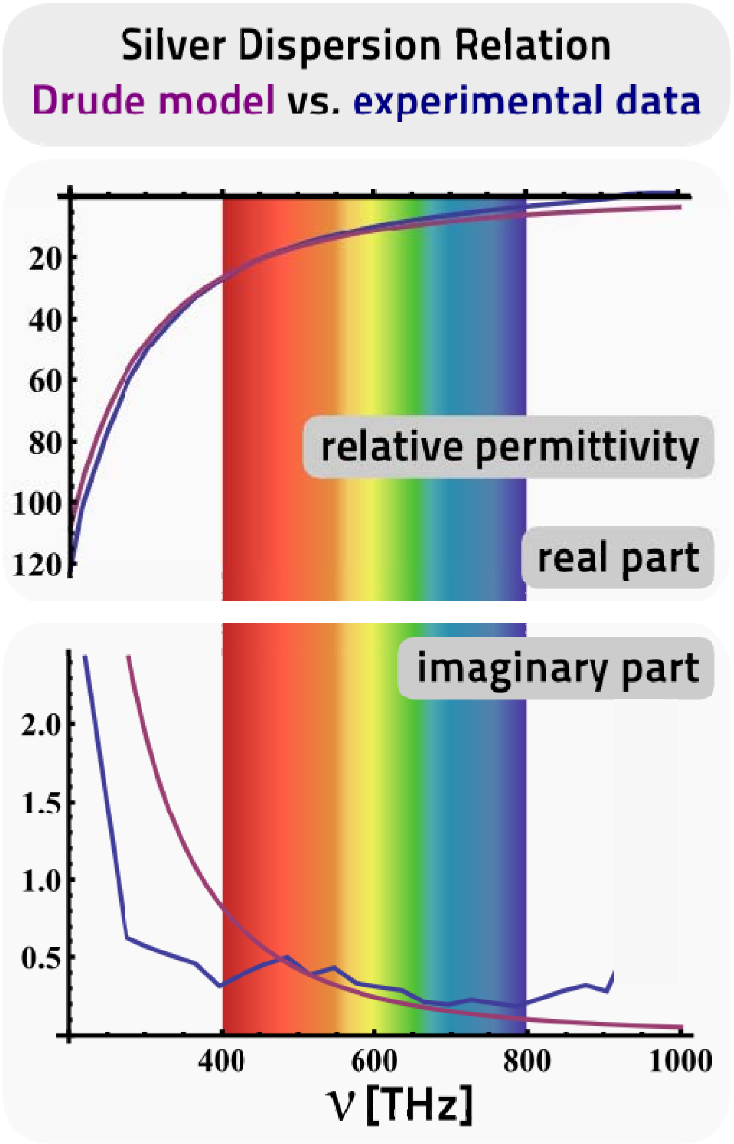
9 & 10. Light-matter interactions - metals

Drude model and the wave equation

Complex dielectric function and complex refractive index

Optical properties of metals
- skin depth
-reflectivity

Plasma frequency of a metal



Drude model and the wave equation

When we are considering metals, there is no "polarization," but instead there is current – flowing electrons!

So what goes here?

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{{c_0}^2}\frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P}{\partial t^2}$$

We make a guess that $P = N \cdot e \cdot x(t)$
can be related to $J = N \cdot e \cdot v(t)$

and we use Drude's result which is equivalent to Ohm's Law: $J(t) = \sigma_0 E(t)$

Wave equation for an E-field in a metal

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{{c_0}^2}\frac{\partial^2 E}{\partial t^2} = \mu_0 \sigma_0 \frac{\partial E}{\partial t}$$

This wave equation is similar to what we encountered previously. So we solve it in a similar way.

Assume that E has a time dependence of $e^{-j\omega t}$, and that the z dependence is a function $E_0(z)$ multiplied by e^{jkz}:

$$E(z,t) = E_0(z)\, e^{j(kz-\omega t)}$$

Plug this into the above equation, and take the necessary t and z derivatives. The resulting equation is:

$$\frac{\partial^2 E_0(z)}{\partial z^2} = -\frac{\omega^2}{c_0^2}\left(1 + j\frac{\sigma_0}{\varepsilon_0 \omega}\right) E_0(z)$$

Optical properties of metals

$$\frac{\partial^2 E_0(z)}{\partial z^2} = -\frac{\omega^2}{c_0^2}\left(1 + j\frac{\sigma_0}{\varepsilon_0 \omega}\right) E_0(z)$$

This is the same as the usual wave equation in empty space, except for the factor in the parentheses.

Thus, the solution is the same as the wave in empty space if we let the speed of light be modified:

modified speed: $c = \dfrac{c_0}{\sqrt{1 + j\dfrac{\sigma_0}{\varepsilon_0 \omega}}}$

This looks just like $c = \dfrac{c_0}{\underset{\sim}{n}}$. So we interpret the denominator as the (complex) refractive index.

complex index: $\underset{\sim}{n} = \sqrt{\dfrac{\varepsilon}{\varepsilon_0}} = \sqrt{1 + j\dfrac{\sigma_0}{\varepsilon_0 \omega}}$

Wait a sec… complex refractive index??

Complex refractive index

Go back for a minute to the expression for waves in a dielectric medium (lecture 7):

$$E(z,t) = E_0(z=0)\cdot e^{-\alpha z/2}\cdot e^{jnk_0 z}\cdot e^{-j\omega t}$$

k_0 = k-vector in vacuum

$$= E_0(z=0)\cdot e^{j\boxed{\left[\frac{j\alpha}{2k_0}+n\right]}k_0 z}\cdot e^{-j\omega t}$$

Define the complex refractive index:

$$\underset{\sim}{n} \equiv n + \frac{j\alpha}{2k_0}$$

Note the tilde!

$$= n + j\boxed{\frac{\alpha\lambda_0}{4\pi}}$$

$$= n + j\boxed{\frac{\alpha c_0}{2\omega}}$$

Often denoted with the Greek letter kappa, κ, a dimensionless quantity!
Note: $\kappa \geq 0$ always!

So $\underset{\sim}{n} \equiv n + j\kappa$

So a complex refractive index is just a convenient way of grouping n and α together into one quantity.

Complex dielectric function

If the refractive index is complex, what about ε?

$$\varepsilon_{med} = \varepsilon_0 \underset{\sim}{n}^2$$

$$= \varepsilon_0 \left(n + j\kappa\right)^2$$

$$\varepsilon_R + j\varepsilon_I = \varepsilon_0 \left(n^2 - \kappa^2 + j2n\kappa\right)$$

Note: this sign convention is not universal - be careful!

This can also be inverted to give equations for n,κ :

$$n = \sqrt{\frac{1}{2\varepsilon_0}\left(|\varepsilon| + \varepsilon_R\right)}$$

$$\kappa = \sqrt{\frac{1}{2\varepsilon_0}\left(|\varepsilon| - \varepsilon_R\right)}$$

Optical properties of metals

For metals we found: $\underset{\sim}{n} = \sqrt{\dfrac{\varepsilon}{\varepsilon_0}} = \sqrt{1 + j\boxed{\dfrac{\sigma_0}{\varepsilon_0 \omega}}}$ what is this value?

From this we can find the values of n and α:

$$n = \mathrm{Re}\left(\underset{\sim}{n}\right) \qquad \alpha = \frac{4\pi}{\lambda_0}\mathrm{Im}\left(\underset{\sim}{n}\right)$$

which together tell us how waves propagate inside a metal:

$$E\left(z,t\right) = E_0^{\ z=0} \cdot e^{j\left[\frac{j\alpha}{2k_0}+n\right]k_0 z} \cdot e^{-j\omega t}$$

$\underset{\sim}{k} = \underset{\sim}{n} k_0$

$$= E_0^{\ z=0} \cdot e^{j\underset{\sim}{n}k_0 z} \cdot e^{-j\omega t} = E_0^{\ z=0} \cdot e^{j\underset{\sim}{k}z} \cdot e^{-j\omega t}$$

For a k-vector with a real and an imaginary part, the imaginary part gives rise to attenuation.

Absorption coefficient is: $\alpha = 2\,\mathrm{Im}\left(\underset{\sim}{k}\right) = 2k_0\,\mathrm{Im}\left(\underset{\sim}{n}\right)$

For real metals, $\sigma_0/\varepsilon_0\omega$ is a big number, especially at lower frequencies

Example: Consider copper at a frequency of 100 MHz:

$$\frac{\sigma_0}{\varepsilon_0\omega} \approx 10^{10}$$

In that case, the complex refractive index is given by:

$$\underset{\sim}{n} = \sqrt{1 + j\frac{\sigma_0}{\varepsilon_0\omega}} \approx \sqrt{j\frac{\sigma_0}{\varepsilon_0\omega}} = e^{j\pi/4}\sqrt{\frac{\sigma_0}{\varepsilon_0\omega}} = \frac{(1+j)}{\sqrt{2}}\sqrt{\frac{\sigma_0}{\varepsilon_0\omega}}$$

in which case, the real and imaginary parts are equal:

$$n = \kappa = \sqrt{\frac{\sigma_0}{2\varepsilon_0\omega}} = 7.32\times 10^4 \text{ for Cu at 100 MHz.}$$

Skin depth

If the conductivity is high (i.e., $\sigma_0/\varepsilon_0\omega$ >>1) then from κ we derive the absorption coefficient:

$$\alpha = \frac{2\omega\kappa}{c_0} \approx \sqrt{\frac{2\sigma_0\omega}{\varepsilon_0 c_0^2}}$$

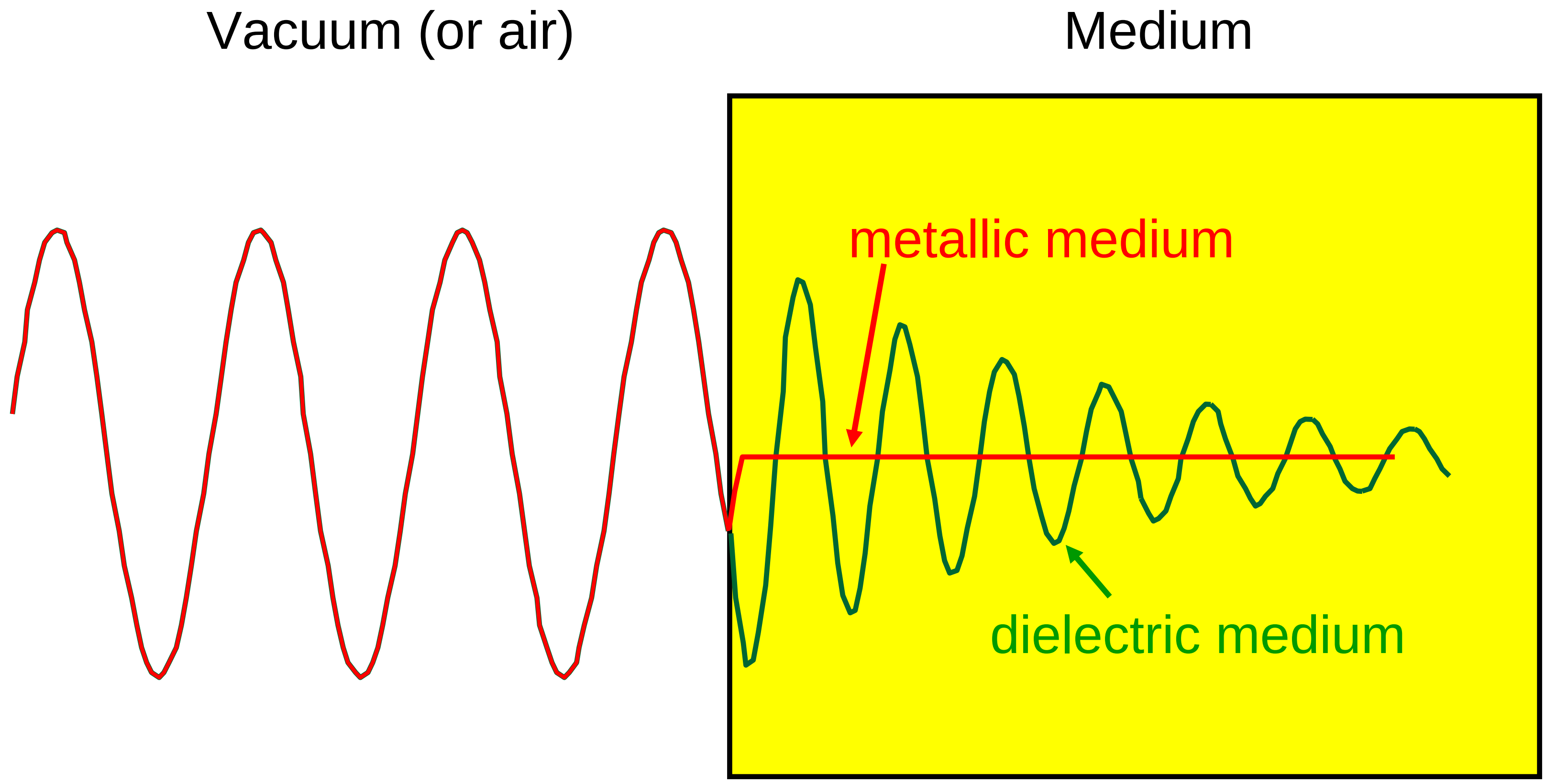
As we have seen, this is a very large number for metals.

"Skin depth" or "penetration depth":
depth of propagation of light into a metallic surface = $1/\alpha$

For metals, this depth is much less than the wavelength.

Example: copper at ν = 100 MHz
$\longrightarrow$ skin depth is 3.2 μm, about λ_0/900,000

Attenuation of waves entering a medium



Real refractive index of metals

If the conductivity is high (i.e., $\sigma_0/\varepsilon_0\omega$ >>1) then the refractive index is also large:

$$n = \kappa = \sqrt{\frac{\sigma_0}{2\varepsilon_0\omega}} = 7.32\times10^4 \text{ for Cu at 100 MHz.}$$

We will soon learn that the reflectance of an object depends on its refractive index:

$$R = \left(\frac{n-1}{n+1}\right)^2$$

For Cu at 100 MHz, we find: $R = 0.9999455$

$\longrightarrow$ metals make very good mirrors

When is our guess likely to be wrong?

Because σ_0 is REAL, our guess, $J(t) = \sigma_0 E(t)$, implies that the current is always in phase with the incident electromagnetic wave!

Recall: in the Drude model, the electrons are free to move - they are not bound to atoms by “springs”.

So, for low or moderate frequencies, this guess is ok.

But at a high enough frequency, it MUST fail.

So, we are back to the inhomogeneous wave equation.

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c_0^{\ 2}} \frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P}{\partial t^2}$$

what goes here?

The "polarization" when there is current

Let's go back to our forced oscillator model.
Newton's law F=ma gave us:

$$\frac{d^2 x_e(t)}{dt^2} = -\omega_0^2 x_e(t) + 2\Gamma \frac{dx_e(t)}{dt} + \frac{eE_0}{m_e} e^{-j\omega t}$$

$$\omega_0{}^2 = \frac{\text{spring constant}}{m_e}$$

resonant frequency of the spring

frictional damping

force due to the incident light field

From this, we found the polarization:

(see lecture 7)

$$P(t) = \left[\frac{Ne^2 / m_e}{\omega_0^2 - \omega^2 - j2\omega\Gamma} \right] E(t)$$

For a Drude metal, there is no spring holding the electrons. So what if we take $\omega_0 = 0$?

The plasma frequency

$$P(t) = -\left[\frac{Ne^2 / m_e}{\omega^2 + j2\omega\Gamma} \right] E(t)$$

Define a new constant, the “plasma frequency” ω_P:

$$\omega_P{}^2 = \frac{Ne^2}{\varepsilon_0 m_e}$$

Thus $$P(t) = -\varepsilon_0 \frac{\omega_P{}^2}{\omega^2 + j2\omega\Gamma} E(t)$$

We now use this as a ‘new and improved’ guess: plug this in for the polarization term in the wave equation.

Back to the wave equation

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c_0^{\ 2}}\frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P}{\partial t^2} = -\mu_0 \varepsilon_0 \frac{\omega_P^{\ 2}}{\omega^2 + j2\omega\Gamma}\frac{\partial^2 E}{\partial t^2}$$

But this is the same as a problem we solved earlier:

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c_0^{\ 2}}\left(1 - \frac{\omega_P^{\ 2}}{\omega^2 + j2\omega\Gamma}\right)\frac{\partial^2 E}{\partial t^2} = 0$$

This is the wave equation for a wave propagating in a uniform medium, if we define the refractive index of the medium as:

$$n_{metal}^{\ 2}(\omega) = 1 - \frac{\omega_P^{\ 2}}{\omega^2 + j2\omega\Gamma}$$

So this must be the (complex) refractive index for a metal.

The new and improved result

Instead of making a ‘guess’ that $\dfrac{dP}{dt} = \sigma_0 E(t)$,

we make the better guess that $P(t) = -\varepsilon_0 \dfrac{{\omega_P}^2}{\omega^2 + j2\omega\Gamma} E(t)$

and then the Drude model (plus the wave equation) predict the optical properties of metals as:

$$n_{metal}(\omega) = \sqrt{1 - \frac{{\omega_P}^2}{\omega^2 + j2\omega\Gamma}} \quad \text{or} \quad \varepsilon_{metal}(\omega) = \varepsilon_0 \left(1 - \frac{{\omega_P}^2}{\omega^2 + j2\omega\Gamma}\right)$$

At low frequencies, $\omega << \Gamma$, this new result gives the same answer as our first guess, as long as we identify τ (the Drude scattering time) with $1/2\Gamma$.

High frequency dielectric of metals

How does this dielectric function behave at higher frequencies, e.g., $\omega >> \Gamma$?

$$\varepsilon(\omega) = \varepsilon_0 \left(1 - \frac{\omega_P{}^2}{\omega^2 + j2\omega\Gamma} \right)$$

In this limit, we find:

$$\varepsilon(\omega) \approx \varepsilon_0 \left(1 - \frac{\omega_P{}^2}{\omega^2} \right)$$

The dielectric function becomes purely a real number.

And, it is negative below the plasma frequency and positive above the plasma frequency.

Some numbers:

Recall from Drude theory, that $\tau \sim 10^{-14}$ sec, so $\Gamma \sim 1/\tau \sim 10^{14}$ Hz.

(corresponding to the frequency of infrared light)

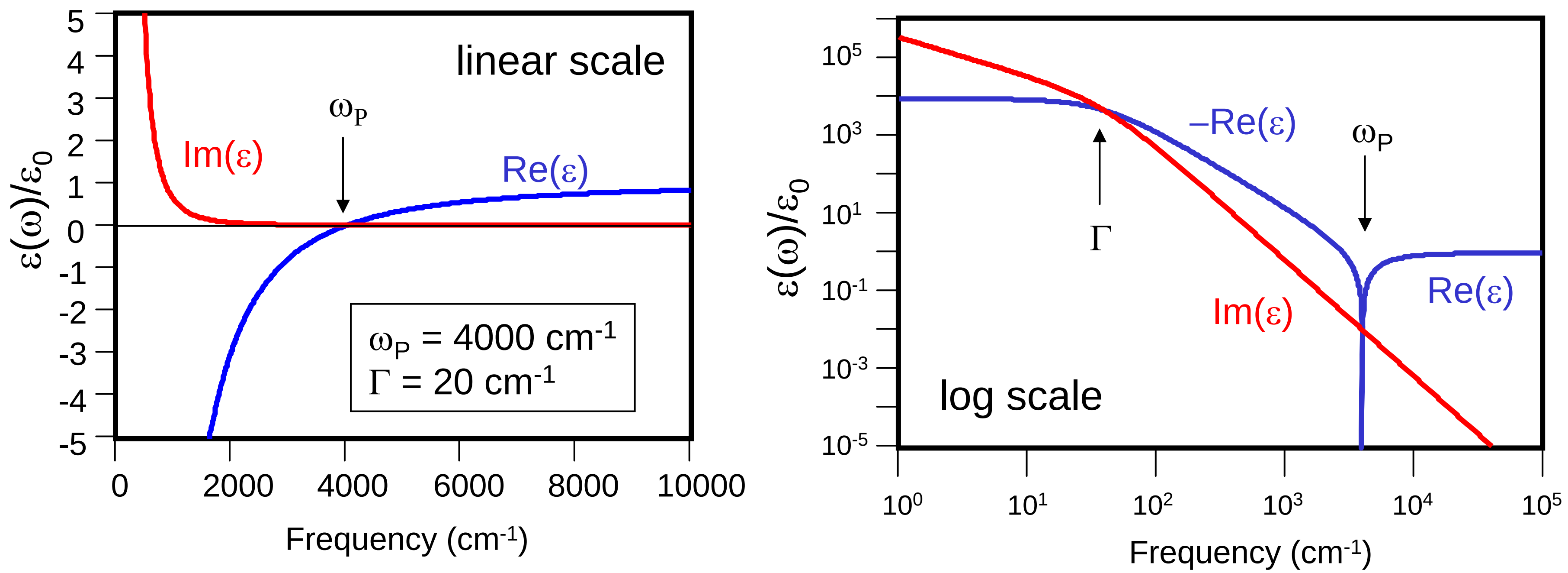
For a typical metal, ω_P is 100 or even 1000 times larger.

(corresponding to the frequency of ultraviolet light)

Dielectric function of metals

$$\varepsilon(\omega) = \varepsilon_0 \left(1 - \frac{\omega_P^{\,2}}{\omega^2 + j2\omega\Gamma} \right)$$

A plot of Re(ε) and Im(ε) for illustrative values:



- imaginary part gets very small for high frequencies
- real part has a zero crossing at the plasma frequency
- real and imaginary parts are equal in magnitude at $\omega = \Gamma$

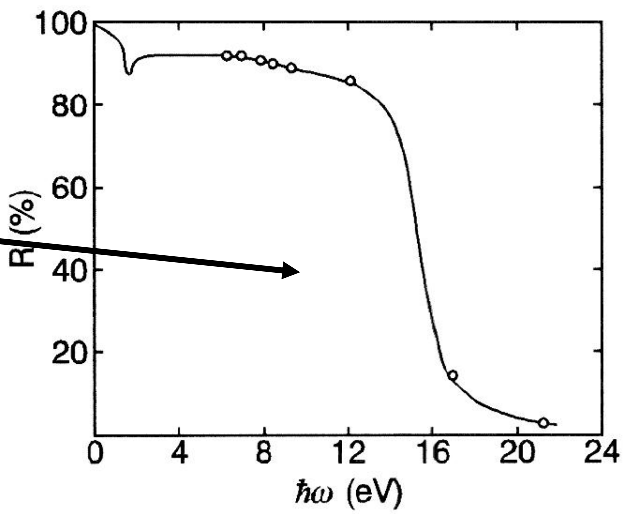
High frequency optical properties

In the regime where $\omega >> \Gamma$, we find: $n(\omega) = \sqrt{1 - \frac{{\omega_P}^2}{\omega^2}}$

For frequencies below the plasma frequency, n is complex, so the wave is attenuated and does not propagate very far into the metal.

For high frequencies above the plasma frequency, n is real. The metal becomes transparent! It behaves like a non-absorbing dielectric medium.

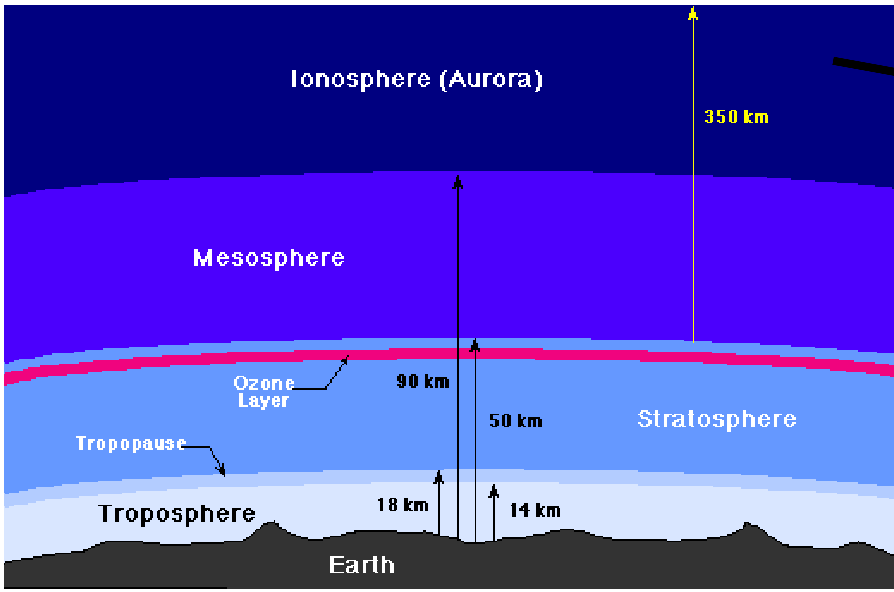
Aluminum
(H. Raether, Springer Tracts in Mod. Phys. Vol 38 (1965)



reflectivity drops abruptly at the plasma frequency

This is why x-rays can pass through metal objects.

Another example: the ionosphere



the uppermost part of the atmosphere, where many of the atoms are ionized. There are a lot of free electrons floating around here…

For N ~ 10^{12} m^{-3}, the plasma frequency is:

$$\omega_P = \sqrt{\frac{Ne^2}{\varepsilon_0 m_e}} = 2\pi \times 9 \text{ MHz}$$

Radiation above 9 MHz is transmitted, while radiation at lower frequencies is reflected back to earth.

That's why AM radio broadcasts can be heard very far away.

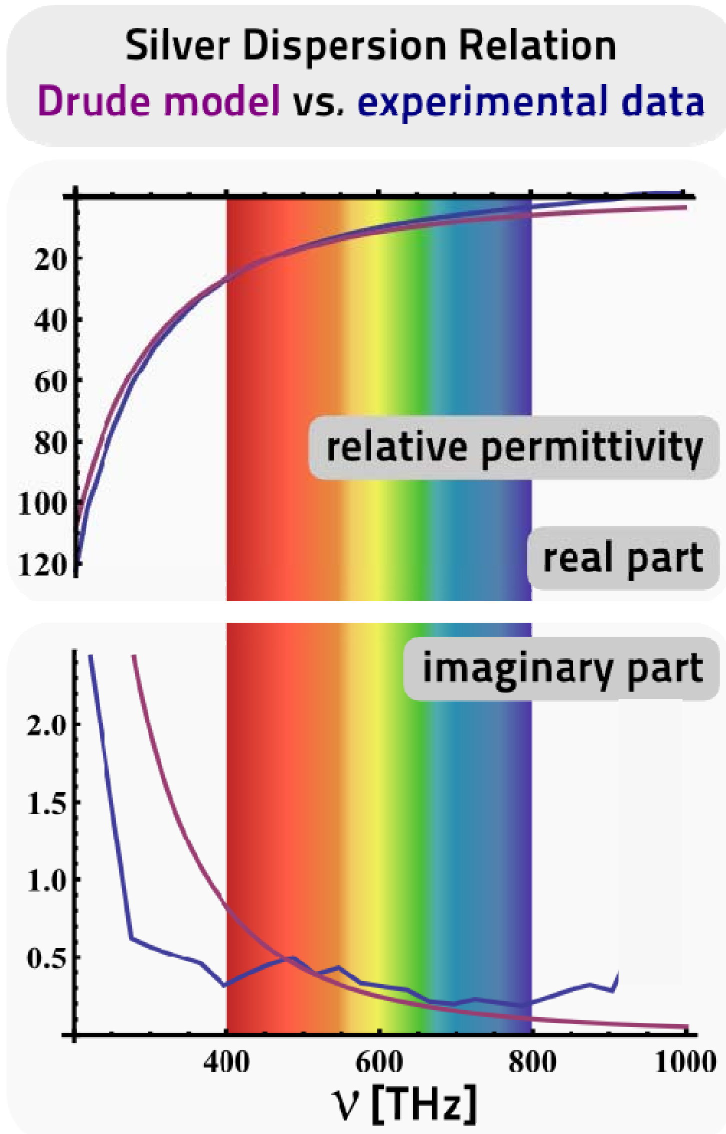
Drude theory: it works pretty well

large and negative (below ω_P)

$\varepsilon_{real} / \varepsilon_0$

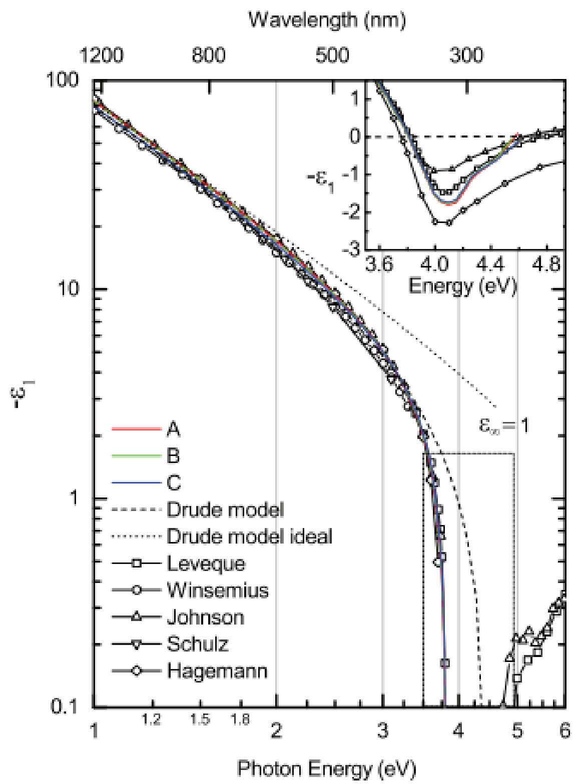
$\varepsilon_{imag} / \varepsilon_0$

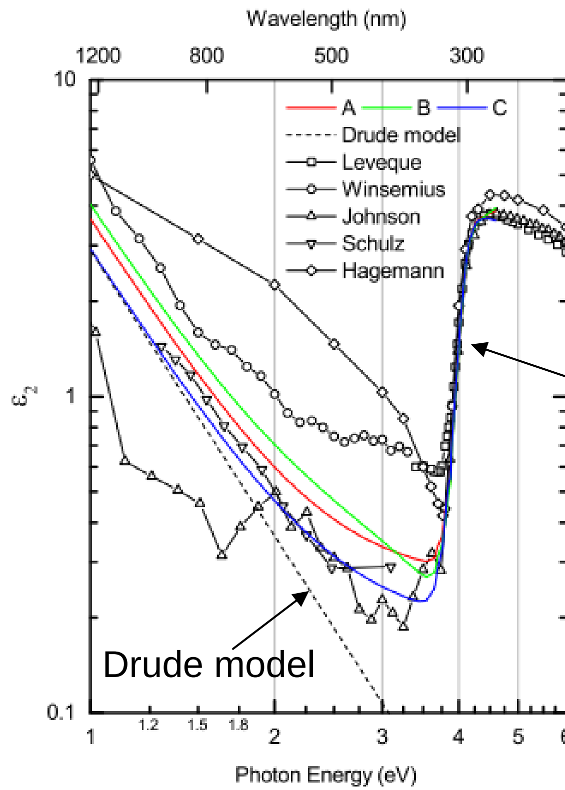
small and positive



… but not perfectly

“Optical dielectric function of silver,” Phys. Rev. B vol. 91, 235137 (2015)





dramatic departure from Drude model

Plasma frequency ~ 9 eV. So the stuff at ~4 eV is not due to ω_P.

They are due to inter-band (valence-to-conduction band) transitions.
This stuff is why gold is gold-colored and silver is silver-colored.

The full explanation of this requires the quantum theory of solids.