Journal of Low Temperature Physics, Vol. 137, Nos. 3/4, November 2004 (© 2004)

The Shape of Electron Bubbles in Liquid Helium and the Line Width of Optical Transitions

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We present a calculation of the effect of zero-point and thermal fluctuations on the shape of electron bubbles in liquid helium. The results are used to determine the line shape for the 1S to 1P optical transition. The calculated line shape is in very good agreement with the experimental measurements of Grimes and Adams (Phys. Rev. **B45**, *2305* (1992)).

PACS numbers: 67.40.Yv, 43.35.+d.

1. INTRODUCTION

When an electron is injected into liquid helium, it forces open a cavity in the liquid, the so-called "electron bubble"[1]. The energy of this cavity is given by the approximate expression

$$E = \frac{h^2}{8mR^2} + 4\pi R^2\alpha + \frac{4\pi}{3}R^3P \tag{1}$$

where m is the mass of the electron, R is the radius of the cavity (assumed for the moment to be spherical), α is the surface tension, and P is the pressure. At zero applied pressure, the energy is a minimum when the radius R is approximately 19 Å. In writing Eq. 1, it has been assumed that the penetration of the electron wave function into the liquid is small, and that the finite width of the liquid-vapor interface does not have a significant effect on the bubble energy[2]. In addition, the polarization of the liquid by the electric field of the electron has been ignored[2,3]. Investigations of electron bubbles include mobility measurements[1], experiments in which light is absorbed and the electron raised to a higher energy state[4–8], and ultrasonic cavitation studies[2,9]. The results of these experiments are in good agreement with predictions based on Eq. 1. For example, once the radius R that minimizes the energy of the bubble has been determined from Eq. 1, it is

0022-2291/04/1100-0491/0

possible to calculate the energy of the photons that can cause a transition between different quantum states of the electron in the bubble. The results of this calculation give good agreement with experiment[6,7]. However, a number of intriguing mysteries remain. For example, time-of flight mobility measurements[10,11] have revealed the existence of a family of electron bubbles (at least 12 different species!) that have significantly higher mobility than the normal electron bubble. From the measured mobility, one can deduce that the radius of these bubbles lies between the radius of the normal bubble and about one half of this radius. So far it has not been possible[12] to provide an explanation of the physical nature of these objects. A second, and possibly related, mystery concerns the effect of light on the mobility of the bubbles[4–6]. It has been found that when the liquid is illuminated with light of the correct wavelength to excite the electron in a bubble from the ground state to an excited state, the mobility increases. It is possible that this comes about because the bubble is initially attached to a vortex and is able to escape after optical excitation. However, there are a number of difficulties with this explanation[13].

In this paper, we consider in some detail the fluctuations in shape of these bubbles. A bubble has a spectrum of normal modes for shape oscillations. The amplitude of these modes is non-zero due to both zero-point and thermal fluctuations. We apply the results to calculate the shape of the optical absorption line for the 1S→1P optical transition. The details of this transition has been studied in beautiful experiments performed by Grimes and Adams[7] and by Parshin and Pereversev[8]. At zero pressure the line width found by Grimes and Adams at 1.3 K is approximately 0.02 eV. The theory of the line shape has been discussed previously by Fomin[14], Fowler and Dexter[15], and Lerner *et al*[16]. Fomin[14] gave only an order of magnitude estimate of the line width, specifically 10^{12} s^{-1} (=0.0007 eV), which is about 30 times smaller than the experimental result. Fowler and Dexter[15] gave a more careful analysis and obtained the result 0.013 eV. Lerner *et al*[16] obtained agreement with the experimental results of Grimes and Adams but have included in their calculation a substantial contribution to the line width (roughly 50%) arising from "homogeneous broadening", presumably meaning broadening due to the finite lifetime of the 1P state. We believe that in fact the contribution to the line width from this mechanism is negligible.

The final results of our calculations are in very good agreement with the experimental results of Grimes and Adams[7].

2. SHAPE FLUCTUATIONS

Before considering the shape fluctuations of an electron bubble, it is of interest to consider the fluctuations of a free surface. The long wavelength fluctuations in the surface height can be considered to arise from capillary waves. At low temperatures ($T \leq 1$K), the capillary waves make the main contribution to the entropy of the surface and hence determine the temperature-dependence of the surface free energy[17,18]. If the liquid is taken to be incompressible, the frequency ω_k of a capillary wave with wave number k is given by

$$\omega_k = (\alpha k^3/\rho)^{1/2} \tag{2}$$

At higher temperatures, the dispersion relation becomes more complicated because it is necessary to take the compressibility of the liquid into account[19], and because the wavelength becomes comparable to the interatomic spacing. Consider an area of the surface of the liquid that is a square of side w. Write the vertical displacement of the surface δz as

$$\delta z = \sum_{k_x,k_y} h_k cos[k_x x + k_y y + \phi(k_x, k_y)], \tag{3}$$

where $\phi(k_x, k_y)$ is a random phase. Applying periodic boundary conditions, we have $k_x = 2\pi n_x/w$ and $k_y = 2\pi n_y/w$, where n_x and n_y are positive or negative integers. At temperature T, the average energy associated with the mode k_x, k_y is

$$E_k = \frac{1}{2}\hbar\omega_k + \frac{\hbar\omega_k}{\exp(\hbar\omega_k/k_B T) - 1} \tag{4}$$

One half of the energy of the mode comes from the kinetic energy in the liquid and the other half from the increase of the energy of the distorted surface. The average value of h_k^2 is found to be

$$< h_k^2 >= \frac{2}{\alpha w^2 k^2}\left[\frac{1}{2}\hbar\omega_k + \frac{\hbar\omega_k}{\exp(\hbar\omega_k/k_B T) - 1}\right] \tag{5}$$

The total squared fluctuation of the height is then

$$< (\delta z)^2 >= \sum_{k_x,k_y} < h_k^2 > \tag{6}$$

We would like to use these equations to get an idea of the topology of the surface at different temperatures, but this cannot be done in a rigorous way. One difficulty is that the zero-point energy makes the dominant contribution to the sum $< (\delta z)^2 >$ for large wave numbers. The contributions

to the sum from fluctuations in the range k, $k + dk$ vary as $k^{1/2}dk$ for large k. This divergence is an artifact of the model we are using. In the first place, there must be an effective upper limit on k since the wavelength of the surface vibrations cannot be less than the interatomic spacing. Secondly, as already mentioned the relation between ω_k and k that we are using is only valid for sufficiently long wavelengths. The temperature dependent part of $< h_k^2 >$, on the other hand results in a divergence at small k if an infinite area of surface is considered. Recognizing these limitations, in order to give some sense of the topology of the surface at different temperatures, we have performed a computer simulation of the surface profile at different temperatures with the zero-point fluctuations neglected. Results are shown in Fig. 1 as contour plots. An area of dimensions 50 Å by 50 Å was considered and periodic boundary conditions applied as described above. The phase of each mode was taken to be random. The amplitude h_k was chosen as a random variable with a Gaussian distribution and average square as given by Eq. 5 with the zero-point term removed[20]. The surface tension was taken as the experimentally measured value at each temperature[21]. Since we have only used the dispersion relation for capillary waves on the surface of an incompressible liquid to calculate the fluctuations in surface height, the results shown in Fig. 1 do not accurately represent the fluctuations at very short wavelengths.

Since the same random number generator was used for each temperature, the phases and the value of $h_k/< h_k^2 >^{1/2}$ is the same for each plot. Thus the *pattern* of the fluctuations is identical for each temperature, but the amplitude increases as the temperature goes up.

We now perform the same type of calculation for the surface of an electron bubble. We consider normal modes with angular dependence of the form

$$U_{lm} \equiv (Y_{lm} + Y_{lm}^*)/\sqrt{2} = \sqrt{\frac{2l+1}{2\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta)\cos(m\phi) \qquad (7)$$

and

$$\tilde{U}_{lm} \equiv -i(Y_{lm} - Y_{lm}^*)/\sqrt{2} = \sqrt{\frac{2l+1}{2\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta)\cos(m\phi) \qquad (8)$$

where $m > 0$. For $m = 0$, we can take the mode pattern to vary as Y_{l0}. Thus, if R_0 is the radius of the bubble in the absence of fluctuations, the effect of fluctuations is to make the distance from the origin to the surface in the direction θ, ϕ become

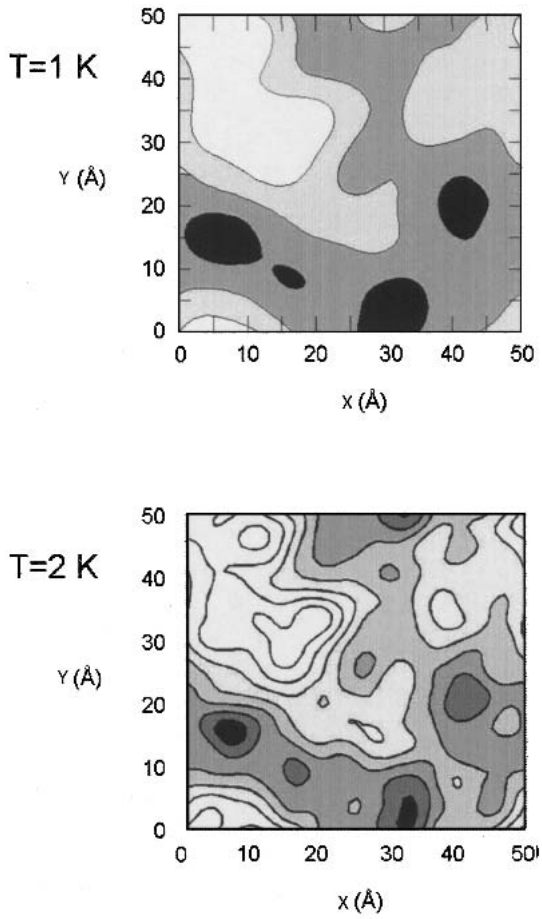

Fig. 1. Fluctuations in height of a 50 Å by 50 Å of the free surface of liquid helium at 1 and 2 K. Each contour line represents a height difference of 1 Å.

$$R(\theta,\phi) = R_0 + \sum_l f_l Y_{l0} + \sum_{l,m} f_{lm} U_{lm} + \sum_{l,m} \tilde{f}_{lm} \tilde{U}_{lm} \tag{9}$$

where f_l , f_{lm} and $\tilde{f}_{lm}$ are amplitudes to be determined.

The frequency of a mode depends only on l. To calculate ω_l we fol-

low the method of Gross and Tung-Li[22], in which the liquid is treated as incompressible[23]. We generalize their calculation to allow for a non-zero pressure in the liquid. The calculation is straightforward and the results are most conveniently expressed in terms of the equilibrium radius $R_0(P)$, which can be found from the equation

$$-\frac{\pi^2\hbar^2}{mR_0^3} + 8\pi R_0\alpha + 4\pi R_0^2 P = 0 \tag{10}$$

Then

$$\omega_0^2 = \frac{(5\pi\hbar^2/4mR_0^2) - 2\alpha R_0^2}{\rho R_0^5} \tag{11}$$

and for $l \neq 0$,

$$\omega_l^2 = (l+1)\frac{\alpha(l^2+l-2) + (\pi\hbar^2/2mR_0^4)(1+\pi S_l)}{\rho R_0^3} \tag{12}$$

In this equation

$$S_l = \frac{j_l'}{j_l} - \frac{j_0''}{2j_0'} \tag{13}$$

where j_l denotes the spherical Bessel function, and the derivatives of this function are evaluated when the argument is π. The mode $l = 1$ corresponds to a translation of the bubble without deformation. For this mode $S_l = -1/\pi$ and so $\omega_1 = 0$ as expected. At zero pressure, the above equations simplify to give

$$\omega_0^2 = \frac{8\alpha}{\rho R_0^3} \tag{14}$$

and for $l \neq 0$,

$$\omega_l^2 = (l+1)\frac{\alpha(l^2+l+2+4\pi S_l)}{\rho R_0^3} \tag{15}$$

Thus[24]

$$\omega_2 = 2\pi\sqrt{\frac{\alpha}{\rho R_0^3}} \tag{16}$$

$$\omega_3 = \sqrt{8\frac{7\pi^2-60}{15-\pi^2}+36}\ \sqrt{\frac{\alpha}{\rho R_0^3}} = 7.08\sqrt{\frac{\alpha}{\rho R_0^3}} \tag{17}$$

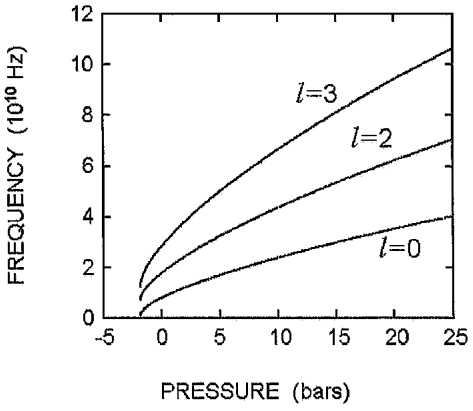

Fig. 2. The frequencies of the normal modes of the bubble with $l = 0, 2$ and 3 as a function of pressure. It is assumed that the liquid is incompressible.

Lerner *et al*[16] have used an approximate method to arrive at the formula[25] $\omega_2 = (72\alpha/\rho R_0^3)^{1/2}$. It can be seen that this overestimates ω_2 by a factor of 1.35. The frequencies of the lowest modes are plotted as a function of pressure in Fig. 2. Note that when the pressure is reduced below the critical pressure P_c given by

$$P_c = -\frac{16}{5}\left(\frac{m\alpha^5}{10\pi\hbar^2}\right)^{1/4}, \tag{18}$$

the bubble becomes unstable and grows without limit. At this pressure the frequency of the $l = 0$ mode goes to zero. The frequencies of the modes with $l \geq 2$ decrease rapidly as the pressure approaches P_c , but remain finite at P_c.

Proceeding as for the free surface, we find

$$< {f_0}^2 >= \left(\frac{5\pi\hbar^2}{4mR_0^4} - 2\alpha\right)^{-1}\left[\frac{1}{2}\hbar\omega_0 + \frac{\hbar\omega_0}{\exp(\hbar\omega_0/k_{\mathrm{B}}T) - 1}\right], \tag{19}$$

and for $l \neq 0$

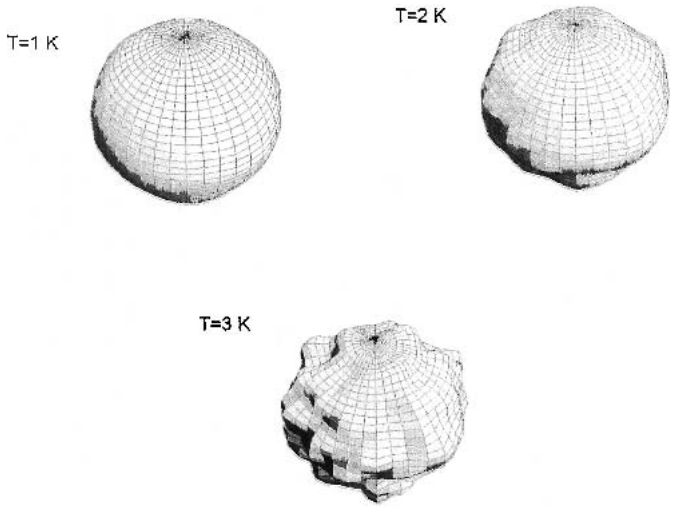

Fig. 3. Shapes of electron bubbles at temperatures of 1, 2 and 3 K. These results are for $P = 0$.

$$< f_l{}^2 >=< f_{lm}{}^2 >=< \tilde{f}_{lm}^2 >$$
$$= \left[\frac{\hbar\omega_l}{\exp(\hbar\omega_l/k_BT)-1} + \frac{1}{2}\hbar\omega_l\right] / \left[\alpha(l^2+l-2) + \frac{\pi\hbar^2}{2mR_0^4}(1+\pi S_l)\right] \tag{20}$$

As for the planar surface, the zero-point contribution gives a divergence at short wavelengths, while the divergence at long wavelengths no longer occurs.

To illustrate the effect of fluctuations on the bubble shape we show in Fig. 3 the results of calculations of bubble shapes at different temperatures. As for the free surface, the contribution from zero-point fluctuations has been dropped. We take the mode amplitude to have a Gaussian distribution with average square as given by the above formulas, and choose the phase of each mode to be random. The same random number generator was used for each temperature, and the calculations use the experimentally measured values of the surface tension.

3. CALCULATION OF THE LINESHAPE

Consider first the situation when the electron is in the ground state and the bubble is spherical with the radius R_0 corresponding to the minimum energy from Eq.10. The electron energy is then

$$E_{1S} = \frac{\pi^2\hbar^2}{2mR_0^2} \tag{21}$$

According to the Franck-Condon principle, when light is absorbed the size and shape of the bubble does not change until *after* the state of the electron has changed. The energy of the electron in the first excited state inside the same bubble is

$$E_{1P} = \frac{\lambda_{1P}^2\hbar^2}{2mR_0^2} \tag{22}$$

where $\lambda_{1P} = 4.493409$ is the argument at which the spherical Bessel function j_1 has its first zero. Thus, the photon energy required to excite from 1S to 1P is[26]

$$E_{1S\rightarrow 1P} = \frac{\hbar^2}{2mR_0^2}(\lambda_{1P}^2 - \pi^2) \tag{23}$$

and so the cross-section for optical absorption should exhibit a delta-function peak at the energy $E_{1S\rightarrow 1P}$.

We now consider the effect of the shape fluctuations on the line shape for this transition. A rigorous calculation of this involves the following steps. In the initial state, the electron is in the 1S state and the state of the bubble can be described by a set of quantum numbers specifying the excitation of each of the normal modes for shape oscillation. The probability of a particular initial vibrational state is proportional to $\exp(-E/k_BT)$, where E is the energy of the state. In each of these states the energy of the electron is slightly different. Optical absorption can occur from any one of these states to a state in which the electron wave function has changed to 1P and the bubble is in one of many different possible vibrational states. The line shape is then calculated by adding up the probabilities of transition from each possible initial state to each possible final state.

In order for the matrix element for the transition to be large, there has to be an appreciable overlap of the wave function of the bubble in the initial and final states. It is straightforward to show that, as a consequence, the vibrational quantum numbers for the bubble in the final state must be very large, specifically greater than 1000. When this condition holds, the calculation of the line shape can be greatly simplified[27–29] since the final

vibrational state can be treated classically. The initial vibrational state still needs to be treated quantum mechanically, but the correct result for the line shape can be obtained by the following method. Consider a particular vibrational mode i with displacement q, frequency ω, and spring constant K. At a finite temperature, the probability of the displacement of the mode being in the range between q and $q + dq$ is $P(q)dq$ where

$$P(q) = \sum_{n=0}^{\infty} \Psi_n^2(q) \exp\left[-(n + 1/2)\hbar\omega/k_B T\right] / \sum_{n=0}^{\infty} \exp\left[-(n + 1/2)\hbar\omega/k_B T\right] \tag{24}$$

where $\Psi_n(q)$ are the eigenfunctions for the harmonic oscillator. It can be shown[28] that Eq. 24 can be simplified to give the expression

$$P(q) = (K/2\pi k_B T_{\text{eff}})^{1/2} \exp\left(-Kq^2/2k_B T_{\text{eff}}\right) \tag{25}$$

where

$$T_{\text{eff}} = \theta / \tanh(\theta/T) \tag{26}$$

and $\theta = \hbar\omega/2k_B$. Thus $P(q)$ is the same as the distribution of displacements of a classical oscillator with an effective temperature T_{eff}. For each vibrational mode we can calculate the probability of a particular displacement. We then find the shift in the 1S and 1P energy levels due to these displacements. Averaging over all possible displacements of the modes then gives the line shape.

To lowest order, the only contribution to the shift of the 1S state arises from shape fluctuations with $l = 0$ symmetry. The shift is

$$-\frac{E_{1S} f_0}{\sqrt{\pi} R_0} \tag{27}$$

The $l = 0$ fluctuations give the same fractional shift for the 1P state as for the 1S state. Hence, the shift in the transition energy due to this fluctuation is

$$-\frac{(E_{1P} - E_{1S}) f_0}{\sqrt{\pi} R_0} \tag{28}$$

It is clear that the $l = 0$ fluctuations alone would result in a Gaussian line shape.

The only other contribution to the shift of the 1P state comes from fluctuations with $l = 2$. These fluctuations do not lead to any shift in the 1S state in lowest order, but they remove the 3-fold degeneracy of the 1P level. Lerner *et al*[16] have treated the effect of these fluctuations on the 1P

states in an approximate way by assuming that the main contribution to the shift in the energy levels comes from axially-symmetric vibrations. Here we treat the problem without approximation using the method originally developed by Rayleigh[30] for finding frequencies of acoustic modes in objects that have a shape close to a circular form. Suppose that at some instant the displacements due to the 5 vibrational modes with $l = 2$ are such that

$$R(\theta, \phi) = R_0 + \sum_{m=-2}^{2} \varepsilon_{2m} Y_{2m}(\theta, \phi) \tag{29}$$

where

$$\begin{aligned} \varepsilon_{20} &= \frac{f_{20}}{R_0} \\ \varepsilon_{21} &= \frac{f_{21} - i\tilde{f}_{21}}{\sqrt{2}R_0}, \quad \varepsilon_{2\bar{1}} = \frac{-f_{21} - i\tilde{f}_{21}}{\sqrt{2}R_0} \\ \varepsilon_{22} &= \frac{f_{22} - i\tilde{f}_{22}}{\sqrt{2}R_0}, \quad \varepsilon_{2\bar{2}} = \frac{f_{22} + i\tilde{f}_{22}}{\sqrt{2}R_0} \end{aligned} \tag{30}$$

Let the shift in the energy level of the 1P state be $\triangle E_{1P}$ and set

$$x \equiv \triangle E_{1P}/E_{1P} \tag{31}$$

Then from Rayleigh's method we find that the values of are the solutions of the eigenvalue equation:

$$\begin{vmatrix} \sqrt{\frac{1}{5\pi}}\varepsilon_{20} & -\sqrt{\frac{3}{5\pi}}\varepsilon_{21} & \sqrt{\frac{6}{5\pi}}\varepsilon_{22} \\ -\sqrt{\frac{3}{5\pi}}\varepsilon_{2\bar{1}} & -\sqrt{\frac{4}{5\pi}}\varepsilon_{20} & -\sqrt{\frac{3}{5\pi}}\varepsilon_{21} \\ \sqrt{\frac{6}{5\pi}}\varepsilon_{2\bar{2}} & -\sqrt{\frac{3}{5\pi}}\varepsilon_{2\bar{1}} & \sqrt{\frac{1}{5\pi}}\varepsilon_{20} \end{vmatrix} \begin{vmatrix} A_{11} \\ A_{10} \\ A_{1\bar{1}} \end{vmatrix} = x \begin{vmatrix} A_{11} \\ A_{10} \\ A_{1\bar{1}} \end{vmatrix} \tag{32}$$

In this equation A_{11}, A_{10} and $A_{1\bar{1}}$ are the amplitudes of the Y_{11}, Y_{10} and $Y_{1\bar{1}}$ components of the wave function. Given these results, the calculation of the line shape and width is a straightforward numerical problem. Results were obtained by considering 10^6 configurations of a bubble, each configuration corresponding to particular values of the six parameters $f_0, f_2, f_{22}, f_{21}, \tilde{f}_{22}, \tilde{f}_{21}$ chosen from a Gaussian distribution with appropriate rms value. The calculation of the line shape includes a weighting factor to allow for the different values of the matrix element between the 1S state and

the three 1P states. Thus, for example, if the light is taken to be polarized along the z-axis, the weighting factor is A_{10}^2.

It is important to note that the only contributions to the line shape come from the shape fluctuations with $l = 0$ and $l = 2$. For these modes, it is a good approximation to treat the liquid as incompressible, to use Eqs. 11 and 12 for the mode frequency, and Eqs. 19 and 20 for the mode amplitudes. Thus, the calculation of the line shape is much more reliable than the calculation of the bubble shape.

The calculated line shape is shown in Fig. 4. along with the experimental data[31] of Grimes and Adams at pressures of 0, 12 and 24.4 atmospheres and 1.3 K. The experimentally-measured surface tension at 1.3 K was used (0.361 erg cm^{-2}). The position of the center of the line and the magnitude of the absorption have been adjusted so as to coincide with the experimental data. Otherwise there are no adjustable parameters. It can be seen that the agreement with experiment is very good. Surprisingly, the theoretical line shape is not Gaussian. The fluctuations with $l = 0$ clearly give a Gaussian line shape; the non-Gaussian shape arises entirely from the fluctuations with $l = 2$.

The theory also predicts a small shift in the center of the line with temperature. One can readily understand the contribution of $l = 0$ fluctuations to this shift. A fluctuation in which the radius of the bubble increases by δR produces a fractional shift in the photon energy of

$$\frac{\delta E_+}{E} = \frac{R_0^2}{(R_0 + \delta R)^2} - 1 \tag{33}$$

while a decrease in radius by the same amount gives

$$\frac{\delta E_-}{E} = \frac{R_0^2}{(R_0 - \delta R)^2} - 1 \tag{34}$$

The average of the shifts due to fluctuations of this magnitude is therefore

$$\frac{\delta E}{E} = \frac{1}{2}\left(\frac{\delta E_+}{E} + \frac{\delta E_-}{E}\right) = 3\left(\frac{\delta R}{R_0}\right)^2 \tag{35}$$

Given the success of the theory in fitting the experimental data, it is of interest to make calculations to predict the line shape and optical absorption at other temperatures and pressures. It is straightforward to show that for the present model of the electron bubble in which the electron wave function does not penetrate into the helium, the integral of the absorption cross-section for the 1S→1P transition has the value 1.06×10^{-16} cm^2 eV, independent of the pressure and the temperature[32]. As a result, once the shape of the absorption line is known, the absorption cross-section can easily

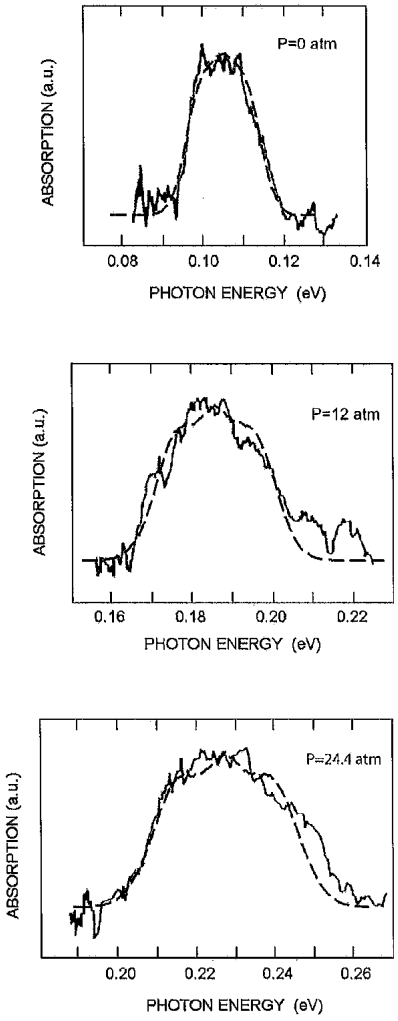

Fig. 4. The shape of the absorption line for the 1S→1P optical transition at pressures of $P = 0$, 12 and 24.4 atmospheres. The solid line is the result of the experiment of Grimes and Adams and the dashed line shows the theory. The details of the theory are described in the text.

be found. Results at a series of pressures for temperatures of 0, 1 and 2 K are shown in Fig. 5. Since as $P \rightarrow P_c$ the frequency of the $l = 0$ mode goes to zero, we would expect the width of the absorption line to become very large in this limit. However, the numerical calculations indicate that this occurs only close to P_c, and that the width at -1 bars is less than it is at $P = 0$. In the region with P very close to P_c the $l = 0$ mode should become anharmonic.

4. DISCUSSION

We note that the calculation we have performed neglects three possible contributions to the line width. The first of these is the contribution from the finite lifetime of the 1P state. The calculated lifetime due to spontaneous emission has been calculated to be around 44 μs when the pressure is zero. This corresponds to a width $\hbar/\tau$ of 1.5×10^{-11} cm^2 cV which is totally negligible compared to the width that is measured experimentally. Experimental measurements of the lifetime of the 1P state[33] indicate that the actual lifetime is considerably less than the calculated lifetime due to spontaneous emission, thus indicating the importance of some sort of non-radiative decay process. However, the contribution to the line width is still negligible.[34]

The second possible contribution comes from the rotons and phonons in the liquid. We have implicitly assumed that the fluctuations in shape of an electron bubble arise only from the normal modes of the bubble, and that these modes are uncoupled to the phonons and rotons in the liquid. The shape of the bubble should also fluctuate when phonons or rotons collide with the bubble. We have not attempted to calculate the magnitude of this effect. The third contribution comes about from the possible effect of vortices. Grimes and Adams considered it likely that the electron bubbles that they studied were attached to quantized vortices. This should cause a splitting of the 1P levels, into a singlet and a doublet. However, they estimate that this splitting was less that 2% and so its contribution to the line shape should be small compared to the effects of shape oscillations.

ACKNOWLEDGMENTS

It is a great pleasure to be able to contribute to this special issue of the Journal of Low Temperature Physics recognizing the work of Professor Siegfried Hunklinger. We thank D.O. Edwards, A. Ghosh and F. Yang for helpful discussions. This work was supported in part by the National Science

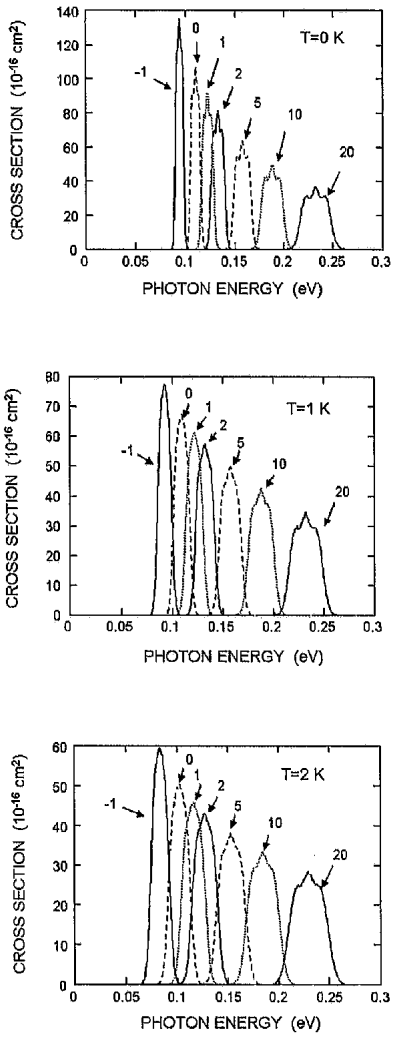

Fig. 5. Calculated shape of the absorption cross-section for the 1S→1P optical transition as a function of pressure at temperatures of 0, 1 and 2 K. The different curves are labeled by the pressure in bars.

Foundation through grant DMR-0305115.

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23. The effect of a finite compressibility of the liquid has been investigated by V. Celli, M.H. Cohen and M.J. Zuckerman, Phys. Rev. **173**, 253 (1968). When the liquid is compressible the modes of the bubble are damped because sound is radiated into the liquid. At zero pressure, the ratio of the imaginary part of the frequency to the real part is found to be 0.28 for the $l = 0$ mode and 0.027 for $l = 2$.
24. The frequency of ω_3 is given incorrectly in ref. 22.
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