Spatio-Temporal Characteristics of Light Pulses and How to Model Them

Spatio-temporal distortions

Angular dispersion

Spatial chirp

Pulse front tilt

A first step in how to model them:
Ray matrices
Spatio-Temporal Distortions

Ordinarily, we assume that the pulse-field spatial and temporal factors (or their Fourier-domain equivalents) separate:

\[ E(x, y, z, t) = E_{xyz}(x, y, z) E_t(t) \]

\[ \tilde{E}(x, y, z, \omega) = E_{xyz}(x, y, z) \tilde{E}_t(\omega) \]

\[ \hat{E}(k_x, k_y, k_z, t) = \hat{E}_{xyz}(k_x, k_y, k_z) E_t(t) \]

\[ \hat{E}(k_x, k_y, k_z, \omega) = \hat{E}_{xyz}(k_x, k_y, k_z) \hat{E}_t(\omega) \]

where the tilde and hat mean FTs with respect to \( t \) and \( x, y, z \)

Sometimes, this separation of variables is not possible.
Angular dispersion is an example of a spatio-temporal distortion.

In the presence of angular dispersion, the off-axis k-vector component $k_x$ depends on $\omega$:

$$\frac{dk_{x0}}{d\omega}$$

where $k_{x0}(\omega)$ is the mean $k_x$ of light of frequency $\omega$. $$E(k_x, k_y, k_z, \omega) = E_0[k_x - \frac{dk_{x0}}{d\omega}(\omega - \omega_0), k_y, k_z, \omega]$$
**Spatial chirp** is a spatio-temporal distortion in which the color varies spatially across the beam.

Propagation through a prism pair produces a beam with no angular dispersion, but with *spatial* dispersion, often called spatial chirp.

Prism pairs are inside nearly every ultrafast laser.
Spatial chirp is difficult to avoid.

Simply propagating through a tilted window causes spatial chirp!

Because ultrashort pulses are so broadband, this distortion is very noticeable—and often problematic!
How to think about spatial chirp

Suppose we send the pulse through a set of monochromatic filters and find the beam center position, $x_0$, for each frequency, $\omega$.

$$\tilde{E}(x, y, z, \omega) \rightarrow \tilde{E}[x - \frac{dx_0}{d\omega}(\omega - \omega_0), y, z, \omega]$$

where $x_0$ is the center of the beam component of frequency $\omega$. 
Pulse fronts vs. phase fronts

Light always propagates in a direction perpendicular to the phase fronts.

Pulse fronts: refers to the intensity not the phase

These lines represent surfaces of constant phase.

This line represents a surface of constant intensity.
Pulse-front tilt is another common spatio-temporal distortion.

Because the group velocity is usually less than phase velocity, pulse fronts tilt when light traverses a prism.
Diffraction gratings also yield pulse-front tilt.

The path is simply shorter for rays that impinge on the near side of the grating.

Of course, angular dispersion and spatial chirp occur, too.

Gratings yield about ten times more pulse-front tilt than prisms do.
Pulse-Front Tilt from a Grating

For a diffraction grating, use a grazing (large) incidence angle (for largest PFT).

In the limit of grazing incidence: The extra distance traveled by the ray that impinges on the back edge of the grating is \( d \), where \( d \) is the length of the grating.

But, in the time it takes for this ray to travel this extra distance, the distance traveled by the ray that impinges on the front edge is also \( d \).

So the maximum pulse-front tilt angle achievable using a grating is given by:

\[
\tan(\varphi) = d/d, \text{ or } \varphi = \sim45^\circ.
\]
Pulse-Front Tilt from an Etalon

Let the incidence angle be $\theta$, which must be small.

The lowest ray passes straight through the etalon and experiences minimal delay.

The distance $x$ traveled upward on the etalon in one round trip is $2t \sin \theta \approx 2t \theta$, where $t$ is the thickness of the etalon (neglecting the refractive index of the etalon glass for now). So, if $d$ is the width of the etalon, the number of round trips before the pulse bumps into the top edge of the etalon is:

$$N = \frac{d}{x} = \frac{d}{(2t \theta)}$$

So the total extra distance traveled by the highest (and most delayed) ray is:

$$N \times 2t = \left[ \frac{d}{(2t \theta)} \right] \times 2t = \frac{d}{\theta}$$
The tangent of the total pulse-front tilt angle $\varphi$ is then the total extra distance traveled by the uppermost ray divided by the width of the etalon $d$:

$$\tan(\varphi) = \left( \frac{d}{\theta} \right) / d = 1/\theta$$

$$\Rightarrow \quad \tan(\varphi) = 1/\theta$$

Interestingly, the etalon’s width and thickness cancel out.

We can take into account the refractive index $n$ of the etalon material by noting that, inside the etalon, $\theta \rightarrow \theta/n$, and also the extra distance (actually time) traveled effectively increases by a factor of $n$ (assuming that the group velocity is slower, about $c/n$), yielding:

$$\tan(\varphi) = n^2/\theta$$

Note that, as the tangent $\rightarrow \infty$, the tilt angle $\varphi$ goes to 90º.
Pulse-front tilt involves coupling between the space and time domains:

\[ E(x, y, z, t) \rightarrow E[x, y, z, t - \frac{dt_0}{dx} (x - x_0)] \]

For a given transverse position in the beam, \( x \), the pulse mean time, \( t_0 \), varies in the presence of pulse-front tilt.
Angular dispersion always causes pulse-front tilt!

Angular dispersion means that the off-axis k-vector depends on $\omega$:

$$\hat{E}(k_x, k_y, k_z, \omega) = \hat{E}_0[k_x - \gamma(\omega - \omega_0), k_y, k_z, \omega]$$

where $\gamma = dk_x/\omega$

Inverse Fourier-transforming with respect to $k_x$, $k_y$, and $k_z$ yields:

$$\tilde{E}(x, y, z, \omega) = \tilde{E}_0(x, y, z, \omega) e^{-i\gamma(\omega-\omega_0)x}$$

using the shift theorem

Inverse Fourier-transforming with respect to $\omega$ (or $\omega - \omega_0$) yields:

$$\Rightarrow E(x, y, z, t) \propto E_0(x, y, z, t - \gamma x)$$

using the shift theorem again

which is just pulse-front tilt!
The combination of spatial and temporal chirp also causes pulse-front tilt.

The theorem we just proved assumed no spatial chirp, however. So it neglects another contribution to the pulse-front tilt.

The total pulse-front tilt is the sum of that due to angular dispersion and that due to this effect.
A pulse with temporal chirp, spatial chirp, and pulse-front tilt.

Suppressing the y-dependence, we can plot such a pulse:

\[ \psi = 11.3 \text{ mrad} \]

where the pulse-front tilt angle is:

\[ \psi \equiv c \frac{\partial t_0}{\partial x} \]

We need a nice formalism for calculating these distortions!
Spatio-temporal distortions can be useful or inconvenient.

Good:

- They allow pulse compression.
- They help to measure pulses (tilted pulse fronts).
- They allow pulse shaping.
- They can increase bandwidth and conversion efficiency in some nonlinear-optical processes.

Bad:

- They usually increase the pulse length.
- They reduce intensity.
- They can be hard to measure.
The Optic Axis

A mirror deflects the optic axis into a new direction with the angle of reflection equal to the angle of incidence.

This ring laser has an optic axis that scans out a rectangle.

We define all rays relative to the relevant optic axis.
Choosing the Optic Axis

We always try to choose the optic axis to make the problem as simple as possible. Fortunately, we have the freedom to do so.

Here, the beam propagates back and forth inside a laser, so we can use two different coordinate systems, one for the beam propagating to the right with $z$ increasing to the right, and another for the beam propagating to the left with $z$ increasing to the left.
At every position, $z$, along the optic axis, a light ray can be defined by two co-ordinates:

- its position, $x$
- its slope, $\theta$

These parameters define a \textbf{ray vector}, which will change with distance, $z$, as the ray propagates through optics.
Ray Matrices

An optical element’s effect on a ray is found by multiplying the ray vector by the element’s ray matrix.

\[
\begin{bmatrix}
    x_{after\ lens} \\
    \theta_{after\ lens}
\end{bmatrix} =
\begin{bmatrix}
    A & B \\
    C & D
\end{bmatrix}
\begin{bmatrix}
    x_{before\ lens} \\
    \theta_{before\ lens}
\end{bmatrix}
\]

We can do the same for the other lenses and the distances.

For many optical components, we can define 2 x 2 ray matrices.
Ray Matrices as Derivatives

Since the displacements, $x_{in}$ and $x_{out}$, and angles, $\theta_{in}$ and $\theta_{out}$, are all assumed to be small, we can think in terms of partial derivatives.

$$x_{out} = \frac{\partial x_{out}}{\partial x_{in}} x_{in} + \frac{\partial x_{out}}{\partial \theta_{in}} \theta_{in}$$

$$\theta_{out} = \frac{\partial \theta_{out}}{\partial x_{in}} x_{in} + \frac{\partial \theta_{out}}{\partial \theta_{in}} \theta_{in}$$

We can write these equations in matrix form.
For cascaded elements, we simply multiply together all the individual ray matrices.

\[
\begin{bmatrix}
    x_{in} \\
    \theta_{in}
\end{bmatrix}
\xrightarrow{O_1} \begin{bmatrix}
    O_1
\end{bmatrix}
\xrightarrow{O_2} \begin{bmatrix}
    O_2
\end{bmatrix}
\xrightarrow{O_3} \begin{bmatrix}
    x_{out} \\
    \theta_{out}
\end{bmatrix}
\]

Notice that the order looks opposite to what it should be, but it actually does make sense.
Ray Matrix for Free Space or a Medium

If $x_{in}$ and $\theta_{in}$ are the position and slope upon entering, let $x_{out}$ and $\theta_{out}$ be the position and slope after propagating an arbitrary distance, $z$.

Rewriting these expressions in matrix notation:

$$
\begin{pmatrix}
  x_{out} \\
  \theta_{out}
\end{pmatrix} =
\begin{bmatrix}
  1 & z \\
  0 & 1
\end{bmatrix}
\begin{pmatrix}
  x_{in} \\
  \theta_{in}
\end{pmatrix}
$$

Small angle approximation: $\sin \theta \approx \theta$
Ray Matrix for an Interface

At the interface:

\[ x_{\text{out}} = x_{\text{in}} \]

Now calculate \( \theta_{\text{out}} \):

Snell's Law says:

\[ n_1 \sin(\theta_{\text{in}}) = n_2 \sin(\theta_{\text{out}}) \]

which, for small angles, becomes:

\[ n_1 \theta_{\text{in}} = n_2 \theta_{\text{out}} \]

\[ \Rightarrow \theta_{\text{out}} = \left[ n_1 / n_2 \right] \theta_{\text{in}} \]

\[ O_{\text{interface}} = \begin{bmatrix} 1 & 0 \\ 0 & n_1 / n_2 \end{bmatrix} \]
Ray Matrix for a Curved Interface

At the interface, again:

\[ x_{out} = x_{in}. \]

To calculate \( \theta_{out} \), we must calculate \( \theta_1 \) and \( \theta_2 \).

\( \theta_s \) is the surface slope at the height \( x_{in} \).

\[ \theta_1 = \theta_{in} + \theta_s \quad \text{and} \quad \theta_2 = \theta_{out} + \theta_s \]

\[ \theta_1 = \theta_{in} + x_{in} / R \quad \text{and} \quad \theta_2 = \theta_{out} + x_{in} / R \]

Snell's Law: \( n_1 \theta_1 = n_2 \theta_2 \quad \Rightarrow \quad n_1 (\theta_{in} + x_{in} / R) = n_2 (\theta_{out} + x_{in} / R) \)

\[ \Rightarrow \theta_{out} = \frac{(n_1 / n_2) (\theta_{in} + x_{in} / R) - x_{in} / R}{n_1 / n_2} \]

\[ \Rightarrow \theta_{out} = (n_1 / n_2) \theta_{in} + (n_1 / n_2 - 1) x_{in} / R \]

\[
O_{\text{curved interface}} = \begin{bmatrix}
1 & 0 \\
\frac{(n_1 / n_2 - 1)}{R} & n_1 / n_2
\end{bmatrix}
\]
A thin lens is just two curved interfaces.

We’ll neglect the glass in between (it’s a really thin lens!), and we’ll take $n_1 = 1$.

$$O_{\text{curved interface}} = \begin{bmatrix} 1 & 0 \\ (n_1 / n_2 - 1) / R & n_1 / n_2 \end{bmatrix}$$

$$O_{\text{thin lens}} = O_{\text{curved interface}} \cdot O_{\text{curved interface}} = \begin{bmatrix} 1 & 0 \\ (n-1) / R_2 & n \end{bmatrix} \begin{bmatrix} 1 & 0 \\ [(1 / n) - 1] / R_1 & 1 / n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ (n-1) / R_2 + n[(1 / n) - 1] / R_1 & n(1 / n) \end{bmatrix}$$

This can be written:

$$\begin{bmatrix} 1 & 0 \\ -1 / f & 1 \end{bmatrix}$$

The Lens-Maker’s Formula

where: $1 / f = (n-1)(1 / R_1 - 1 / R_2)$
Ray Matrix for a Lens

\[
1/f = (n-1)(1/R_1 - 1/R_2)
\]

The quantity, \( f \), is the **focal length** of the lens. It’s the single most important parameter of a lens. It can be positive or negative.

\[
O_{\text{lens}} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}
\]

If \( f > 0 \), the lens deflects rays toward the axis.

If \( f < 0 \), the lens deflects rays away from the axis.

It’s easy to extend the Lens Maker’s Formula to real lenses of greater thickness.

Sign convention:
\( R > 0 \) if the sphere center is to the right \((z > 0)\), and \( R < 0 \) if the sphere center is to the left \((z < 0)\).
Types of lenses

Lens nomenclature

Which type of lens to use (and how to orient it) depends on the aberrations and application.
Ray Matrix for a Curved Mirror

Consider a mirror with radius of curvature, $R$, with its optic axis perpendicular to the mirror:

$$\theta_1 = \theta_{in} - \theta_s \quad \theta_s \approx \frac{x_{in}}{R}$$

$$\theta_{out} = \theta_1 - \theta_s = (\theta_{in} - \theta_s) - \theta_s \approx \theta_{in} - 2\frac{x_{in}}{R}$$

$$\Rightarrow O_{mirror} = \begin{bmatrix} 1 & 0 \\ -2/R & 1 \end{bmatrix}$$

Like a lens, a curved mirror will focus a beam. Its focal length is $R/2$.

Note that a flat mirror has $R = \infty$ and hence an identity ray matrix.
Laser Cavities

Mirror curvatures matter in lasers.

Two flat mirrors, the “flat-flat” laser cavity, is difficult to align and maintain aligned.

Two concave curved mirrors, the “stable” laser cavity, is easy to align and maintain aligned.

Two convex mirrors, the “unstable” laser cavity, is impossible to align!
A lens focuses parallel rays to a point one focal length away.

A lens followed by propagation by one focal length:

\[
\begin{bmatrix}
  x_{\text{out}} \\
  \theta_{\text{out}}
\end{bmatrix} =
\begin{bmatrix}
  1 & f \\
  0 & 1
\end{bmatrix}
\begin{bmatrix}
  1 & 0 \\
  -1/f & 1
\end{bmatrix}
\begin{bmatrix}
  x_{\text{in}} \\
  0
\end{bmatrix} =
\begin{bmatrix}
  0 & f \\
  -1/f & 1
\end{bmatrix}
\begin{bmatrix}
  x_{\text{in}} \\
  0
\end{bmatrix} =
\begin{bmatrix}
  0 \\
  -x_{\text{in}}/f
\end{bmatrix}
\]

At the focal plane, all rays converge to the \( z \) axis (\( x_{\text{out}} = 0 \)) independent of input position.

This illustrates the big problem with Ray Optics: how big is a focal spot?

For all rays

\( x_{\text{out}} = 0 \)!