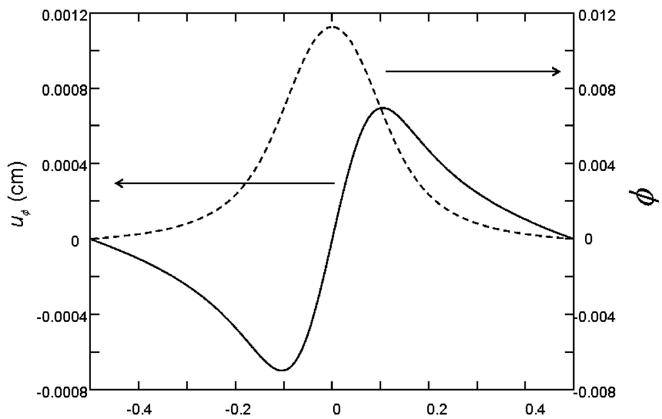

FIG. 2. The tangential displacement u_ϕ and the angular rotation ϕ of the top surface of the disk as a function of the distance r from the axis. The lower end of the torsion rod has been rotated by a unit angle. These results were obtained using the finite-difference relaxation method.

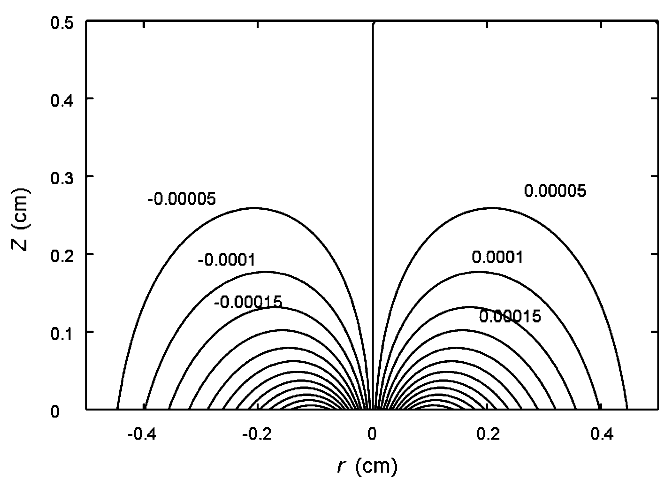

FIG. 3. Contour plot of the tangential displacement u_ϕ in solid helium resulting from the displacement of the top surface of the disk as shown in Fig. 2. In the right-hand part of the figure the contours are equally spaced between 0.000 05 and 0.0006 cm, and in the left-hand part of the figure between $-$0.0006 and 0.000 05 cm. The magnitude of the displacements is based on a rotation of the lower end of the torsion rod of 1 radian.

Since the disk and helium are springs connected in parallel, we have that $\delta K_{\rm disk} = 2.7 \times 10^6$. We can now use this value in Eq. (5) along with the values $K_{\rm rod} = 4.43 \times 10^7$ cgs and $K_{\rm disk} = 4.06 \times 10^9$ cgs, to obtain the final result

$$\frac{\delta \Omega}{\Omega} \sim 3.6 \times 10^{-6} \frac{\delta c_{\rm He}}{c_{\rm He}}, \tag{17}$$

i.e., about a factor of 1.5 larger than the order of magnitude estimate in Eq. (12). This frequency shift is comparable to the frequency shift seen in many experiments and attributed to NCRI.[9]

IV. SUMMARY

We have not attempted to make quantitative estimates of the magnitude of this effect for the many different torsional oscillators which have been used in different laboratories. The construction details of most of these TOs are not available. We can, however, make some general comments on how the magnitude of this effect will vary according to the design of the cell.

(1) The effect described here will decrease rapidly as the thickness of the bottom disk of the cell increases. Increase of the thickness of the disk makes $K_{\rm disk}$ larger and decreases $\delta \Omega$ (see Eq. (5)). In addition, if the radius of the torsion rod is small compared to the plate thickness, the top surface of the plate will undergo pure rotation without distortion.

(2) If the cell contains porous Vycor glass (or other porous material), the effect considered here continues to be important. A change in the shear modulus of the helium in the Vycor will cause a change in the shear modulus $c_{\rm V-He}$ of the "Vycor plus helium." The change in $c_{\rm V-He}$ should be of the order of the porosity p times the change in $c_{\rm He}$.[13] The finite porosity of the Vycor makes the frequency shift $\delta \Omega$ due to a change in the helium shear modulus smaller. However, if this change in frequency is attributed to the NCRI of the helium, the NCRI fraction will be comparable to the value that would be obtained in a helium cell without Vycor. Of course, the temperature variation of the shear modulus of helium within Vycor may be quite different from the temperature variation for bulk solid helium.

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